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SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR IN SPHERES

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Abstract. Let M^n be a compact submanifold of a sphere with parallel mean curvature vector h. We introduce a tensor ϕ, related to h and the second fundamental form, and show that if ϕ satisfies a certain inequality, then $\phi\equiv 0$ and M^n is totally umbilic or equality holds. The estimate is sharp in the sense that we describe all M^n that satisfy this equality.

1. Introduction. Let M^n be an n-dimensional oriented manifold and let S_c^{n+p} be the $(n+p)$-sphere of constant sectional curvature c. Consider an isometric immersion $f: M^n \to S_c^{n+p}$. Fix a point $x \in M^n$ and a local orthonormal frame $\{e_1, \dots, e_{n+p}\}$ of S_c^{n+p} around x such that $\{e_1, \dots, e_n\}$ are tangent fields. For each α, $n+1 \le \alpha \le n+p$, define a linear map $A_\alpha: T_xM \to T_xM$ by

$$\langle A_\alpha X, Y\rangle = \langle \bar{\nabla}_X Y, e_\alpha\rangle ,$$

where X, Y are tangent fields and $\bar{\nabla}$ is the Riemannian connection of S_c^{n+p}. The square $|A^2|$ of the norm of the second fundamental form and the mean curvature vector h are defined by:

$$|A|^2 = \sum_{\alpha=n+1}^{n+p} \operatorname{tr}(A_\alpha^2) , \qquad h = \frac{1}{n} \sum_{\alpha=n+1}^{n+p} (\operatorname{tr} A_\alpha) e_\alpha .$$

When f is minimal (i.e., $h=0$), the following gap theorem is well-known. We will denote by $S^m(r)$ the m-sphere of radius r.

(1.1) THEOREM (cf. [CdCK]). *Let M^n be compact and let $f: M^n \to S_c^{n+p}$ be a minimal immersion. Assume that*

$$|A|^2 \le \frac{nc}{(2-1/p)} ,$$

for all $x \in M$. Then either

(i) $|A|^2 = 0$ *and M^n is totally geodesic*

or

(ii) $|A|^2 = nc/(2-1/p)$ *and one of the following cases occurs:*

(a) $p=1$ *and M^n is a minimal Clifford hypersurface*

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$$M^n = S^m\left(\left(\frac{m}{nc}\right)^{1/2}\right) \times S^{n-m}\left(\left(\frac{n-m}{nc}\right)^{1/2}\right) \subset S_c^{n+1}.$$

(b) *$n=p=2$ and M^2 is a Veronese surface in S_c^4.*

In this paper we will describe some sharp generalizations of the above theorem for submanifolds of constant mean curvature.

A natural hypothesis to replace the minimal condition is to require the submanifold M^n to have parallel mean curvature vector (i.e., $\nabla^{\perp} h = 0$). Note that this condition implies that $H = |h|$ is constant on M^n and if $p=1$ then these two conditions are equivalent.

For codimension one, Alencar and do Carmo [AdC] were able to generalize Theorem (1.1) to submanifolds with constant mean curvature. Before stating their result, we need some definitions. For each α, $n+1 \le \alpha \le n+p$, define linear maps $\phi_\alpha \colon T_xM \to T_xM$ by

$$(1.2) \qquad \langle \phi_\alpha X, Y\rangle = \langle X, Y\rangle\langle h, e_\alpha\rangle - \langle A_\alpha X, Y\rangle,$$

and a bilinear map $\phi \colon T_xM \times T_xM \to T_xM^{\perp}$ by

$$(1.3) \qquad \phi(X, Y) = \sum_{\alpha=n+1}^{n+p} \langle \phi_\alpha X, Y\rangle e_\alpha .$$

It is easy to check that each map ϕ_α is traceless and that

$$(1.4) \qquad |\phi|^2 := \sum_{\alpha=n+1}^{n+p} \operatorname{tr} \phi_\alpha^2 = |A|^2 - nH^2 .$$

Observe that $|\phi|^2 = 0$ if and only if M^n is a totally umbilic submanifold of S_c^{n+p}.

It turns out that $|\phi|^2$ is a natural invariant to use when extending Theorem (1.1) to submanifolds $M^n \subset S_c^{n+p}$ of constant mean curvature.

An $H(r)$-torus in S_c^{n+1} is obtained by considering canonical immersions $S^{n-1}(r) \to \boldsymbol{R}^n$ and $S^1(r_1) \to \boldsymbol{R}^2$, $r^2 + r_1^2 = c^{-1}$ and taking the product immersion $S^{n-1}(r) \times S^1(r_1) \to \boldsymbol{R}^n \times \boldsymbol{R}^2$. Our choice of the radii r and r_1 implies that this immersion is, in fact, contained in S_c^{n+1} and has constant mean curvature given by

$$H(r) = \frac{nr^2c^2 - (n-1)}{nr(1 - c^2r^2)^{1/2}}.$$

For each $H \in \boldsymbol{R}$, define the polynomial $P_H(x)$ by

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - n(H^2 + c)$$

and let B_H be the square of the positive root of $P_H(x) = 0$. The following result is a natural generalization of Theorem (1.1) for $p = 1$.

(1.5) THEOREM (cf. [AdC]). *Let M^n be a compact oriented hypersurface of S_c^{n+1}. Assume that the mean curvature H is constant and that*

$$|\phi|^2 \leq B_H ,$$

for all $x \in M^n$. Then either

(i) $|\phi|^2 = 0$ *and M^n is totally umbilic*

or

(ii) $|\phi|^2 = B_H$ *and one of the following cases occurs*:

(a) $H=0$ *and M^n is a Clifford hypersurface in S_c^{n+1}*,

(b) $H \neq 0$, $n \geq 3$ *and M^n is an H-torus,*

$$M^n = S^{n-1}(r_1) \times S^1(r_2) \subset S_c^{n+1} ,$$

where $r_1^2 + r_2^2 = c^{-1}$ and $r_1^2 < (n-1)/nc$,

(c) $H \neq 0$, $n=2$ *and M^2 is an H-torus,*

$$M^2 = S^1(r_1) \times S^1(r_2) \subset S_c^3 ,$$

where $r_1^2 + r_2^2 = c^{-1}$ and $r_1^2 \neq 1/2c$.

The purpose of this paper is to extend the above result to the case $p \geq 2$. We will denote by ϕ_h the bilinear map defined by

$$\phi_h(X, Y) = \langle \phi(X, Y), h \rangle ,$$

and by $B_{p,h}$ the function of p and h given by

$$B_{p,h} = \begin{cases} 1/(2-1/p) , & \text{if } p=1 \text{ or } h=0 , \\ 1/(2-(1/(p-1))) , & \text{otherwise} . \end{cases}$$

For simplicity, we will write $S_c^m \underset{u}{\rightarrow} S_c^{m+q}$ to mean that S_c^m is an umbilic submanifold of S_c^{m+q}. The following is the main result of the paper.

(1.6) THEOREM. *Let M^n be a compact orientable submanifold of S_c^{n+p}. Assume the mean curvature vector h is parallel with respect to the normal connection. If $|\phi|$ satisfies*

$$|\phi|^2 \leq B_{p,h} \left\{ n(c+H^2) - \frac{n(n-2)}{\sqrt{n(n-1)}} |\phi_h| \right\} , \tag{1.7}$$

then either

(i) $|\phi| = 0$ *and M^n is totally umbilic*

or

(ii) *the equality holds in* (1.7) *and one if the following cases occurs*:

(a) $H=0$, $p=1$ *and M^n is a minimal Clifford hypersurface*

$$M^n = S^m\left(\left(\frac{m}{nc}\right)^{1/2}\right) \times S^{n-m}\left(\left(\frac{n-m}{nc}\right)^{1/2}\right) \hookrightarrow S_c^{n+1} .$$

(b) *$H=0$, $n=p=2$ and M^2 is a Veronese surface, $M^2 \to S_c^4$.*

(c) *$H \neq 0$, $p=1$ and M^n is an H-torus,*

$$M^n = S^{n-1}(r_1) \times S^1(r_2) \to S_c^{n+1},$$

where $r_1^2 + r_2^2 = c^{-1}$. If $n \geq 3$, we have only those H-tori which satisfy $r_1^2 < (n-1)/nc$; if $n=2$, the only condition is $r_1^2 \neq 1/2c$.

(d) *$H \neq 0$, $p=2$ and M^n is a minimal Clifford hypersurface in a hypersphere*

$$M^n = S^m\left(\left(\frac{m}{n(c+H^2)}\right)^{1/2}\right) \times S^{n-m}\left(\left(\frac{n-m}{n(c+H^2)}\right)^{1/2}\right) \subset S_{c+H^2}^{n+1} \underset{u}{\to} S_c^{n+2}.$$

(e) *$H \neq 0$, $p=2$ and for all H_2, $0 < H_2 \leq H$, M^n is an H_1-torus,*

$$M^n = S^{n-1}(r_1) \times S^1(r_2) \subset S_{c+H_2^2}^{n+1} \underset{u}{\to} S_c^{n+2},$$

where $H_1^2 + H_2^2 = H^2$, $r_1^2 + r_2^2 = (c+H_2^2)$. If $n \geq 3$ we have only those H_1-tori with $r_1^2 < (n-1)/n(c+H_2^2)$; if $n=2$ the only condition is $r_1^2 \neq 1/2(c+H_2^2)$.

(f) *$H \neq 0$, $n=2$, $p=3$ and M^2 is a Veronese surface in a hypersphere*

$$M^2 \subset S_{c+H^2}^4 \underset{u}{\to} S_c^5.$$

(1.8) REMARK. The results in part (ii) of Theorems (1.1), (1.5) and (1.6) are local.

(1.9) REMARK. Observe that if $h=0$, the condition (1.7) reduces to

$$|A|^2 \leq \frac{nc}{(2-1/p)},$$

and for $p=1$, (1.7) is

$$|\phi|^2 \leq n(c+H^2) - \frac{n(n-2)H}{\sqrt{n(n-1)}}|\phi|,$$

which is equivalent to $|\phi|^2 \leq B_H$. Hence Theorem (1.6) generalizes Theorems (1.1) and (1.5).

The paper is divided in two sections. In the first section, we estimate the Laplacian $\Delta|\phi|$ of $|\phi|$ and prove Part (i) of Theorem (1.6). In the second section we prove some classification theorems which will imply Theorem (1.6)-(ii). In particular, we prove a further generalization (Proposition (3.3)) of Theorem (1.5), in which, we assume that the submanifold has flat normal bundle.

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2. Theorem (1.6); part (i). If $H\equiv 0$, the theorem reduces to Theorem (1.1) and if $p=1$, it is Theorem (1.5). So we will consider the case where $H\neq 0$ and $p\geq 2$.

We first compute the Laplacian $\Delta|\phi|$ of $|\phi|$. For that purpose, we choose a local field of orthonormal frames $\{e_1,\ldots,e_{n+p}\}$ in such way that $e_{n+1}=h/H$. With this choice,

$$(2.1)\qquad \begin{cases} \phi_{n+1}=H\,\mathrm{id}-A_{n+1}\,, \\ \phi_\alpha \quad =-A_\alpha\,, \qquad n+2\leq\alpha\leq n+p\,. \end{cases}$$

We observe that

$$\Delta|\phi|^2=\Delta(|A|^2-nH^2)=\Delta|A|^2\,,$$

since the hypothesis implies that $H=|h|$ is constant on M^n.

Erbacher [E] calculated $\Delta|A|^2$ and obtained:

$$(2.2)\qquad \frac{1}{2}\Delta|A|^2=\sum_{\alpha=n+1}^{n+p}|\nabla A_\alpha|^2+nc|A|^2-c\sum_{\alpha=n+1}^{n+p}(\mathrm{tr}\,A_\alpha)^2+\sum_{\alpha,\beta>n+1}^{n+p}\mathrm{tr}([A_\alpha,A_\beta])^2$$
$$+\sum_{\alpha,\beta=n+1}^{n+p}(\mathrm{tr}\,A_\alpha)(\mathrm{tr}\,A_\alpha A_\beta^2)-\sum_{\alpha,\beta=n+1}^{n+p}(\mathrm{tr}\,A_\alpha A_\beta)^2\,,$$

where $[A_\alpha,A_\beta]=A_\alpha A_\beta-A_\beta A_\alpha$. By substituting (2.1) into (2.2), we obtain

(2.3) LEMMA. *With the above notation,*

$$(2.4)\qquad \frac{1}{2}\Delta|\phi|^2=\sum_{\alpha=n+1}^{n+p}|\nabla\phi_\alpha|^2+n(c+H^2)|\phi|^2-nH\sum_{a=n+1}^{n+p}\mathrm{tr}\,\phi_{n+1}\phi_\alpha^2$$
$$-\sum_{\alpha,\beta=n+1}^{n+p}(\mathrm{tr}\,\phi_\alpha\phi_\beta)^2+\sum_{\alpha,\beta>n+1}^{n+p}\mathrm{tr}([\phi_\alpha,\phi_\beta])^2\,.$$

(2.5) REMARK. In the proof of the above lemma we use the fact that if e_α is a parallel direction then $[A_\alpha,A_\beta]=0$, for all β.

In order to estimate the right hand side of (2.4), we need an algebraic lemma.

(2.6) LEMMA. *Let A, B: $\boldsymbol{R}^n\to\boldsymbol{R}^n$ be symmetric linear maps such that* $[A,B]=0$ *and* $\mathrm{tr}\,A=\mathrm{tr}\,B=0$. *Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}(\mathrm{tr}\,A^2)(\mathrm{tr}\,B^2)^{1/2}\leq\mathrm{tr}\,A^2B\leq\frac{n-2}{\sqrt{n(n-1)}}(\mathrm{tr}\,A^2)(\mathrm{tr}\,B^2)^{1/2}$$

and the equality holds on the right hand (resp. left hand) side if and only if $n-1$ of the eigenvalues x_i of A and the corresponding eigenvalues y_i of B satisfy

$$|x_i|=\frac{(\mathrm{tr}\,A^2)^{1/2}}{\sqrt{n(n-1)}}\,,\qquad x_ix_j\geq 0\,,$$

$$y_i = \frac{(\operatorname{tr} B^2)^{1/2}}{\sqrt{n(n-1)}} \left(\text{resp. } y_i = -\frac{(\operatorname{tr} B^2)^{1/2}}{\sqrt{n(n-1)}}\right).$$

In the proof of the above Lemma we need the following

(2.7) SUBLEMMA. *Let x_i, $i=1,\ldots,n$, be real numbers such that $\sum_{i=1}^n x_i=0$ and $\sum_{i=1}^n x_i^2=1$. Then*

$$\sum_{i=1}^n x_i^4 \le \frac{(n-2)^2}{n(n-1)} + \frac{1}{n},$$

and the equality holds if and only if $n-1$ of the x_i's are equal.

PROOF OF SUBLEMMA (2.7). The above upper bound is obtained in [C]. We give a complete proof here since we need it in proving the equality in (2.7).

We first observe that the result follows by direct calculation for $n=2, 3$. Suppose $n \ge 4$ and consider an n-tuple $(x_1,\ldots,x_n)$ such that $\sum_i x_i=0$ and $\sum_i x_i^2=1$.

The polynomial

$$\mathscr{P}(x) = \prod_{i=1}^n (x-x_i) = x^n - \frac{1}{2}x^{n-2} + \frac{1}{3}\left(\sum_i x_i^3\right)x^{n-3} + \frac{1}{8}\left(1 - 2\sum_i x_i^4\right)x^{n-4} + \ldots$$

has n real roots. Hence the $(n-4)$-th derivative $\mathscr{P}^{(n-4)}(x)$ of $\mathscr{P}(x)$ has four real roots. This implies that the equation

$$x^4 - \frac{6}{n(n-1)}x^2 + \frac{8(\sum_i x_i^3)}{n(n-1)(n-2)}x + \frac{3(1-2\sum_i x_i^4)}{n(n-1)(n-2)(n-3)} = 0 \tag{2.8}$$

has only real roots. Since an equation of the form $x^4+6Ax^2+4Bx+C=0$ has only real roots if $C+3A^2 \ge 0$, we get

$$\sum_{i=1}^n x_i^4 \le \frac{(n-2)^2}{n(n-1)} + \frac{1}{n}.$$

Let us now analyze the equality case. If

$$\sum_{i=1}^n x_i^4 = \frac{(n-2)^2}{n(n-1)} + \frac{1}{n},$$

then $\mathscr{P}^{(n-4)}(x)=0$ has three equal roots. This implies that $\mathscr{P}^{(n-3)}(x)=0$ must have at least two equal roots. Hence

$$x^3 - \frac{3}{n(n-1)}x + \frac{2(\sum_i x_i^3)}{n(n-1)(n-2)} = 0$$

has two equal roots. Then

$$\left(\sum_{i=1}^{n} x_i^3\right)^2 = \frac{(n-2)^2}{n(n-1)},$$

and by Lemma (2.6) of [AdC], $n-1$ of the x_i's must be equal. □

PROOF OF LEMMA (2.6). Since $[A, B]=0$. we can choose an orthonomal basis of $\boldsymbol{R}^n$ which simultaneously diagonalizes A and B. So we must show that the eigenvalues $\{x_1, \dots, x_n\}$ of A and $\{y_i, \dots, y_n\}$ of B satisfy

$$\frac{-(n-2)}{\sqrt{n(n-1)}}\left(\sum_i x_i^2\right)\left(\sum_i y_i^2\right)^{1/2} \leq \sum_i x_i^2 y_i \leq \frac{(n-2)}{\sqrt{n-1}}\left(\sum_i x_i^2\right)\left(\sum_i y_i^2\right)^{1/2}.$$

Without loss of generality we may assume

$$\sum_i x_i^2 = \sum_i y_i^2 = 1 .$$

We are looking for the extremals of the function $\mathscr{F} : \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}$ defined by $\mathscr{F}(x_i, y_i) = \sum_{i=1}^n x_i^2 y_i$, with the constraints $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0$ and $\sum_{i=1}^n x_i^2 = \sum_{i=1}^n y_i^2 = 1$. By the method of Lagrange's multipliers, there exist constants C and D such that

$$\begin{cases} x_i y_i = C x_i + D , \\ x_i^2 = C y_i + 1/n \end{cases}$$

for all i. Observe that $C = \sum_{i=1}^n x_i^2 y_i$, i.e., C is the extremal we are looking for. Squaring both sides of the second equation above and summing up for all i, we obtain that

$$C^2 = \sum_{i=1}^{n} x_i^4 - \frac{1}{n}.$$

Sublemma (2.7) implies that

$$C^2 \leq \frac{(n-2)^2}{n(n-1)}.$$

Therefore,

$$-\frac{n-2}{\sqrt{n(n-1)}} \leq \sum_{i=1}^{n} x_i^2 y_i \leq \frac{n-2}{\sqrt{n(n-1)}}.$$

The equality part also follows from Sublemma (2.7). □

Now we are able to estimate $\Delta|\phi|^2$. The following proposition proves Theorem (1.6)-(i).

(2.9) PROPOSITION. *Let M^n be a compact oriented submanifold of S_c^{n+p}, $p \geq 2$. Assume* *that the mean curvature vector h is parallel. If*

$$|\phi|^2 \leq \frac{p-1}{2p-3}\left\{n(c+H^2)-\frac{n(n-2)}{\sqrt{n(n-1)}}|\phi_h|\right\}, \tag{2.10}$$

then $|\phi|$ *is constant and either* $|\phi|^2=0$ *or the equality holds in* (2.10).

PROOF. Let us begin by showing that (2.10) implies that $\Delta|\phi|^2 \geq 0$. For that purpose, we are going to estimate separately each term on the right hand side of (2.4): Assertion:

$$\sum_{\alpha,\beta>n+1} \operatorname{tr}[\phi_\alpha, \phi_\beta]^2 - \sum_{\alpha,\beta>n+1} (\operatorname{tr}\phi_\alpha\phi_\beta)^2 \geq -\left(2-\frac{1}{p-1}\right)(|\phi|^2-|\phi_{n+1}|^2)^2 . \tag{2.11}$$

This assertion follows in the same way as the estimate (3.8) of [CdCK], by using that

$$\operatorname{tr}[\phi_\alpha, \phi_\beta]^2 = -N(\phi_\alpha\phi_\beta - \phi_\beta\phi_\alpha) ,$$

where $N(A)=\operatorname{tr} A^tA$.
Assertion:

$$\sum_{\alpha\geq n+1} \operatorname{tr}\phi_{n+1}\phi_\alpha^2 \leq \frac{n-2}{\sqrt{n(n+1)}}|\phi_{n+1}||\phi|^2 . \tag{2.12}$$

Indeed, for each α, $n+1\leq\alpha\leq n+p$, we can apply Lemma (2.6) to the linear maps ϕ_α and ϕ_{n+1}. (We recall that $[\phi_\alpha, \phi_{n+1}]=0$ and $\operatorname{tr}\phi_\alpha=0$, for all α). So, we get

$$\operatorname{tr}\phi_{n+1}\phi_\alpha^2 \leq \frac{n-2}{\sqrt{n(n-1)}}|\phi_\alpha|^2|\phi_{n+1}| .$$

Summing up in α, we obtain (2.12).
Assertion:

$$\sum_{\alpha>n+1} (\operatorname{tr}\phi_{n+1}\phi_\alpha)^2 \leq |\phi_{n+1}|^2(|\phi|^2-|\phi_{n+1}|^2) . \tag{2.13}$$

This is a consequence of Cauchy-Schwarz's inequality.

Let us now return to the estimate of the right hand side of (2.4). By using (2.11)–(2.13) in (2.4), we obtain

$$\frac{1}{2}\Delta|\phi|^2 \geq n(c+H^2)|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|\phi_h||\phi|^2 - \left(2-\frac{1}{p-1}\right)|\phi|^4$$
$$+\left(1-\frac{1}{p-1}\right)|\phi_{n+1}|^2(2|\phi|^2-|\phi_{n+1}|^2) .$$

Since

$$\left(1-\frac{1}{p-1}\right)|\phi_{n+1}|^2(2|\phi|^2-|\phi_{n+1}|^2)\geq 0\,.$$

We obtain

$$\frac{1}{2}\Delta|\phi|^2\geq|\phi|^2\left\{n(c+H^2)-\frac{n(n-2)}{\sqrt{n(n-1)}}|\phi_h|-\left(\frac{2p-3}{p-1}\right)|\phi|^2\right\}. \tag{2.14}$$

Thus, the hypothesis (2.10) implies that

$$\Delta|\phi|^2\geq 0.$$

It follows, by Hopf's maximum principle, that $|\phi|^2$ is constant. Then $\Delta|\phi|^2=0$ and by (2.14), either $|\phi|^2=0$ or the equality holds in (2.10). □

3. Proof of Theorem (1.6); part (ii). In this section we classify the compact submanifolds of S_c^{n+p} such that $|\phi|^2$ satisfies the equality in (1.7).

We will consider various particular cases of Theorem (1.6) and will show later that the proof reduces to these particular cases. Some of the cases are interesting in their own right.

First we will assume that M^n is pseudo-umbilical. We recall that a submanifold is pseudo-umbilical if the mean curvature vector is non-zero and lies in an umbilical direction. In this case $|\phi_h|=0$. The following classification result is also proved in [Ch].

(3.1) PROPOSITION. *Let M^n be a compact pseudo-umbilical submanifold of S_c^{n+p}, $p\geq 2$, with parallel mean curvature vector h, $H=|h|$. If $|\phi|$ satisfies*

$$|\phi|^2\leq\frac{n(c+H^2)}{2-1/(p-1)}, \tag{3.2}$$

then either

(i) $|\phi|=0$ *and M^n is totally umbilic*

or

(ii) $|\phi|^2=n(c+H^2)/(2-1/(p-1))$ *and M^n is a minimal Clifford hypersurface in $S^{n+1}_{c+H^2}\xrightarrow[u]{}S_c^{n+2}$ or M^2 is a Veronese surface in $S^4_{c+H^2}\xrightarrow[u]{}S_c^5$.*

PROOF. Observe first that if the immersion f has a parallel umbilic direction, it is a composite of the form $f=g_1\circ g_2$ where $g_2: M^n\to S_{\tilde c}^{n+p-1}$ is an isometric immersion and $g_1: S_{\tilde c}^{n+p+1}\to S_c^{n+p}$ is an umbilical immersion. In our case, the umbilic direction is the mean curvature direction; so that g_2 must be minimal. We will denote by α_i the second fundamental form of g_i and by $|A_i|$ the norm of α_i.

Since $|\phi_h|=0$ (h is an umbilical direction) we have $|A_2|=|\phi|$. In terms of $|A_2|$, the hypothesis (3.2) reads

$$|A_2|^2 \leq \frac{n(c+H^2)}{2-1/(p-1)}.$$

The result will follow from Theorem (1.1) if we show that $\tilde{c}=c+H^2$. To see that this is the case, we first observe that

$$\alpha=\alpha_1 \oplus \alpha_2 ,$$

where α is the second fundamental form of f. Since g_2 is a minimal immersion and g_1 is an umbilical one,

$$|\operatorname{tr}\alpha_1|=|\operatorname{tr}(\alpha_1 \oplus \alpha_2)|=nH \qquad \text{and} \qquad |A_1|^2=nH^2 .$$

Gauss' Equation for the immersion g_2 implies then that $\tilde{c}=c+H^2$, as we claimed. □

Next we will consider compact submanifolds of S_c^{n+p} with parallel mean curvature vector and $R^{\perp}\equiv 0$. The following proposition also generalizes Theorem (1.5), since, in the codimension one case, we always have $R^{\perp}\equiv 0$.

(3.3) PROPOSITION. *Let M^n be a compact oriented submanifold of S_c^{n+p}. Assume that the mean curvature vector h is parallel and $R^{\perp}\equiv 0$. If $|\phi|$ satisfies*

$$|\phi|^2 \leq n(c+H^2)-\frac{n(n-2)}{\sqrt{n(n-1)}}|\phi_h| , \tag{3.4}$$

then either

(i) $|\phi|^2=0$ *and M^n is totally umbilic*

or

(ii) *the equality holds in* (3.4) *and one of the following cases occurs*:

(a) *M^n is a minimal Clifford hypersurface*

$$S^m\left(\sqrt{\frac{m}{n(n+H^2)}}\right)\times S^{n-m}\left(\sqrt{\frac{n-m}{n(c+H^2)}}\right)\subset S^{n+1}_{c+H^2} \overrightarrow{u} S_c^{n+p} .$$

(b) *For all H_2, $0\leq H_2<H$, M^n is an H_1-torus $S^{n-1}(r_1)\times S^1(r_2)\subset S^{n+1}_{c+H_2^2} \overrightarrow{u} S_c^{n+p}$, where $H_1^2+H_2^2=H^2$, $r_1^2+r_2^2=(c+H_2^2)^{-1}$. If $n\geq 3$, we only have those tori with $r_1^2<(n-1)/n(c+H_1^2)$; if $n=2$, then the only condition is $r_1^2\neq 1/2(c+H_2^2)$.*

PROOF. The proof of part (i) is similar of the proof of Proposition (2.9), observing that if $R^{\perp}\equiv 0$, then $[\phi_\alpha, \phi_\beta]=0$, for all α, β. Let us analyze what happens when we have the equality in (3.4). In this case, we obtain

$$\sum_{\alpha=n+1}^{n+p}|\nabla\phi_\alpha|^2=0 , \tag{3.5}$$

and

$$\sum_{\alpha>n+1} \operatorname{tr} \phi_{n+1}\phi_\alpha^2 = \frac{n-2}{\sqrt{n(n-1)}} |\phi|^2 |\phi_{n+1}| . \tag{2.6}$$

Let (ϕ_α) be the matrices which define the maps ϕ_α's. By the equality part of Lemma (2.6), there exists an orthonormal frame $\{e_1, \dots, e_n\}$ of TM^n such that, in this frame, ϕ_α's are of the form:

$$(\phi_\alpha) = a_\alpha \begin{pmatrix} 1 & & & \\ & \ddots & & 0 \\ & 0 & 1 & \\ & & & -(n-1) \end{pmatrix},$$

where $a_\alpha = \pm(\operatorname{tr} \phi^2)^{1/2}/\sqrt{n(n-1)}$. Because of (3.5), it follows that a_α is constant for each α. This implies that the first normal space of f, namely

$$N_1(x) = \operatorname{span}\left\{\sum_\alpha A_\alpha(X)e_\alpha\,,\ X \in T_xM\right\},$$

has constant dimension in M^n. It is easy to see that $\dim N_1(x) \le 2$, for all $x \in M^n$. Since $R^\perp = 0$ and $\nabla^\perp h = 0$, N_1 is a parallel normal bundle (see [D1]). Hence it is possible to reduce the codimension of the immersion to two (see Prop. 4.1, 45 in [D2]).

First case: $H = 0$. In this case $\phi_\alpha = -A_\alpha$ for all α and $\dim N_1(x) \equiv 1$. So we reduce the codimension to one. Since $|\phi|^2 = |A|^2$ and $|\phi_h| = 0$, our hypothesis implies that $|A|^2 = nc$. By Theorem (1.1), M^n is a minimal Clifford hypersurface in $S_c^{n+1} \subset S_c^{n+p}$.

Second case: $H \ne 0$. After reducing the codimension, we can assume that $M^n \subset S_c^{n+2}$. First we will show that M^n has a parallel umbilic direction. For this purpose, choose a new orthonormal frame $\{\xi_1, \xi_2\}$ of $TM^\perp$ by

$$\xi_i = \frac{1}{(a_{n+1}^2 + a_{n+2}^2)}(a_{n+2}e_{n+1} + (-1)^i a_{n+1}e_{n+2}) .$$

These vector fields are parallel in the normal connection and ξ_2 is an umbilic direction.

Since we have a parallel umbilic direction, the immersion f is a composite of the form $f = g_1 \circ g_2$ where $g_2 : M^n \to S_{\bar c}^{n+1}$ is an isometric immersion and $g_1 : S_{\bar c}^{n+1} \to S_c^{n+2}$ is an umbilic immersion.

If we define new linear maps $\phi_i \colon T_xM \to T_xM$ by

$$\langle \phi_i(X), Y\rangle = \langle \phi(X, Y), \xi_i\rangle\,, \qquad i = 1, 2\,,$$

then $|\phi|^2 = |\phi_1|^2 + |\phi_2|^2$. Since ξ_2 is an umbilic direction, $|\phi_2| = 0$. Hence

$$|\phi|^2 = |\phi_1|^2 . \tag{3.7}$$

It also holds that

$$|\phi_h|=|H_1||\phi_1|, \tag{3.8}$$

where $H_1=\langle h,\xi_1\rangle$.

By substituing (3.7) and (3.8) into the equality of (3.4), we obtain

$$|\phi_1|=n(c+H^2)-\frac{n(n-2)}{\sqrt{n(n-1)}}|H_1||\phi_1|.$$

Now the result follows from Theorem (1.5), since Gauss' equation for the umbilic immersion of $S_{\tilde{c}}^{n+1}$ in S_c^{n+2} implies that $\tilde{c}=c+H_2^2$. □

(3.9) REMARK. For $H=0$, Proposition (3.3) was proved by Kenmotsu in [K].

PROOF OF THEOREM (1.6)-(ii). If $H=0$ or $p=1$ we have nothing to prove, since these cases reduce to Theorem (1.1) and Theorem (1.5), respectively. Let us suppose $H\neq 0$ and $p\geq 2$.

If the equality holds in (1.7), then $\Delta|\phi|^2=0$, and this implies that all estimates used to obtain (2.9) are equalities. Thus,

$$\left(1-\frac{1}{p-1}\right)|\phi_{n+1}|^2(2|\phi|^2-|\phi_{n+1}|^2)=0.$$

Since $(2|\phi|^2-|\phi_{n+1}|^2)>0$, we have $p=2$ or $|\phi_{n+1}|=0$.

First case: $p=2$. Since M^n has a parallel normal direction, by hypothesis, it is easy to see that $R^{\perp}\equiv 0$. Then the theorem is a consequence of Proposition (3.3).

Second case: $|\phi_{n+1}|=0$. Notice that in this case the immersion is pseudo-umbilical, since when $|\phi_\alpha|=0$, e_α is characterized as an umbilical direction. Hence Proposition (3.1) implies the result. □

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