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MINIMAL SURFACES IN R^3 WITH DIHEDRAL SYMMETRY

WAYNE ROSSMAN

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Abstract. We construct new examples of immersed minimal surfaces with catenoid ends and finite total curvature, of both genus zero and higher genus. In the genus zero case, we classify all such surfaces with at most $2n+1$ ends, and with symmetry group the natural Z_2 extension of the dihedral group D_n.

The surfaces are constructed by proving existence of the conjugate surfaces. We extend this method to cases where the conjugate surface of the fundamental piece is noncompact and is not a graph over a convex plane domain.

1. Introduction. Recently, new examples of immersed minimal surfaces of finite total curvature with catenoid ends have been found. Among these examples are: the genus-zero Jorge-Meeks n-oid with symmetry group $D_n \times Z_2$ [JoMe], the genus-one n-oid with symmetry group $D_n \times Z_2$ [BeRo], the genus-zero Platonoids with symmetry groups isomorphic to the symmetry groups of the Platonic solids [Xu], [Kat], [UmYa], and the higher genus Platonoids with Platonic symmetry groups [BeRo]. (See Figures 1(1)–(4), 2(1).) By $D_n \times Z_2$, we mean the natural Z_2 extension into $O(3)$ of the dihedral group $D_n \subset SO(3)$.

In this present work we find more examples with symmetry group $D_n \times Z_2$ (see Figures 2(2)–(4), 3(1)–(3)), of both genus zero and higher genus. Then, in the genus zero case, we classify all such surfaces that have at most $2n+1$ ends.

To prove existence of these surfaces we use the conjugate surface construction, by an approach similar to that of [BeRo]. Generally speaking, the conjugate surface construction seems to require a high degree of symmetry of the surface. In fact, all of the known techniques for creating examples of minimal surfaces seem to benefit from symmetry assumptions.

The examples we construct here are less symmetric than the examples mentioned in the first paragraph, in the sense that their fundamental pieces have higher total Gaussian curvature. It is therefore harder to prove existence of the conjugate surfaces to these fundamental pieces. Hence the constructions we need are more delicate than those in [BeRo]. For less symmetric surfaces, the conjugates may no longer lie over convex plane domains, thus making Nitsche's theorem no longer applicable. In fact, they may not even be graphs, or may not even be embedded.

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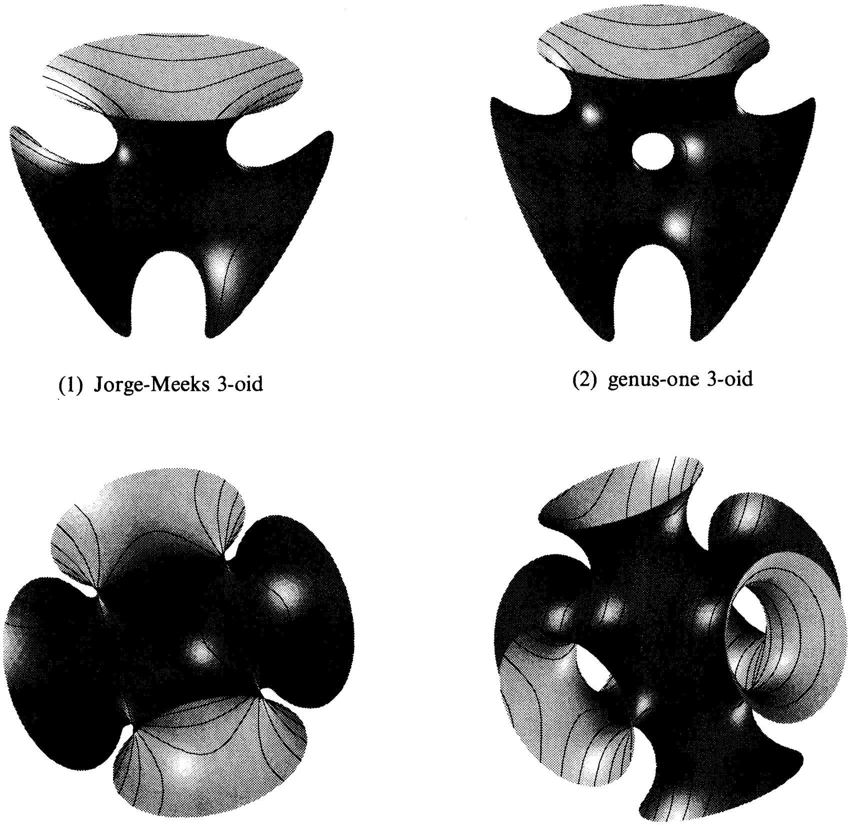


(1) Jorge-Meeks 3-oid

(2) genus-one 3-oid

(3) genus-zero tetroid

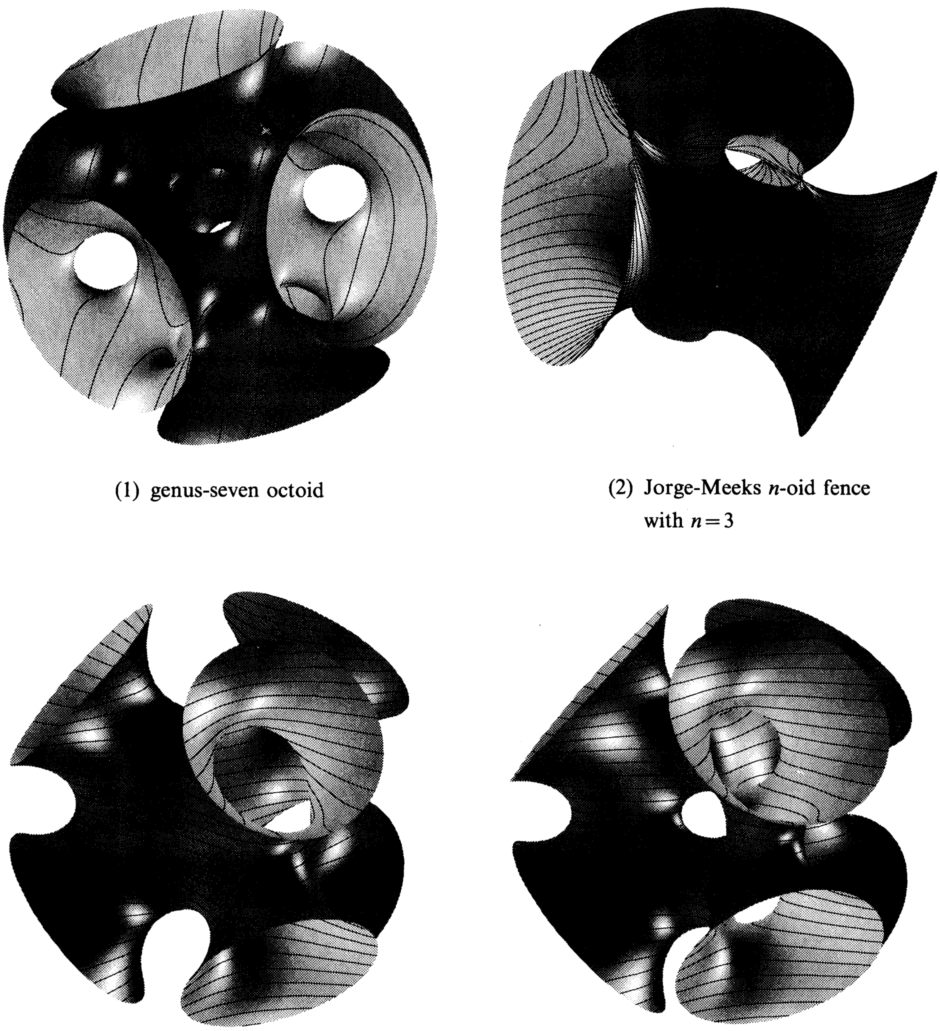
(4) genus-zero octoid

FIGURE 1.

Another consideration is the period problem. In general, integration of the Weierstrass integral (described in the next section) about a nontrivial cycle produces a period vector. Translation by this period vector in $\boldsymbol{R}^3$ will produce an isometry of the surface. Thus, if the surface has finite total curvature, the period vectors must be the zero vector for all cycles. Ensuring that this is the case is called "removing" or "killing" the periods.

Often it is possible to remove one period with an intermediate-value-theorem argument, but for surfaces with more than one period cycle this will not suffice. If there are two periods to remove, it is usually quite difficult to theoretically argue that both can be removed simultaneously. In our case we are able to make an argument to kill two periods (the proof of Theorem 1.2).

Since it is well known [Os] that the Gauss map of a finite-total-curvature minimal surface extends continuously to each end of the surface, we shall simply refer to the



(1) genus-seven octoid

(2) Jorge-Meeks n-oid fence with $n=3$

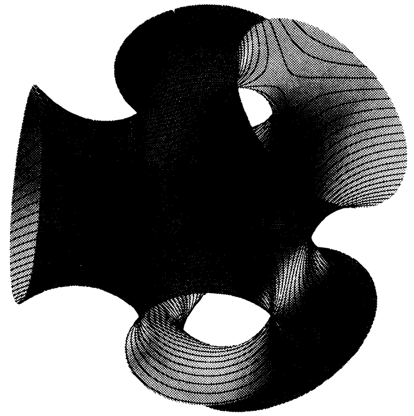
(3) $\mathscr{P}_0(2n, \theta)$ for $n=3$ and $\theta=45$ degrees

(4) $\mathscr{P}_{n-1}(2n, \theta)$ for $n=3$ and $\theta=45$ degrees

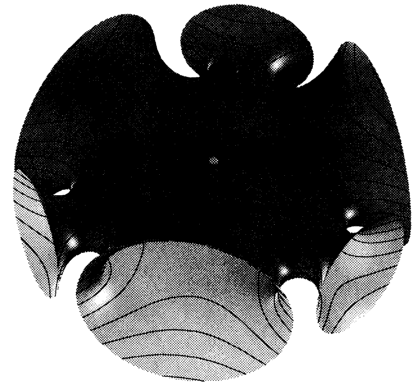
FIGURE 2.

extended Gauss map at an end as the normal vector at that end.

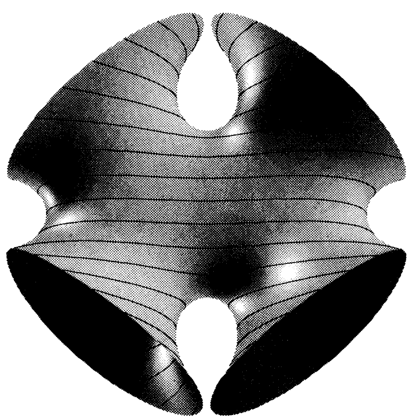
The naming scheme for the surfaces discussed here is $\diamond_A(B, C)$, where: A is the genus of the surface, B is the number of ends of the surface, and C represents the parameter for a one-parameter family of surfaces. C is either an angle θ or a weight w of an end, and is omitted when there is no relevant one-parameter family. In each case $\diamond$ is replaced by hopefully informative lettering. ($\mathscr{P}$ represents "prismoid", $\mathscr{J}\mathscr{M}$



(1) $\mathscr{JMV}_0(n+2, w)$ for $n=3$ and $w \approx 5/2$



(2) $\mathscr{AW}_0(2n, w)$ for $n=3$ and $w \approx 5/2$



(3) $\mathscr{AA}_0(2n, \theta)$ for $n=2$ and $\theta=30$ degrees

FIGURE 3.

represents "Jorge-Meeks surface", $\mathscr{JMV}$ represents "Jorge-Meeks surface with added vertical ends", $\mathscr{AW}$ represents "surfaces with alternately weighted ends", and $\mathscr{AA}$ represents "surfaces with alternating angles between ends".)

THEOREM 1.1 (the prismoids). *For each* $n \geq 2$, *there exists a one-parameter family of immersed minimal surfaces* $\mathscr{P}_0(2n, \theta)$, $0 < \theta < \pi/2$, *satisfying the following*:

(1) $\mathscr{P}_0(2n, \theta)$ *has genus zero.*

(2) $\mathscr{P}_0(2n, \theta)$ *has* $2n$ *catenoid ends, and the normal vector at each end makes an angle* θ *with a horizontal plane.*

(3) *The symmetry group of* $\mathscr{P}_0(2n, \theta)$ *is* $D_n \times \boldsymbol{Z}_2$.

THEOREM 1.2 (the higher genus prismoids). *For each* $n \geq 2$, *there exists a one-parameter family of immersed minimal surfaces* $\mathscr{P}_{n-1}(2n, \theta)$, $0 < \theta < \pi/2$, *satisfying the following*:

(1) $\mathscr{P}_{n-1}(2n, \theta)$ *has genus* $n-1$.

(2) $\mathscr{P}_{n-1}(2n, \theta)$ *has* $2n$ *catenoid ends, and the normal vector at each end makes an angle* θ *with a horizontal plane.*

(3) *The symmetry group of* $\mathscr{P}_{n-1}(2n, \theta)$ *is* $D_n \times \boldsymbol{Z}_2$.

THEOREM 1.3 (the genus-zero n-oids plus two ends). *For each* $n \geq 2$, *there exists a positive constant* $c(n)$ *so that, for any* $w \geq c(n)$, *there exists an immersed minimal surface* $\mathscr{J}\mathscr{M}\mathscr{V}_0(n+2, w)$ *satisfying the following conditions*:

(1) $\mathscr{J}\mathscr{M}\mathscr{V}_0(n+2, w)$ *has genus zero.*

(2) $\mathscr{J}\mathscr{M}\mathscr{V}_0(n+2, w)$ *has* n *catenoid ends with weight one, and the normal vectors at these ends all lie within a horizontal plane and are symmetrically placed.*

(3) $\mathscr{J}\mathscr{M}\mathscr{V}_0(n+2, w)$ *has two catenoid ends of weight* w *with vertical normals pointing in opposite directions.*

(4) *The symmetry group of* $\mathscr{J}\mathscr{M}\mathscr{V}_0(n+2, w)$ *is* $D_n \times \boldsymbol{Z}_2$.

By "symmetrically placed" in the second condition above, we mean that, up to a rotation of $\mathscr{J}\mathscr{M}\mathscr{V}_0(n+2, w)$ if necessary, the n ends with weight one have normal vectors whose stereographic projections to the complex plane are the n-th roots of unity. This arrangement of ends is the same as for the Jorge-Meeks surface. The remaining two ends of $\mathscr{J}\mathscr{M}\mathscr{V}_0(n+2, w)$ have normal vectors whose stereographic projections are $z=0$ and $z=\infty$.

Roughly speaking, the weight of a catenoid end is a measure of the size of the catenoid to which it is asymptotic. We give an exact definition in the next section. In the previous theorem, the condition $w \geq c(n)$ seems to be unnecessarily restrictive, but is necessary for the proof we give here [Xu], [KUY], [Kat].

THEOREM 1.4 (the $2n$-oids with alternating weights at the ends). *For each* $n \geq 2$ *there exists a one-parameter family of immersed minimal surfaces* $\mathscr{A}\mathscr{W}_0(2n, w)$, $0 < w < \infty$, *satisfying the following*:

(1) $\mathscr{A}\mathscr{W}_0(2n, w)$ *has genus zero.*

(2) $\mathscr{A}\mathscr{W}_0(2n, w)$ *has* $2n$ *catenoid ends. These ends have normal vectors all lying within a common plane and symmetrically placed. (That is, up to a rotation of the surface if necessary, these normal vectors stereographically project to the* $2n$*-th roots of unity.)*

(3) *Half of the* $2n$ *ends have weight one, and the other* n *ends have weight* w, *and they alternate between each other.*

(4) *The symmetry group of* $\mathscr{A}\mathscr{W}_0(2n, w)$ *is* $D_n \times \boldsymbol{Z}_2$.

THEOREM 1.5 (the $2n$-oids with alternating angles between the ends). *For each*

$n \geq 2$ there exists a one-parameter family of immersed minimal surfaces $\mathscr{A}\mathscr{A}_0(2n, \theta)$, $0 < \theta < \pi/n$, satisfying the following:

(1) *$\mathscr{A}\mathscr{A}_0(2n, \theta)$ has genus zero.*

(2) *$\mathscr{A}\mathscr{A}_0(2n, \theta)$ has $2n$ catenoid ends. These ends all have weight one, and have normal vectors all lying within a common plane.*

(3) *The angles between adjacent ends alternate between θ and $(2\pi - n\theta)/n$.*

(4) *The symmetry group of $\mathscr{A}\mathscr{A}_0(2n, \theta)$ is $D_n \times \boldsymbol{Z}_2$.*

THEOREM 1.6 (classification). *Any genus-zero catenoid-ended immersed minimal surface with symmetry group $D_n \times \boldsymbol{Z}_2$ and at most $2n+1$ ends is either the Jorge-Meeks n-oid, $\mathscr{P}_0(2n, \theta)$ for some $\theta \in \boldsymbol{R}$, $\mathscr{J}\mathscr{M}\mathscr{V}_0(n+2, w)$ for some $w \in \boldsymbol{R}$, $\mathscr{A}\mathscr{W}_0(2n, w)$ for some $w \in \boldsymbol{R}$, or $\mathscr{A}\mathscr{A}_0(2n, \theta)$ for some $\theta \in \boldsymbol{R}$.*

In this classification we do not need to place any restrictions on the ranges of θ or w. The proof of this theorem given in Section 4 is independent of the values of θ and w. (Note that $\mathscr{J}\mathscr{M}\mathscr{V}_0(n+2, w)$ with $w=0$ and $\mathscr{A}\mathscr{W}_0(2n, w)$ with $w=0$ are simply the Jorge-Meeks n-oid.) We caution, however, that we have not proven existence of $\mathscr{P}_0(2n, \theta)$ (resp. $\mathscr{A}\mathscr{A}_0(2n, \theta)$) when $\theta \notin (0, \pi/2)$ (resp. $\theta \notin (0, \pi/n)$), and of $\mathscr{J}\mathscr{M}\mathscr{V}_0(n+2, w)$ (resp. $\mathscr{A}\mathscr{W}_0(2n, w)$) when $w < c(n)$ (resp. $w < 0$).

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2. Preliminaries.

2.1. The conjugate surface construction. Consider a simply-connected finite-total-curvature immersed minimal surface M in $\boldsymbol{R}^3$ with a boundary consisting of a finite number of piecewise smooth curves. As proven by Enneper and Weierstrass, there exists a meromorphic function g and a holomorphic 1-form η defined on the unit disk in the complex plane such that M has the parametrization

$$\Phi(p) = \operatorname{Re} \int_{p_0}^{p} \begin{pmatrix} (1-g^2)\eta \\ i(1+g^2)\eta \\ 2g\eta \end{pmatrix}, \qquad p \in \{z \in \boldsymbol{C} \text{ such that } |z| \leq 1\}.$$

We refer to $\{g, \eta\}$ as the Weierstrass data for M, and to Φ as the Weierstrass representation of M. The map g is stereographic projection of the Gauss map from the sphere to the complex plane. The conjugate surface M_{conj} of M is the minimal surface with the same parametrization, but with Weierstrass data $\{g, i\eta\}$; that is, η is replaced with $i\eta$ in the parametrization above. We shall call this conjugate Weierstrass representation $\Phi_{\mathrm{conj}}(p)$. Note that the conjugate of the conjugate of M is given by the Weierstrass data $\{g, -\eta\}$, giving us the original surface reflected through the origin.

Thus we have the maps $z \mapsto \Phi(z)$ and $z \mapsto \Phi_{\mathrm{conj}}(z)$ from the unit disk to M and M_{conj}, respectively. This induces a map $\phi = \Phi_{\text{conj}} \circ \Phi^{-1}$, the *conjugate map*, from M to M_{conj}. The conjugate map ϕ is an isometry and preserves the Gauss map. It also has the following property, which we shall use in an essential way: ϕ maps planar principal curves in M to planar asymptotic curves in M_{conj}, and maps planar asymptotic curves in M to planar principal curves in M_{conj}. From this we can conclude that *ϕ maps non-straight planar geodesics to straight lines, and vice versa.* And since the Gauss map is preserved by ϕ, it follows that if ϕ maps a non-straight planar geodesic $\alpha \subset M$ to a line segment $\beta \subset M_{\text{conj}}$, the line segment β must be perpendicular to the plane containing α.

In the cases we consider here, M is bounded by piecewise smooth boundary curves that consist of a finite number of planar geodesics, and hence M_{conj} is bounded by piecewise smooth boundary curves that consist of a finite number of line segments, rays, and complete lines.

Recall from [Scn1] that an end of a complete minimal immersion in $\boldsymbol{R}^3$ is a *regular* end if a neighborhood of this end is a graph f with bounded slope over some plane (without loss of generality, the x_1x_2-plane), so that f has the following asymptotic behavior:

$$f(x_1, x_2) = a \log\left(\sqrt{x_1^2 + x_2^2}\right) + b + \frac{c_1 x_1 + c_2 x_2}{x_1^2 + x_2^2} + \mathcal{O}\left(\frac{1}{x_1^2 + x_2^2}\right).$$

If $a = 0$ we have a planar end, and if $a \neq 0$ we have a catenoid end. In this paper we shall use the terminology more loosely. We shall say that a minimal end is a catenoid end (resp. planar end) if it satisfies the above asymptotic condition with $a \neq 0$ (resp. $a = 0$), even if the minimal immersion has a nonempty boundary and there exist boundary curves which extend to the end. With this more general definition in mind, we shall say that a minimal surface has a *helicoid end* if the corresponding end of the conjugate surface is a catenoid end.

For more detailed information on the conjugate surface construction, see [Ka1], [Ka2], [Ka3], [Ka4], [BeRo].

2.2. Weights. Here we define a useful quantity (see [KKS]) that is a vector associated to each Jordan curve in a minimal surface. We then describe some properties of this weight vector that are pertinent to our situation.

DEFINITION 2.1. Let $\alpha_1, \ldots, \alpha_k$ be the boundary curves of a compact immersed minimal surface $M \subset \boldsymbol{R}^3$. Let υ be the outward pointing unit conormal of M along α_i. Then the *weight* (also called *flux*) of the boundary curve α_i is

$$w(\alpha_i) = \int_{\alpha_i} \upsilon \, ds .$$

It is a well-known application of Stoke's theorem that the weight vectors satisfy a "balancing" condition

$$\sum_{i=1}^{k} w(\alpha_i)=0 .$$

Furthermore, the weight vector can be defined by the same integral for any closed curve α on any complete oriented minimal surface M, up to a sign. The signature depends on the choice of orientation of the conormal along α.

It readily follows that the weight vector $w(\alpha)$ associated to each closed curve $\alpha\subset M$ is an invariant of homology. That is to say, if $\alpha, \beta\subset M$ are homologous closed curves with the same orientation of the conormal, then $w(\alpha)=w(\beta)$ (cf. [HoMe], [KKS]).

Thus, by considering any closed loop about each end of a complete finite-total-curvature immersed minimal surface, there is a well defined weight vector associated to each end. It is easily seen that an embedded finite-total-curvature end is a catenoid end if and only if it has a nonzero weight vector, and is a planar end if and only if its weight vector is zero.

We say that a set of n vectors is in a balanced configuration if their sum is zero. If a minimal surface M with n catenoid ends has these n vectors as the weight vectors of its ends, we say that this configuration of vectors is *realized* by M. Clearly, if the configuration is not balanced, it cannot be realized by any minimal surface.

Suppose a catenoid end E of a minimal surface M is asymptotic to a catenoid $\mathscr{C}$, and E has weight vector $w(E)$. From the Weierstrass representation, we can see that $\mathscr{C}$ is, up to a rigid motion of $\boldsymbol{R}^3$, a catenoid whose Weierstrass data is

$$g=\frac{1}{z}, \qquad \eta=\frac{|w(E)|}{4\pi}dz ,$$

where the base Riemann manifold is $\boldsymbol{C}\setminus\{0\}$. It follows that $|w(E)|$ is proportional to the "size" of E.

We can then see that $|w(E)|$ is the length of the fundamental period vector of the helicoid which is conjugate to $\mathscr{C}$. (If one also considers periods of the helicoid that do not preserve orientation, then the length of the fundamental period vector is $|w(E)|/2$.)

2.3. Known results. Before proving the theorems in this paper, we give some preliminary results that will be needed for the proofs. Here we state some results that come from previous works, and in Subsection 2.4 we prove three lemmas that are designed specifically for our purposes.

The following well-known lemma is the maximum principle for minimal surfaces. It is a special case of a lemma in [Scn1], and is proven there. We apply this lemma later in a variety of situations.

LEMMA 2.1. (Interior maximum principle.) *Let M_1 and M_2 be minimal surfaces in $\boldsymbol{R}^3$. Suppose p is an interior point of both M_1, M_2, and suppose $T_p(M_1)=T_p(M_2)$. If M_1 lies to one side of M_2 near p, then $M_1=M_2$.*

(Boundary point maximum principle.) *Suppose M_1, M_2 have C^2-boundaries C_1,*

C_2, respectively, and suppose p is a point of both C_1, C_2. Furthermore, suppose the tangent planes of both M_1, M_2 and C_1, C_2 agree at p: that is to say, suppose $T_p(M_1)=T_p(M_2)$, $T_p(C_1)=T_p(C_2)$. If, near p, M_1 lies to one side of M_2, then $M_1=M_2$.

The following theorems are special cases of a result by Meeks-Yau [MeYa], and a result by Nitsche [Ni], [JeSe], [BeRo], [MeYa]. These theorems will be needed later to show that the Plateau solutions for certain polygonal contours are embedded.

THEOREM 2.1. *Let $\hat{M}$ be a 3-dimensional compact submanifold of $\boldsymbol{R}^3$ so that $\partial\hat{M}$ is piecewise smooth and consists of the smooth pieces $\{H_1, \ldots, H_l\}$. Assume the following two conditions*:

(1) *Each H_i is a compact subset of some minimal surface in $\boldsymbol{R}^3$.*

(2) *Whenever H_i and H_j meet along a curve, the angle between the two surfaces is at most 180 degrees, with respect to the region $\hat{M}$.*

Let α be a Jordan curve in $\partial\hat{M}$. Then there exists a branched minimal immersion from a disk D into $\hat{M}$ with boundary α, which is smooth in the interior of D and has minimal area among all such maps. Furthermore, any branched minimal immersion of the above type must be an embedding.

THEOREM 2.2. *Let D be a bounded convex domain in a horizontal plane, so that its boundary ∂D is piecewise smooth. Let $\partial\tilde{D}=\partial D\setminus\{p_1, \ldots, p_r\}$. Then there exists a solution (as a graph over D) of the minimal surface equation in D taking on preassigned bounded continuous data on the arcs of $\partial\tilde{D}$. As a surface, this solution contains vertical line segments over the jump discontinuities of the boundary data.*

2.4. Lemmas. Consider a finite-topology minimal surface M (with boundary ∂M) with an end that is a 180 degree arc of a helicoid end. Denote a neighborhood of this end by E. By rotating if necessary, we may assume that the normal vector at this end is vertical. Suppose that outside a compact ball in $\boldsymbol{R}^3$ the boundary ∂E is a pair of straight (necessarily horizontal) rays r_1, r_2. The conjugate surface E_{conj} of E is a surface with a 180 degree arc of a catenoid end that, outside a compact ball in $\boldsymbol{R}^3$, is bounded by two infinite planar geodesics s_1, s_2 asymptotic to catenaries. The curves s_1, s_2 lie in parallel vertical planes. For this situation, we have the following lemma.

LEMMA 2.2. *The two planar geodesics s_1, $s_2\in\partial E_{\mathrm{conj}}$ lie in the same plane if and only if the two conjugate straight boundary rays r_1, $r_2\in\partial E$ lie in a common vertical plane.*

PROOF. Assume that r_1 and r_2 lie in a common vertical plane. Let Rot_1 be the 180 degree rotation about the line containing r_1, and let Rot_2 be the 180 degree rotation about the line containing r_2.

The surface $E\cup\mathrm{Rot}_2(E)$ is a smooth embedded end asymptotic to a 360 degree arc of a helicoid end; and outside of a compact ball in $\boldsymbol{R}^3$, it is bounded by two parallel rays r_1, $\mathrm{Rot}_2(r_1)$, which also lie in a common vertical plane.

We choose the orientation for $E\cup\mathrm{Rot}_2(E)$ so that the normal vector at the end is

$(0, 0, -1)$. Thus the Weierstrass data for this end can be given, in a punctured neighborhood of the origin in $\boldsymbol{C}$, as

$$g=c_1 z+\mathcal{O}(z^2)\,, \qquad \eta=\left(\frac{c_2}{z^2}+\frac{c_3}{z}+\mathcal{O}(1)\right)dz\,.$$

The conformal transformation $z\mapsto z/c_1$ preserves the origin, so we may therefore assume that $c_1=1$.

Since $\mathrm{Rot}_2\circ\mathrm{Rot}_1$ is a vertical translation, the surface $E\cup\mathrm{Rot}_2(E)$ is a portion of a helicoid end that is periodic in the x_3 direction. Therefore, in the Weierstrass representation with this data, integrating around a small circle $\{z\in\boldsymbol{C}$ such that $|z|=\varepsilon\}$ about the origin results in a vertical period. From an examination of the first two coordinates of the Weierstrass representation, we see that c_3 must be 0.

Now consider E_{conj} and its reflection $\mathrm{Ref}(E_{\mathrm{conj}})$ through the plane containing s_2. We wish to conclude that $\mathrm{Ref}(s_1)$ and s_1 are the same curve. The Weierstrass data for this catenoid end is

$$g=z+\mathcal{O}(z^2)\,, \qquad \eta=\left(\frac{ic_2}{z^2}+\mathcal{O}(1)\right)dz\,.$$

Since this is a catenoid end with vertical normal vector, it satisfies the asymptotic condition in Subsection 2.1, therefore it cannot have any periodicity in the x_3 direction. Examining the third coordinate of the Weierstrass representation for this data shows that c_2 is purely imaginary. It follows that integrating around $\{z\in\boldsymbol{C}$ such that $|z|=\varepsilon\}$ produces the zero vector. Thus $\mathrm{Ref}(s_1)$ and s_1 are indeed the same curve. Hence s_1 and s_2 lie in the same plane.

The above argument can be reversed to produce the converse conclusion. □

The following lemma will be needed later to extend compact embedded Plateau solutions to stable noncompact embedded minimal surfaces. We use the term stable in the following sense: A noncompact minimal surface M (possibly with boundary) is *stable* if the second derivative of area is nonnegative at M for all smooth variations of the surface with compact support (and fixing the boundary ∂M).

LEMMA 2.3. *Let $\{C_i\}_{i=1}^{\infty}$ be a sequence of compact Jordan contours in $\boldsymbol{R}^3$ so that the following conditions hold*:

(1) *Each C_i is a piecewise smooth contour consisting of a finite number of line segments.*

(2) *Each C_i bounds a least-area minimal disk M_i.*

(3) *For any ball B_R of radius R in $\boldsymbol{R}^3$, there exists $N_R\in\boldsymbol{Z}$ so that $C_i\cap B_R=C_j\cap B_R$ for any $i,j\geq N_R$.*

(4) *There exists a fixed $\delta>0$ and a compact 3-dimensional region $\hat{M}\subset B_{1/\delta}\subset\boldsymbol{R}^3$ so that $M_i\cap\hat{M}\neq\phi$ and* $\mathrm{dist}(C_i,\hat{M})>\delta$ *for all i.*

(5) *$\{C_i\}_{i=1}^{\infty}$ converges (in the topology of compact uniform convergence) to a*

noncompact contour C, and C is a piecewise smooth contour consisting of a finite number of line segments, rays, and complete lines.

Then a subsequence of $\{M_i\}_{i=1}^{\infty}$ converges to a nonempty stable minimal surface M (possibly disconnected) with boundary C. Furthermore, if each M_i is embedded, then M is embedded.

PROOF. Schoen [Scn2] has proven that the Gaussian curvature on a stable minimal surface $M \subset \boldsymbol{R}^3$ is bounded by $|K(p)| \leq c/r^2$, where r is the distance within M from the point $p \in M$ to the boundary ∂M, and c is a universal constant. Let $\mathcal{N}_\varepsilon(C)$ be an ε-neighborhood of C. From Schoen's estimate and the fact that $C_i \cap B_R = C \cap B_R$ for i large enough, we see that the function $|K|$ is bounded by c/ε^2 on $M_i \cap (B_R \setminus \mathcal{N}_\varepsilon(C))$ for i large enough. Thus by a well-known compactness theorem for surfaces with uniformly bounded Gaussian curvature (see, for example, [An]), there exists a subsequence of the sequence $\{M_i\}_{i=1}^{\infty}$ which converges in $B_{1/\varepsilon} \setminus \mathcal{N}_\varepsilon(C)$. The limit of this sequence is nonempty if $\varepsilon < \delta$, by the fourth assumption of the lemma. Also, by the compactness theorem in [An], if each M_i is embedded, the limit surface is embedded.

By considering a sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ so that $\varepsilon_j \searrow 0$ as $j \to \infty$, and by repeatedly applying the above argument, we can create a nested sequence of convergent subsequences. The first subsequence $\{M_{1i}\}_{i=1}^{\infty}$ converges in $B_{1/\varepsilon_1} \setminus \mathcal{N}_{\varepsilon_1}(C)$; the second subsequence $\{M_{2i}\}_{i=1}^{\infty}$ is a subsequence of $\{M_{1i}\}_{i=1}^{\infty}$ and converges in $B_{1/\varepsilon_2} \setminus \mathcal{N}_{\varepsilon_2}(C)$; the third subsequence $\{M_{3i}\}_{i=1}^{\infty}$ is a subsequence of $\{M_{2i}\}_{i=1}^{\infty}$ and converges in $B_{1/\varepsilon_3} \setminus \mathcal{N}_{\varepsilon_3}(C)$; and so on. By a Cantor diagonalization argument, the subsequence $\{M_{ii}\}_{i=1}^{\infty}$ of the sequence $\{M_i\}_{i=1}^{\infty}$ converges in $\boldsymbol{R}^3$. The limit surface M is a surface with boundary C. Furthermore, M must be stable, for if it were not, if follows that some M_{ii} would not be least-area. □

The following lemma will be used to prove the classification theorem.

LEMMA 2.4. *Suppose that M is a genus-zero finite-total-curvature complete immersed minimal surface. Suppose that ψ is a nontrivial orientation-preserving isometry of M. Then the set of points and ends of M that are fixed by ψ contains at most two elements.*

PROOF. The surface M is conformally the sphere with a finite number of points removed [Os]; that is, we have a bijective conformal map

$$\Phi: \boldsymbol{C} \cup \{\infty\} \setminus \{p_1, \ldots, p_l\} \to M .$$

The points $\{p_1, \ldots, p_l\}$ represent the ends of M, and Φ can be extended conformally to the points $\{p_1, \ldots, p_l\}$. The map $(\Phi)^{-1} \circ \psi \circ \Phi$ extends to a bijective conformal map of $\boldsymbol{C} \cup \{\infty\}$ to itself. If ψ were to fix three or more points or ends of M, then the extension of $(\Phi)^{-1} \circ \psi \circ \Phi$ would have three fixed points, and thus would be the identity map. Therefore ψ would be the identity map, a contradiction. □

3. Previously known minimal surfaces. In this section, using results from the last section, we prove existence of some previously known minimal surfaces. We do this to introduce the methods that will be later used to prove existence of the new examples, and because nowhere in the literature have these old examples explicitly been proven to exist via the conjugate surface construction.

The proofs in [BeRo] depend on the fact that the Jorge-Meeks n-oids and genus-zero Platonoids exist. The known existence of these genus zero surfaces is used to prove the existence of higher genus analogues [BeRo]. We prove here the existence of these genus-zero examples.

THEOREM 3.1 (the Jorge-Meeks n-oids). *For each $n \geq 2$, there exists an immersed minimal surface $\mathcal{JM}_0(n)$, satisfying the following*:

(1) *$\mathcal{JM}_0(n)$ has genus zero.*

(2) *$\mathcal{JM}_0(n)$ has n catenoid ends with equal weight, and the normal vectors at these ends all lie within a plane and are symmetrically placed.*

(3) *The symmetry group of $\mathcal{JM}_0(n)$ is $D_n \times \boldsymbol{Z}_2$.*

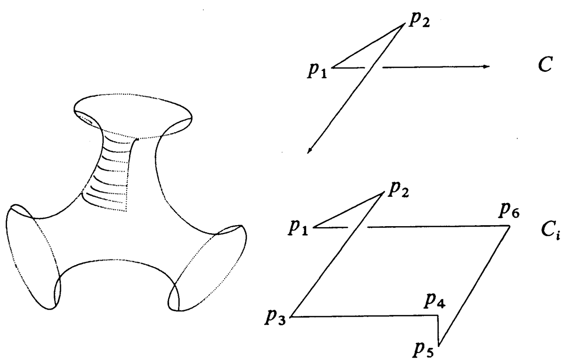

FIGURE 4. A fundamental piece of the Jorge-Meeks surface, boundary contour C of the conjugate of the fundamental piece, and the compact boundary contours C_i.

PROOF. Let M be any immersed smooth minimal surface with a planar geodesic α in its boundary, and let Ref(M) be the reflection of M across the plane containing α. It is well known, by analytic continuation properties of minimal surfaces, that $M \cup \text{Ref}(M)$ is a smooth surface along α, which is now an interior curve of the surface [Ka2]. Therefore the surface $\mathcal{JM}_0(n)$ exists if its fundamental piece exists, and its fundamental piece would look as in Figure 4.

The fundamental piece exists if its conjugate surface exists. If the conjugate surface exists, it would be a surface with an end which is a 90 degree arc of a helicoid end. The boundary C of the conjugate surface, up to a homothety and rigid motion of $\boldsymbol{R}^3$, consists of a line segment from $p_1 = (0, 0, 0)$ to $p_2 = (0, \cos(\pi/n), \sin(\pi/n))$, a ray pointing in the direction of the positive x_1-axis with endpoint p_2, and a ray pointing in the direction of the positive x_2-axis with endpoint p_1. This follows from the properties of the conjugate map, as described in Subsection 2.1. The points p_1 and p_2 are the singular points of the boundary C, and the angles that C forms at these two points are determined, since the conjugate map preserves angles.

Thus we only need to prove existence of a minimal surface with a 90 degree arc of a helicoid end and boundary C. We do this by finding a sequence $\{C_i\}_{i=1}^{\infty}$ of compact contours converging to C and satisfying all the conditions of Lemma 2.3 (see Figure 4).

We now describe the finite contour C_i. Consider the additional points $p_3=(i, \cos(\pi/n), \sin(\pi/n))$, $p_4=(i, i, \sin(\pi/n))$, $p_5=(i, i, 0)$, $p_6=(0, i, 0)$ in $\boldsymbol{R}^3$. Let l_i be the line segment connecting p_i to p_{i+1} for $i=1, \ldots, 5$, and let l_6 be the line segment connecting p_6 and p_1. Then $C_i=l_1 \cup l_2 \cup l_3 \cup l_4 \cup l_5 \cup l_6$.

The fact that each C_i bounds an embedded least-area disk follows from either Theorem 2.1 or Theorem 2.2. Thus, by Lemma 2.3, we have a minimal surface M_{conj} which is bounded by C. By Theorem 2.2, each M_i is a connected graph over a convex domain R_i in the x_2x_3-plane, and $R_i \subset R_j$ for $j>i$. It follows that M_{conj} is a connected graph, and is therefore conformally a disk.

We do not yet know that the end of M_{conj} is a 90 degree arc of a helicoid end. To show this we first show that M_{conj} has finite total curvature. Choose an orientation on M_{conj}, and consider the Gauss map $G\colon M_{\text{conj}} \mapsto S^2$. Let P be the plane containing the points p_1, p_2 and p_3. Let $\operatorname{Im}(M_{\text{conj}}) \subset S^2$ be the image of M_{conj} under G. Note that since M_{conj} is a graph, the image $\operatorname{Im}(M_{\text{conj}})$ must lie within a hemisphere. Let N be the normal vector to P, chosen so that $G(p_2)=+N$. Note that M_{conj} lies to one side of P and that C makes a 90 degree angle at p_2. It follows that G cannot be branched at p_2. Furthermore, by comparing the plane P and the surface M_{conj} along C and applying the boundary point maximum principle, we can conclude that the set $G^{-1}(N) \cap C$ consists only of the point p_2. Thus the branched covering map $G\colon M_{\text{conj}} \mapsto \operatorname{Im}(M_{\text{conj}})$ must be a finite covering map, in fact it must have degree one. In particular, M_{conj} has finite total curvature.

Since conjugation is an isometry, we know that the fundamental piece M also has finite total curvature. We can extend M by reflection to a complete smooth finite-total-curvature surface M_{comp}. Since M is a graph over both the x_1x_3-plane and the x_2x_3-plane (cf. [Ka3]), we can see that the ends of M_{comp} are embedded. Thus they must be of either catenoid or planar type [Scn1]. Suppose they are of planar type. Then M_{conj} must have an end which is asymptotic to a plane. But the two boundary rays of M_{conj} do not lie in a common plane, so the end of M_{conj} cannot be asymptotic to a plane. Hence the ends of M_{comp} must be of catenoid type. This shows that the end of M_{conj} is a 90 degree arc of a helicoid end.

By setting $\mathcal{J}\mathcal{M}_0(n)=M_{\text{comp}}$, the proof is completed. □

The same method will be used in all of the following proofs (except the proof of Theorem 1.6). Therefore, in the following proofs, we shall only emphasize the differences from the previous proof. We ask the reader to refer to the proof of Theorem

3.1 to find information that is left unstated in the following arguments. The proofs of Theorems 1.1 and 1.2 also include additional period-killing arguments.

THEOREM 3.2 (the Platonoids). *The following genus-zero minimal surfaces with catenoid ends exist*:

(1) (The genus-zero tetroid.) *A surface with four ends and symmetry group isomorphic to the symmetry group of a tetrahedron.*

(2) (The geus-zero cuboid.) *A surface with eight ends and symmetry group isomorphic to the symmetry group of a cube.*

(3) (The genus-zero octoid.) *A surface with six ends and symmetry group isomorphic to the symmetry group of an octahedron.*

(4) (The genus-zero dodecoid.) *A surface with twenty ends and symmetry group isomorphic to the symmetry group of a dodecahedron.*

(5) (The genus-zero icosoid.) *A surface with twelve ends and symmetry group isomorphic to the symmetry group of an icosahedron.*

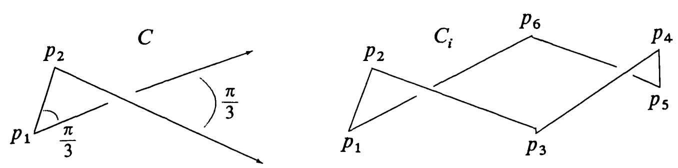

FIGURE 5. The boundary contour C of the conjugate of a fundamental piece of the tetroid, and the compact boundary contours C_i.

PROOF. We give here the proof only for the tetroid, as the other four cases are similar.

The surface exists if its fundamental piece exists. The fundamental piece exists if its conjugate surface exists; that is, if the noncompact contour C (see Figure 5) bounds a minimal surface with a 60 degree arc of a helicoid end. Again, there exists a sequence $\{C_i\}_{i=1}^{\infty}$ of compact contours which bound least-area embedded disks, and which

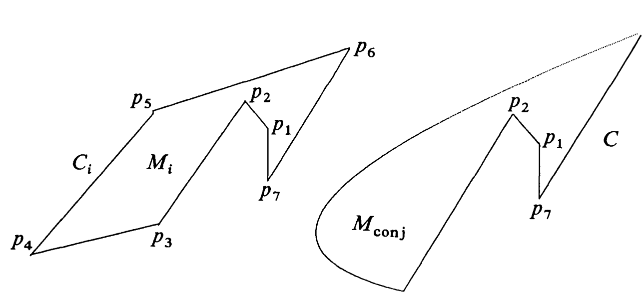

FIGURE 6. The conjugate of a fundamental piece for the prismoids, and the corresponding finite contours C_i.

converge to C in the sense of Lemma 2.3. The curve C_i can be chosen to be a polygonal contour with vertices $p_1=(0,0,0)$, $p_2=(-\sqrt{1/8}, \sqrt{3/8}, 1)$, $p_3=p_2+(i, i/\sqrt{3}, 0)$, $p_4=(i, i, 1)$, $p_5=(i, i, 0)$, and $p_6=(0, i, 0)$, connected in the same way as for the proof of Theorem 3.1. Note that C_i makes an angle of 60 degrees at p_1 and an angle of 90 degrees at p_2.

The result follows just as in the previous proof. □

4. Proofs of the main results.

PROOF (of Theorem 1.1, prismoids). Using either Theorem 2.1 or Theorem 2.2, there exists a sequence of compact contours C_i bounding least-area embedded disks M_i, so that the compact contours converge to the noncompact boundary C of the conjugate of a fundamental piece. All the conditions of Lemma 2.3 are satisfied. The Jordan contour C_i can be chosen to consist of straight line segments from $p_1=(0,0,0)$ to $p_2=(-t\cos(\pi/n), -t\sin(\pi/n), 0)$, then to $p_3=(i, -t\sin(\pi/n), 0)$, then to $p_4=(i, -i, t\sin(\pi/n)\cot(\theta)-i\cot(\theta))$, then to $p_5=(-i, -i, -i\cot(\theta)-s)$, then to $p_6=(-i, 0, -s)$, then to $p_7=(0, 0, -s)$, and then back to p_1. Thus the noncompact contour C consists of a line segment from p_1 to p_2, a line segment from p_1 to p_7, a ray with endpoint p_2 pointing in the direction of the positive x_1-axis, and a ray with endpoint p_7 pointing in the direction of the negative x_1-axis.

Therefore by Lemma 2.3 the conjugate surface M_{conj} of the potential fundamental piece exists (see Figure 6). M_{conj} is conformally a disk, and has an end that is a 180 degree arc of a helicoid end.

We now know that the fundamental piece M exists, and has a 180 degree arc of a catenoid end. The boundary of this fundamental piece ∂M consists of two finite planar geodesics and two infinite planar geodesics. The two infinite planar geodesics lie in parallel planes. If these two infinite planar geodesics lie in the same plane, then the entire complete surface exists. Thus there is one period to kill. The numbers $s, t>0$ can be chosen so that the two infinite boundary rays of M_{conj} lie in a common plane that is perpendicular to the end of M_{conj}, thus satisfying the conditions of Lemma 2.2 (up to a rigid motion). We conclude by Lemma 2.2 that, for these values of s and t, M extends to a complete finite-total-curvature minimal surface. Thus the period is zero, and the proof is completed. □

REMARK (Jorge-Meeks n-oid fence). By the method of the first two paragraphs in the previous proof, we can construct embedded minimal disks M_s, $s\in(0,\infty)$ with the following properties: M_s is bounded by a straight line segment from $(0,0,0)$ to $(0, -\cos(\pi/n), -\sin(\pi/n))$, a straight line segment from $(0,0,0)$ to $(-s, 0, 0)$, a ray with the endpoint $(0, -\cos(\pi/n), -\sin(\pi/n))$ pointing in the direction of the negative x_1-axis, and a ray with the endpoint $(-s, 0, 0)$ pointing in the direction of the negative x_2-axis. Furthermore, M_s has a single end that is 90 degree arc of a helicoid end, and M_s is a graph over the region $\{(0, x_2, x_3) \mid -\sin(\pi/n)<x_3<0, x_2<\cot(\pi/n)\cdot x_3\}$.

Consider the conjugate surface to M_s and this conjugate surface's extension by reflection across boundary planar geodesics to a complete minimal surface. We call the resulting surface the Jorge-Meeks n-oid fence (see Figure 2(2)). It is a surface with translational symmetry in one direction. The portion of the surface which generates the entire surface under the translation has n symmetrically placed ends, just as for the Jorge-Meeks n-oid. The complete surface looks like an infinite collection of n-oids regularly spaced along a single direction, with each pair of adjacent "n-oids" connected by a handle.

There is a one-parameter family of Jorge-Meeks n-oid fences, one surface for each value of $s>0$. In the case $n=2$ we have the catenoid fence (cf. [Ka3]). □

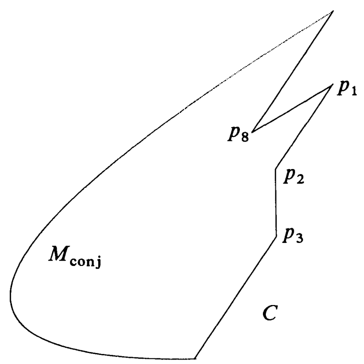

FIGURE 7. The conjugate of a fundamental piece for the higher genus prismoids.

PROOF (of Theorem 1.2, higher genus prismoids). Again the conjugate surface of the fundamental piece exists by Theorem 2.1 or Theorem 2.2, followed by Lemma 2.3. The finite polygonal contours C_i can be chosen to consist of straight line segments from $p_1=(-s,0,0)$ to $p_2=(0,0,0)$, then to $p_3=(0,0,-t)$, then to $p_4=(i,0,-t)$, then to $p_5=(i,-i,-t-i\cot(\theta))$, then to $p_6=(-i,-i,-i\cot(\theta)+u\sin(\pi/n)\cot(\theta))$, then to $p_7=(-i,-u\sin(\pi/n),0)$, then to $p_8=(u\cos(\pi/n)-s,-u\sin(\pi/n),0)$, and then back to p_1. Thus the limit contour C consists of a line segment from p_1 to p_8, a line segment from p_1 to p_2, a line segment from p_2 to p_3, a ray with endpoint p_8 pointing in the direction of the negative x_1-axis, and a ray with endpoint p_3 pointing in the direction of the positive x_1-axis. Furthermore, the limit surface M_{conj} bounded by C has a normal vector at its end which makes an angle of θ with a horizontal plane (see Figure 7). The conjugate surface M_{conj} is conformally a disk, and has an end that is a 180 degree arc of a helicoid end.

Here we have two periods to kill. Let α_1 be the unbounded boundary curve on the fundamental piece M that corresponds (via the conjugate map) to the boundary ray of M_{conj} with endpoint p_8. Let α_2 be the unbounded boundary curve of M corresponding to the boundary ray of M_{conj} with endpoint p_3. Let α_3 be the bounded boundary curve of M corresponding to the boundary line segment of M_{conj} with endpoints p_1 and p_2. To kill both periods we must show there exist choices of $s, t, u>0$ so that α_1, α_2, and α_3 all lie within a common plane.

Values can be chosen for u, t so that the two boundary rays of M_{conj} lie in a common plane that is perpendicular to the end of M_{conj}. Then, by Lemma 2.2, we conclude that α_1 and α_2 lie in a common plane P. Note that these values of u, t are independent of the value of s.

We now show that for some value of $s>0$, the curve α_3 also lies in P. The surface M_{conj} varies smoothly in $s\in[0, \infty)$, and is a graph over a fixed region in the x_2x_3-plane for all $s\in[0, \infty)$. As $s\to 0$, M_{conj} converges smoothly to the conjugate surface of a fundamental piece of $\mathscr{P}_0(2n, \theta)$. We shall refer to this fundamental piece of $\mathscr{P}_0(2n, \theta)$ as FM.

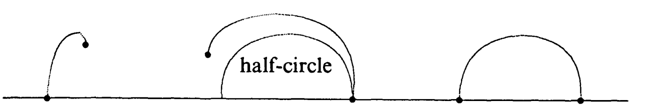

FIGURE 8. The conjugate images of the line segment from p_2 to p_3 for various values of s, in the proof of Theorem 1.2.

To describe the behavior of M_{conj} as $s\to\infty$, we first describe a portion of a helicoid. Let H be a portion of a helicoid with a single end that traverses 180 degrees, and has boundary consisting of the line segment from p_2 to p_3, the ray with endpoint p_3 pointing the direction of the positive x_1-axis, and the ray with endpoint p_2 pointing the direction of the negative x_1-axis. Choose H so that it is a nonempty graph over $\{(x_1, x_2, 0)\in \boldsymbol{R}^3 \mid x_2<0\}$, thus H is unique. As $s\to\infty$, $\mathscr{P}_0(2n, \theta)$ converges smoothly to a surface which looks similar to H, except that its end is "slanted" by the angle θ, hence we call this limit surface SH. Note that $\partial SH=\partial H$.

By the maximum principle, H and SH are disjoint in their interiors, and H lies above SH. (This can be argued rigorously by a "sliding" argument, see [BeRo].) Thus, as one travels downward along the boundary line from p_2 to p_3, the normal vector of H must be turning ahead of the normal vector of SH. The same is then true of the corresponding boundary curves of the conjugate surfaces (with respect to arc length along these two planar geodesics). Since the catenoid is the conjugate surface of the helicoid, the conjugate image of the boundary line from p_2 to p_3 with respect to H is a half-circle. The conjugate image of the boundary line from p_2 to p_3 with respect to SH is not a half-circle, but it has the same length as the half-circle, since conjugation is an isometry.

It follows that for large values of s, the curve α_3 lies strictly to one particular side of P. By an examination of the placement of FM in $\boldsymbol{R}^3$, we see that for values of s

close to zero, the curve α_3 lies strictly to the other side of P. Thus, by the intermediate-value-theorem, there exists some value of s so that $\alpha_3 \subset P$ (see Figure 8). □

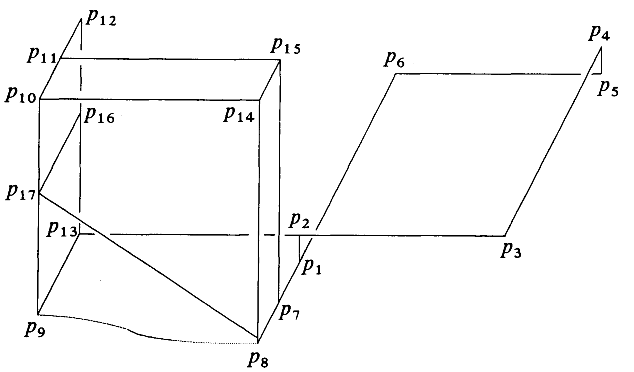

FIGURE 9. The construction of $\hat{M}_i$ in the proof of the genus-zero n-oids plus two ends.

PROOF (of Theorem 1.3, genus-zero n-oids plus two ends). For this proof there is no period problem, but since the conjugate of the fundamental piece is not a graph over a *convex* plane domain, Nitsche's theorem cannot be applied to show existence of the conjugate piece. Thus Theorem 2.1 must be used for this, followed by Lemma 2.3.

We describe now the construction of compact 3-manifolds $\hat{M}_i \subset \boldsymbol{R}^3$ and the finite polygonal contours $C_i \subset \partial \hat{M}_i$. We construct $\hat{M}_i$ so that it satisfies the hypotheses of Theorem 2.1, and thus C_i bounds a least-area embedded disk M_i. The result follows as in the previous proofs.

Assume that $w \geq n/2$, thus $w/2n \geq 1/4$. Later we shall need to assume that $w > c(n) \geq n/2$, for some constant $c(n)$ depending only on n.

The skeletal structure of $\hat{M}_i$ is given in Figure 9. Let $p_1=(0, 0, 0)$, $p_2=(0, 0, 1/4)$, $p_3=(0, i, 1/4)$, $p_4=(-i+1/4, i, 1/4)$, $P_5=(-i+1/4, i, 0)$, $p_6=(-i+1/4, 0, 0)$, $p_7=(1/4, 0, 0)$, $p_8=(w/2n, 0, 0)$, $p_9=(w/2n, -i, 1/4)$, $p_{10}=(w/2n, -i, i)$, $p_{11}=(1/4, -i, i)$, $p_{12}=(0, -i, i)$, $p_{13}=(0, -i, 1/4)$, $p_{14}=(w/2n, 0, i)$, $p_{15}=(1/4, 0, i)$, $p_{16}=(0, -i, i\tan(\pi/n))$, and $p_{17}=(w/2n, -i, i\tan(\pi/n))$.

Consider the polygonal contour defined by connecting the following vertices by line segments: p_2 to p_7, p_7 to p_6 (through p_1), p_6 to p_5, p_5 to p_4, p_4 to p_3, and p_3 back to p_2. By Theorem 2.1 or Theorem 2.2, this contour bounds a minimal graph M'. Let $\mathrm{Rot}(M')$ be the surface that results from rotating M' by 180 degrees about the line through p_2 and p_7. Then $M' \cup \mathrm{Rot}(M')$ is a smooth disk, which we shall call M_{i1}. Let C_{i1} be the boundary of M_{i1}. Thus C_{i1} is the polygonal contour defined by connecting the following vertices by line segments: p_3 to p_4, p_4 to p_5, p_5 to p_6, p_6 to p_7 (through p_1), p_7 to p_{15}, p_{15} to p_{11}, p_{11} to p_{12}, p_{12} to p_{13} (through p_{16}), and p_{13} back to p_3 (through p_2).

Consider the polygonal contour defined by connecting the following vertices: p_2 to p_1, p_1 to p_6, p_6 to p_5, p_5 to p_4, p_4 to p_3, and p_3 back to p_2. Again by either Theorem 2.1 or Theorem 2.2, this contour bounds a minimal graph M'. Let $\mathrm{Rot}(M')$ be the surface that results from rotating M' by 180 degrees about the line through p_2 and p_1. Thus we have the smooth disk $M' \cup \mathrm{Rot}(M')$. Let P be the plane $\{x_1 = w/2n\}$. Then $(M' \cup \mathrm{Rot}(M')) \setminus P$ has two components. Consider the component in the region $\{x_1 \leq w/2n\}$, and name it M_{i2}. Let C_{i2} be the boundary of M_{i2}. Thus C_{i2} is the polygonal contour defined by connecting p_3 to p_4, p_4 to p_5, p_5 to p_6, p_6 to p_8 (through p_1 and p_7), p_8 to p_9, p_9 to p_{13}, and p_{13} back to p_3 (through p_2). All these connections are made by straight lines, except for the connection from p_8 to p_9, which is made by the nonstraight planar curve $\alpha = P \cap \mathrm{Rot}(M')$.

For our construction to be valid, we need that the line segment from p_8 to p_{17} lies above α, with respect to the positive x_3 direction. Note that the surface $M' \cup \mathrm{Rot}(M')$ in the previous paragraph converges to a portion of a helicoid as $i \to \infty$. Thus the function dx_3/dx_1 along α approaches zero as w and i become large. It follows that there exists a positive number $c(n)$ so that if $w > c(n)$, then the line segment from p_8 to p_{17} lies above α.

Let M_{i3} be the plane rectangle with vertices p_7, p_8, p_{14}, and p_{15}. Let M_{i4} be the plane rectangle with vertices p_9, p_{10}, p_{12}, and p_{13}. Let M_{i5} be the plane rectangle with vertices p_{10}, p_{11}, p_{14}, and p_{15}. Let M_{i6} be the plane region in P bounded by the line segment from p_8 to p_{14}, the line segment from p_{14} to p_{10}, the line segment from p_{10} to p_9, and the curve α from p_8 to p_9.

By the maximum principle, the surfaces M_{ij}, $j = 1, \ldots, 6$, are pairwise disjoint in their interiors. Their union forms the boundary of a compact 3-manifold $\hat{M}_i$, and it is clear that $\hat{M}_i$ satisfies all the conditions of Theorem 2.1. Let C_i be the polygonal Jordan curve connecting the following vertices by line segments: p_3 to p_4, p_4 to p_5, p_5 to p_6, p_6 to p_8 (through p_1 and p_7), p_8 to p_{17}, p_{17} to p_{16}, p_{16} to p_{13}, and p_{13} back to p_3 (through p_2). Since C_i is a curve in the boundary of $\hat{M}_i$, we have by Theorem 2.1 that C_i bounds a least-area embedded disk, as desired. □

Proof (of Theorem 1.4, $2n$-oids with alternating weights). If $w < 1$, we can apply a homothety with dilation factor $1/w$ to get an equivalent surface, but with $w > 1$. So we may assume $w \geq 1$. And since the case $w = 1$ is the known Jorge-Meeks $2n$-oid, we may further assume $w > 1$.

There is no period problem here, but Nitsche's theorem again does not apply, so Theorem 2.1 followed by Lemma 2.3 is necessary. We describe now the construction of compact 3-manifolds $\hat{M}_i \subset \boldsymbol{R}^3$ and the finite polygonal contours $C_i \subset \partial \hat{M}_i$. We show that $\hat{M}_i$ satisfies the hypotheses of Theorem 2.1, and thus C_i bounds a least-area embedded disk M_i, and the result follows as before.

The skeletal structure of $\hat{M}_i$ is given in Figure 10. Let $p_2 = (0, 0, 0)$, $p_3 = (-i, 0, 0)$, $p_4 = (-i, i, 0)$, $p_5 = (-i, i, 1/4)$, $p_6 = (-i, i, w/4)$, $p_7 = (0, i, w/4)$, $p_{12} = (0, i, 1/4)$, $p_{16} = (i, 0, 0)$, $p_9 = (0, -\cot(\pi/2n)/4, 1/4)$, $p_1 = p_9 + s(0, \cos(\pi/n), -\sin(\pi/n))$ with $s =$

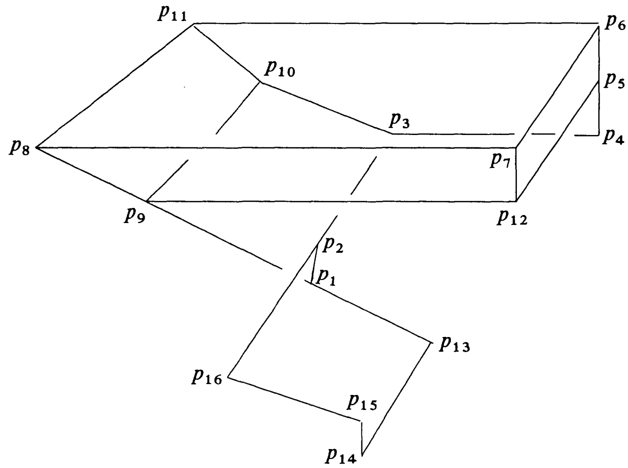

FIGURE 10. The construction of $\hat{M}_i$ in the proof of the $2n$-oids with alternating weights.

$(4\sin(\pi/2n)\cos(\pi/2n))^{-1}$, $p_8=p_9+(1-w)(4\sin(\pi/n))^{-1}(0, \cos(\pi/n), -\sin(\pi/n))$, $p_{13}=p_9+(i+(1/4)\cos(\pi/2n))(0, \cos(\pi/n), -\sin(\pi/n))$, $p_{14}=p_{13}+(i, 0, 0)$, $p_{15}=p_{16}+i(0, \cos(\pi/n), -\sin(\pi/n))$, $p_{10}=p_9+(-i, 0, 0)$, and $p_{11}=p_8+(-i, 0, 0)$.

Let C_{i1} be the polygonal contour defined by connecting the following vertices by line segments: p_8 to p_9, p_9 to p_{10}, p_{10} to p_{11}, and p_{11} back to p_8. Let C_{i2} be the contour connecting p_3 to p_4, p_4 to p_6 (through p_5), p_6 to p_{11}, p_{11} to p_{10}, and p_{10} back to p_3. Let C_{i3} be the contour connecting p_9 to p_{12}, p_{12} to p_7, p_7 to p_8, and p_8 back to p_9. Let C_{i4} be the contour connecting p_{12} to p_5, p_5 to p_6, p_6 to p_7, and p_7 back to p_{12}. Let C_{i5} be the contour connecting p_7 to p_6, p_6 to p_{11}, p_{11} to p_8, and p_8 back to p_7. These five contours all bound plane regions, which we shall call M_{i1}, M_{i2}, M_{i3}, M_{i4}, and M_{i5}, respectively.

Let C_{i6} be the contour connecting p_1 to p_2, p_2 to p_3, p_3 to p_{10}, p_{10} to p_9, and p_9 back to p_1. Let C_{i7} be the contour connecting p_1 to p_2, p_2 to p_{16}, p_{16} to p_{15}, p_{15} to p_{14}, p_{14} to p_{13}, and p_{13} back to p_1. By Theorem 2.1 or Theorem 2.2, these two contours bound minimal graphs M_{i6}, M_{i7}, respectively.

Consider the contour connecting p_2 to p_3, p_3 to p_4, p_4 to p_5, p_5 to p_{12}, p_{12} to p_9, and p_9 back to p_2. By Theorem 2.1 or Theorem 2.2, this contour bounds a minimal graph M'. We rotate M' by 180 degrees about the line through p_2 and p_9 to obtain the surface $\mathrm{Rot}(M')$. Then $M_{i8}=M'\cup\mathrm{Rot}(M')$ is a smooth embedded minimal disk. Let C_{i8} be the contour connecting p_3 to p_4, p_4 to p_5, p_5 to p_{12}, p_{12} to p_9, p_9 to p_{13} (through p_1), p_{13} to p_{14}, p_{14} to p_{15}, p_{15} to p_{16}, and p_{16} back to p_3 (through p_2). Then $\partial M_{i8}=C_{i8}$.

The surfaces $M_{ij}, j=1, \ldots, 8$ are pairwise disjoint in their interiors, by the maximum principle. Therefore, the union of these eight surfaces forms the boundary of a region $\hat{M}_i$ in $\boldsymbol{R}^3$. Let C_i be the contour connecting p_3 to p_4, p_4 to p_6 (through p_5), p_6 to p_7,

p_7 to p_8, p_8 to p_{13} (through p_9 and p_1), p_{13} to p_{14}, p_{14} to p_{15}, p_{15} to p_{16}, and p_{16} back to p_3 (through p_2). We wish to show that $\hat{M}_i$ satisfies the conditions of Theorem 2.1, thus C_i would bound an embedded least-area disk M_i. $\hat{M}_i$ clearly satisfies the conditions of Theorem 2.1, except possibly along the line segment between p_1 and p_2, and possibly along the line segment between p_2 and p_3.

Let l_2 be the line through p_2 and p_3. We now start to rotate M_{i6} about l_2 (in the clockwise direction with respect to the vector from p_2 to p_3). By the maximum principle, the first moment of contact between M_{i8} and the interior of M_{i6} cannot occur as a tangential contact along l_2 and cannot occur at a point in the interior of M_{i8}. Thus it must occur as a nontangential contact along the line segment from p_{12} to p_9, which occurs only after the rotation has traversed an arc of 180 degrees. This implies that the angles between M_{i6} and M_{i8} along l_2 must be at most 180 degrees with respect to the interior of $\hat{M}_i$. Thus the conditions of Theorem 2.1 are satisfied along the line segment from p_2 to p_3.

Let l_1 be the line through p_1 and p_2. Note that l_1 is perpendicular to the line through p_2 and p_9. Here we start to rotate M_{i6} about l_1 (in the clockwise direction with respect to the vector from p_1 to p_2). By arguing just as in the previous paragraph, we conclude that the angles between M_{i6} and M_{i7} along l_1 are at most 180 degrees with respect to the interior of $\hat{M}_i$. Thus the conditions of Theorem 2.1 are satisfied along the line segment from p_1 to p_2. □

PROOF (of Theorem 1.5, the $2n$-oids with alternating angles between the ends). The proof of Theorem 1.5 is identical to the proof of Theorem 1.1, once we replace the points $p_1, \ldots, p_7$ in the proof of Theorem 1.1 by the points $p_1=(0,0,0)$, $p_2=(0, -t\cos(\pi/n), -t\sin(\pi/n))$, $p_3=(i, -t\cos(\pi/n), -t\sin(\pi/n))$, $p_4=(i, -i, (t\cos(\pi/n)-i)\tan(\theta/2)-t\sin(\pi/n))$, $p_5=(-i, -i, (s-i)\tan(\theta/2))$, $p_6=(-i, -s, 0)$, and $p_7=(0, -s, 0)$. □

PROOF (of Theorem 1.6, classification). Let M be any complete genus-zero immersed catenoid-ended minimal surface with symmetry group $D_n \times \boldsymbol{Z}_2$ and at most $2n+1$ ends. We can place M in $\boldsymbol{R}^3$ so that its planes of reflectional symmetry are $P_0=\{x_3=0\}$, $P_i=\{x_1=\cot((i\pi/n)x_2)\}$ for $i=1, \ldots, n-1$, and $P_n=\{x_2=0\}$. Thus the x_3-axis is the axis for the rotational symmetry of order n of the surface M.

Choose an orientation on M. Consider an end E of M with limiting normal vector $\vec{v}$. Let $l(E)$ be the central axis line of the end E. Let Orb(E) be the orbit of E under the symmetry group $D_n \times \boldsymbol{Z}_2$ of M.

If E is not invariant under any element of the symmetry group, then Orb(E) would consist of $4n$ distinct ends, which contradicts our hypothesis. So E must be invariant under reflection through P_i for some i. It follows that $\vec{v} \in P_i$ for some i. Clearly E cannot be invariant under reflection through all P_i, so Orb(E) must contain at least two ends. In fact, the following list represents all possibilities for Orb(E):

(1) If the limiting normal vector $\vec{v}$ of E is neither vertical nor horizontal, then

Orb(E) consists of $2n$ ends.

(2) If $\vec{v} \in P_0$ but $\vec{v} \notin P_i$ for all $i \geq 1$, then Orb(E) consists of $2n$ ends.

(3) If $\vec{v} \in P_0$ and $\vec{v} \in P_i$ for some $i \geq 1$ but the central axis $l(E) \notin P_i$ for that value of $i \geq 1$, then Orb(E) consists of $2n$ ends.

(4) If $\vec{v} \in P_0$ and $\vec{v} \in P_i$ for some $i \geq 1$ and $l(E) \in P_i$ for that value of $i \geq 1$, then Orb(E) consists of n ends.

(5) If $\vec{v}$ is vertical and $l(E)$ is not the x_3-axis, then Orb(E) consists of $2n$ ends.

(6) If $\vec{v}$ is vertical and $l(E)$ is the x_3-axis, then Orb(E) consists of 2 ends.

Claim 1: E is the unique end of M with normal vector v and axis $l(E)$.

Suppose that E' is another end of M with normal vector v, and that $l(E)=l(E')$. In all six cases above, the only end contained in Orb(E) that has both the same normal vector and same central axis as E is E itself. Therefore, $E' \notin$ Orb(E).

If Orb(E) contains $2n$ ends, then Orb(E) $\cup$ Orb(E') contains at least $2n+2$ ends, a contradiction. So in cases 1, 2, 3, and 5 above, E' cannot exist.

Consider case 6. In this case Orb(E) $\cup$ E' consists of three ends all with central axis the x_3-axis. Thus all three of these ends are invariant by a nontrivial rotation about the x_3-axis. This is impossible by Lemma 2.4, so E' cannot exist.

Consider case 4. In this case we may assume (by rotating in $\boldsymbol{R}^3$ about the x_3-axis if necessary) that $l(E)$ is the x_1-axis and $\vec{v}=(1, 0, 0)$. Let R be rotation by 180 degrees about the x_1-axis. Note that Orb(E) $\cup$ Orb(E') contains all the ends of M. By Osserman's inequality, since we have a genus zero minimal surface with $2n$ embedded ends, we see that the Gauss map is a branched covering from M to the unit sphere with order $2n-1$. Let S be the set of $2n-1$ points on M with Gauss map $(1, 0, 0)$ (including ends, and counting with multiplicity). S is invariant under R as a set, and there are an odd number of points in S. Since $R \circ R$ is the identity map, it follows that there must be an odd number of fixed points of R contained in S. The ends E and E' represent two such points, so there must be a third. But R cannot have three fixed points, by Lemma 2.4. Thus E' cannot exist.

This proves Claim 1.

Claim 2: The ends of M can have at most one orbit consisting of exactly two ends.

Suppose $\{E_1, E_2\}$ is one orbit consisting of two ends, and suppose $\{E_3, E_4\}$ is another orbit consisting of two ends. Then E_1, E_2, E_3, and E_4 all have central axis the x_3-axis. Thus two of these ends must have the same normal vector, but that contradicts Claim 1.

This proves Claim 2.

So the only possibilities are the following:

(1) The ends of M have a single orbit consisting of n ends. Then M is equivalent to $\mathscr{J}\mathscr{M}_0(n)$.

(2) The ends of M have two orbits, each consisting of n ends. The M is equivalent to $\mathscr{A}\mathscr{W}_0(2n, w)$ with $w \neq 0, 1$.

(3) The ends of M have two orbits, one consisting of two ends, and one consisting

of n ends. Then M is equivalent to $\mathcal{JMV}_0(n+2, w)$.

(4) The ends of M have a single orbit consisting of $2n$ ends. Then M is equivalent to $\mathcal{P}_0(2n, \theta)$ or $\mathcal{AA}_0(2n, \theta)$.

M cannot have only two ends, for in this case M would be a catenoid (cf. [Scn1]), whose symmetry group is not $D_n \times \mathbf{Z}_2$. This completes the proof. $\square$

5. Open problems. One can ask whether the following immersed minimal surfaces with catenoid ends and symmetry group $D_n \times \mathbf{Z}_2$ exist:

(1) The prismoids $\mathcal{P}_0(2n, \theta)$ and $\mathcal{P}_{n-1}(2n, \theta)$ plus two vertical ends (analogous to the way $\mathcal{JMV}_0(n+2, w)$ is the Jorge-Meeks surface plus two vertical ends).

(2) The surface $\mathcal{AW}_0(2n, w)$ plus two vertical ends.

(3) A $3n$-oid with weight one at every third end as one travels around the circle of ends, and weight w at the other $2n$ ends, $w \in (0, \infty)$.

(4) The example mentioned in the previous item, plus two vertical ends (assuming the surface is placed so that the first $3n$ ends have horizontal normal vectors).

(5) Prismoids with k layers of catenoid ends, still with symmetry group $D_n \times \mathbf{Z}_2$, both genus zero and higher genus. If k is odd, this surface would have n ends with horizontal normal vectors, otherwise it would have no ends with horizontal normal vectors. In either case, it would have n ends with normal vectors pointing upward making an angle θ_1 with a horizontal plane, and n ends with normal vectors pointing downward making an angle θ_1 with a horizontal plane, $0 < \theta_1 < \pi/2$. The same would then be true for some angle θ_2 with $\theta_1 < \theta_2 < \pi/2$. And again this holds for some angle θ_3 with $\theta_2 < \theta_3 < \pi/2$. This continues up to the angle $\theta_{[k/2]}$, where $[k/2]$ is the greatest integer less than or equal to $k/2$.

(6) Prismoids with k layers of catenoid ends plus two vertical ends.

Solving some of the conjectures above might lead to a generalization of Theorem 1.6 to higher numbers of ends.

A broader open question is the following:

CONJECTURE 5.1 (Kusner's conjecture). *Any balanced configuration* $\{v_1, \ldots, v_n\}$ *of n vectors, such that for all i and j,* $v_i \neq r \cdot v_j$ *for any positive real r, can be realized as a genus-zero immersed minimal surface with n catenoid ends.*

Kapouleas [Kap] has some corresponding results for the nonminimal constant-mean-curvature case.

Shin Kato, Masaaki Umehara, and Kotaro Yamada have some results in the direction of these open questions [KUY], [Kat], [UmYa].

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MATHEMATICAL INSTITUTE
FACULTY OF SCIENCE
TOHOKU UNIVERSITY
SENDAI 980-77
JAPAN