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MINIMAL HYPERSURFACES OF UNIT SPHERE

HUAFEI SUN AND KOICHI OGIUE

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Abstract. In this paper, a pinching theorem for minimal hypersurfaces in a unit sphere is given and Peng-Terng's Theorem is improved.

Let S^{n+1} be an $(n+1)$-dimensional unit sphere and M a closed minimally immersed hypersurface in S^{n+1}. S denotes the square of the length of the second fundamental form of M. It is well known that if $0 \leq S \leq n$, then $S \equiv 0$ or $S \equiv n$ (cf. [1]). Chern-do Carmo-Kobayashi [2] and Lawson [3] independently proved that the Clifford tori are the only minimal hypersurfaces with $S \equiv n$. In [2] Chern conjectured that for minimal hypersurfaces in the unit sphere S^{n+1} with constant scalar curvature, the values S are discrete. On this conjecture, Peng and Terng [5] made a breakthrough, and proved:

THEOREM A. *Let M be an n-dimensional closed minimally immersed hypersurface in S^{n+1} with constant scalar curvature. Then there exists a constant $\varepsilon(n)$ such that if $n \leq S \leq n+\varepsilon(n)$, then $S \equiv n$ so that M is a Clifford torus.*

It is natural to consider the problem without assuming that the scalar curvature is constant. For $n \leq 5$, Peng and Terng [4] proved:

THEOREM B. *Let M be an n-dimensional closed minimally immersed hypersurface in S^{n+1}, $n \leq 5$. Then there exists a constant $\varepsilon(n)=(6-1.13n)/(5+\sqrt{17})$ such that if $n \leq S \leq n+\varepsilon(n)$, then $S \equiv n$ so that M is a Clifford torus.*

In this paper, we remove Peng-Terng's assumption $n \leq 5$ and improve Theorem B. Our result is:

THEOREM. *Let M be an n-dimensional closed minimally immersed hypersurface in S^{n+1}. Then there exists a constant $\varepsilon(n)=2n^2(n+4)/[3(n+2)^2]$ such that if $n \leq S \leq n+\varepsilon(n)$, then $S \equiv n$ so that M is a Clifford torus.*

It is clear that our Theorem is better than Theorem B and this $\varepsilon(n)$ is the same as that in Cheng [6].

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1. Fundamental Formulas. We use the same notation and terminologies as in [4], [5] unless otherwise stated.

Let S^{n+1} be an $(n+1)$-dimensional unit sphere and M a closed minimally immersed hypersurface in S^{n+1}. $\{e_1, \ldots, e_n, e_{n+1}\}$ is an orthonormal frame field on S^{n+1}, $\{e_1, \ldots, e_n\}$ is tangent to M, and let $\{\omega_1, \ldots, \omega_{n+1}\}$ be the dual frame field. We use of the following convention on the range of the indices:

$$A, B, \ldots = 1, \ldots, n+1\,; \quad i, j, \ldots = 1, \ldots, n\,.$$

Then the structure equations of S^{n+1} are given by

$$d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B\,, \quad \omega_{AB} + \omega_{BA} = 0\,,$$

$$d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2}\sum_{CD} K_{ABCD}\omega_C \wedge \omega_D\,,$$

$$K_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}\,.$$

Restricting these forms to M, we get

$$\omega_{n+1} = 0\,, \quad \omega_{n+1\,i} = \sum_j h_{ij}\omega_j\,, \quad h_{ij} = h_{ji}\,,$$

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j\,, \quad \omega_{ij} + \omega_{ji} = 0\,,$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{kl} R_{ijkl}\omega_k \wedge \omega_l\,,$$

where

$$R_{ijkl} = K_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}\,.$$

We call $h = \sum_{ij} h_{ij}\omega_i\omega_j$ and $H = (1/n)\sum_i h_{ii}$ the second fundamental form and the mean curvature of the immersion, respectively. If $H \equiv 0$, then M is said to be minimal. Denote by $S = \sum_{ij} h_{ij}^2$ the square of the length of h. For a minimal hypersurface M in S^{n+1}, we have

$$S = n(n-1) - R\,,$$

where R is the scalar curvature of M. It is clear that S is constant if and only if so is R. Define h_{ijk}, h_{ijkl} and h_{ijkln} respectively by

$$\sum_{ijk} h_{ijk}\omega_k = dh_{ij} - \sum_m h_{mj}\omega_{mi} - \sum_m h_{im}\omega_{mj}\,,$$

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_m h_{mjk}\omega_{mi} - \sum_m h_{imk}\omega_{mj} - \sum_m h_{ijm}\omega_{mk}\,,$$

$$\sum_m h_{ijklm}\omega_m = dh_{ijkl} - \sum_m h_{mjkl}\omega_{mi} - \sum_m h_{imkl}\omega_{mj} - \sum_m h_{ijml}\omega_{mk} - \sum_m h_{ijkm}\omega_{ml}\,.$$

Here

$$h_{ijk}=h_{ikj}\,,$$

$$h_{ijkl}-h_{ijlk}=\sum_m h_{mj}R_{mikl}+\sum_m h_{im}R_{mjkl}\,,$$

and

$$h_{ijkln}-h_{ijknl}=\sum_m h_{mjk}R_{miln}+\sum_m h_{imk}R_{mjln}+\sum_m h_{ijm}R_{mkln}\,.$$

We can choose a local frame field $\{e_1, \ldots, e_n\}$ such that

$$h_{ij}=h_{ii}\delta_{ij}\,.$$

From now on, we assume that M is minimal. By calculation we have (cf. [4], [5])

$$\Delta h_{ij}=(n-S)h_{ij}\,, \tag{1.1}$$

$$\int_M |\nabla h|^2=\int_M S(S-n)\,, \tag{1.2}$$

$$\int_M \sum_{ijkl} h_{ijkl}^2=\int_M\left[(S-2n-3)|\nabla h|^2+3(A-2B)+\frac{3}{2}|\nabla S|^2\right], \tag{1.3}$$

where

$$A=\sum_{ijk} h_{ijk}^2h_{ii}^2\,,\quad B=\sum_{ijk} h_{ijk}^2h_{ii}h_{jj}\,.$$

$$\sum_{ijkl} h_{ijkl}^2\geq\sum_i h_{iiii}^2+3\sum_{i\neq j} h_{ijij}^2\,. \tag{1.4}$$

Set

$$g_3=\sum_{ijk} h_{ij}h_{jk}h_{ki}\,,\quad g_4=\sum_{ijkl} h_{ij}h_{jk}h_{kl}h_{li}\,.$$

Then

$$\int_M (A-2B)=\int_M\left(Sg_4-g_3^2-S^2-\frac{1}{4}|\nabla S|^2\right). \tag{1.5}$$

$$\int_M |\nabla S|^2=2\int_M [S^2(S-n)-S|\nabla h|^2]\,. \tag{1.6}$$

2. Proof of the Theorem. In order to prove our Theorem, we need the following:

LEMMA. *Let M be an n-dimensional closed minimally immersed hypersurface in S^{n+1}. If $S\geq n$, then we have*

$$\sum_{ijkl} h_{ijkl}^2 \geq \frac{3(n+2)}{n(n+4)} S(S-n)^2 - \frac{3}{n} S^2(S-n) + 3(Sg_4 - g_3^2 - S^2)\,.$$

PROOF. Since M is minimal, we have $\sum_i h_{ii}=0$ and

$$\sum_{ij} h_{iijj} h_{jj} = 0\,.$$

From (1.1) we get $\Delta h_{ii} = (n-S)h_{ii}$ and

$$\sum_{ij} h_{iijj} h_{ii} = S(n-S)\,.$$

Let $f_{ij} = h_{ijij}$. We consider $f = \sum_i f_{ii}^2 + 3\sum_{i \neq j} f_{ij}^2 + 6\sum_{ij}(h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij}$ as a function of f_{ij}. Solve the following problem for the conditional extremum:

$$F = \sum_i f_{ii}^2 + 3\sum_{i \neq j} f_{ij}^2 + 6\sum_{ij}(h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij} + \lambda\left(\sum_{ij} f_{ij} h_{ii} - S(n-S)\right) + \mu \sum_{ij} f_{ij} h_{jj}\,, \tag{2.1}$$

where λ and μ are the Lagrange multipliers. It is clear that the critical point of F is the minimum point of f. Taking derivatives of F with respect to f_{ij}, we get

$$F_{f_{ii}} = 2f_{ii} + \lambda h_{ii} + \mu h_{ii} = 0\,, \qquad i=j\,, \tag{2.2}$$

$$F_{f_{ij}} = 6f_{ij} + 6(h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) + \lambda h_{ii} + \mu h_{jj} = 0\,, \qquad i \neq j\,, \tag{2.3}$$

and they satisfy

$$\sum_{ij} h_{jj} f_{ij} = 0\,, \quad \sum_{ij} h_{ii} f_{ij} = S(n-S)\,, \quad \sum_i h_{ii}^2 = S\,, \quad \sum_i = h_{ii} = 0\,. \tag{2.4}$$

From (2.2) and (2.3) we get

$$\sum_i f_{ii} F_{f_{ii}} = 2\sum_i f_{ii}^2 + (\lambda+\mu)\sum_i f_{ii} h_{ii} = 0\,, \tag{2.5}$$

$$\sum_{i \neq j} f_{ij} F_{f_{ij}} = 6\sum_{i \neq j} f_{ij}^2 + 6\sum_{i \neq j}(h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij} + \lambda \sum_{i \neq j} f_{ij} h_{ii} + \mu \sum_{i \neq j} f_{ij} h_{jj} = 0\,, \tag{2.6}$$

$$\sum_{i \neq j} h_{ii} F_{f_{ij}} = 6\sum_{i \neq j} h_{ii} f_{ij} + 6\sum_{i \neq j}(h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) h_{ii} + \lambda \sum_{i \neq j} h_{ii}^2 + \mu \sum_{i \neq j} h_{ii} h_{jj} = 0\,, \tag{2.7}$$

$$\sum_{i \neq j} h_{jj} F_{f_{ij}} = 6\sum_{i \neq j} h_{jj} f_{ij} + 6\sum_{i \neq j}(h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) h_{jj} + \lambda \sum_{i \neq j} h_{ii} h_{jj} + \mu \sum_{i \neq j} h_{jj}^2 = 0\,, \tag{2.8}$$

$$\sum_i h_{ii} F_{f_{ii}} = 2\sum_i h_{ii} f_{ii} + \lambda \sum_i h_{ii}^2 + \mu \sum_i h_{ii}^2 = 0\,. \tag{2.9}$$

From (2.5) and (2.6) we get

$$2\Big(\sum_i f_{ii}^2+3\sum_{i\neq j} f_{ij}^2+6\sum_{i\neq j}(h_{jj}^2h_{ii}-h_{ii}^2h_{jj})f_{ij}\Big)$$
$$-6\sum_{i\neq j}(h_{jj}^2h_{ii}-h_{ii}^2h_{jj})f_{ij}+\lambda\sum_{ij}h_{ii}f_{ij}+\mu\sum_{ij}f_{ij}h_{jj}=0$$

and so, in view of (2.4),

$$\begin{aligned}\sum_i f_{ii}^2+3\sum_{i\neq j} f_{ij}^2+6\sum_{i\neq j}(h_{jj}^2h_{ii}-h_{ii}^2h_{jj})f_{ij}\\ =3\sum_{i\neq j}(h_{jj}^2h_{ii}-h_{ii}^2h_{jj})f_{ij}-\frac{1}{2}\lambda\sum_{ij}h_{ii}f_{ij}\\ =3\sum_{i\neq j}(h_{jj}^2h_{ii}-h_{ii}^2h_{jj})f_{ij}+\frac{\lambda}{2}S(S-n)\,.\end{aligned}\tag{2.10}$$

Using (2.4), from (2.8) and (2.9) we get

$$-4\sum_i h_{ii}f_{ii}+6\sum_{i\neq j}(h_{jj}^2h_{ii}-h_{ii}^2h_{jj})h_{jj}+n\mu S=0\,,\tag{2.11}$$

and from (2.7), (2.9) we get

$$-4\sum_i h_{ii}f_{ii}+6\sum_{i\neq j}(h_{jj}^2h_{ii}-h_{ii}^2h_{jj})h_{ii}+n\lambda S+6S(n-S)=0\,.\tag{2.12}$$

Notice that $S\geq n$. Thus by (2.11) and (2.12) we obtain

$$n(\mu-\lambda)S=6S(n-S)+12S^2\,,$$

that is,

$$\mu-\lambda=\frac{6}{n}(n-S)+\frac{12}{n}S\,.\tag{2.13}$$

On the other hand, $(2.11)+(2.12)+4\times(2.9)$ shows

$$4(\lambda+\mu)S+n(\lambda+\mu)S+6S(n-S)=-6\sum_{i\neq j}(h_{jj}^2h_{ii}-h_{ii}^2h_{jj})h_{jj}-6\sum_{i\neq j}(h_{jj}^2h_{ii}-h_{ii}^2h_{jj})h_{ii}=0\,,$$

that is,

$$\lambda+\mu=\frac{6}{4+n}(S-n)\,.\tag{2.14}$$

Combining (2.13) with (2.14), we get

$$\lambda=\frac{6(n+2)}{n(n+4)}(S-n)-\frac{6}{n}S\,.\tag{2.15}$$

(2.10) and (2.15) show that

$$\sum_i f_{ii}^2 + 3\sum_{i\neq j} f_{ij}^2 + 6\sum_{i\neq j}(h_{jj}^2 h_{jj} - h_{ii}^2 h_{jj}) f_{ij}$$
$$= \frac{3(n+2)}{n(n+4)} S(S-n)^2 - \frac{3}{n} S^2(S-n) + 3\sum_{i\neq j}(h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij},$$

and so,

$$\text{(2.16)}\quad \sum_i h_{iiii}^2 + 3\sum_{i\neq j} h_{ijij}^2 \geq \frac{3(n+2)}{n(n+4)} S(S-n)^2 - \frac{3}{n} S^2(S-n) - 3\sum_{i\neq j}(h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) h_{ijij}$$
$$= \frac{3(n+2)}{n(n+4)} S(S-n)^2 - \frac{3}{n} S^2(S-n) + 3(Sg_4 - g_3^2 - S^2).$$

Combining (1.4) with (2.16), we get the Lemma.

Proof of the Theorem. Using (1.3) and the Lemma we get

$$\text{(2.17)}\quad \int_M \left[(S-2n-3)|\nabla h|^2 + 3(A-2B) + \frac{3}{2}|\nabla S|^2\right]$$
$$\geq \int_M \left[\frac{3(n+2)}{n(n+4)} S(S-n)^2 - \frac{3}{n} S^2(S-n) + 3(Sg_4 - g_3^2 - S^2)\right].$$

Noting (1.5), from (2.17) we obtain

$$\int_M \left[(S-2n-3)|\nabla h|^2 + 3(A-2B) + \frac{3}{2}|\nabla S|^2\right]$$
$$\geq \int_M \left[\frac{3(n+2)}{n(n+4)} S(S-n)^2 - \frac{3}{n} S^2(S-n) + 3(A-2B) + \frac{3}{4}|\nabla S|^2\right],$$

namely,

$$\text{(2.18)}\quad \int_M \left[(S-2n-3)|\nabla h|^2 - \frac{3(n+2)}{n(n+4)} S(S-n)^2 + \frac{3}{n} S^2(S-n) + \frac{3}{4}|\nabla S|^2\right] \geq 0.$$

Substituting (1.6) into (2.18), we get

$$\int_M \left[(S-2n-3)|\nabla h|^2 - \frac{3(n+2)}{n(n+4)} S(S-n)^2 + \frac{3}{n} S^2(S-n)\right.$$
$$\left. + \frac{3}{2} S^2(S-n) - \frac{3}{2} S|\nabla h|^2\right] \geq 0,$$

that is,

$$\text{(2.19)}\quad \int_M \left[\left(-\frac{1}{2} S - 2n - 3\right)|\nabla h|^2 + \frac{3(n+2)^2}{2n(n+4)} S^2(S-n) + \frac{3(n+2)}{n+4} S(S-n)\right] \geq 0.$$

Now, we assume that $n \leq S \leq n+\varepsilon$. Then using (1.2) we get

$$\int_M S^2(S-n) \leq (n+\varepsilon) \int_M S(S-n) = (n+\varepsilon) \int_M |\nabla h|^2 . \tag{2.20}$$

By (2.20), (2.19) yields

$$\int_M \left[-\frac{1}{2} S - \frac{1}{2} n + \frac{3(n+2)^2}{2n(n+4)} \varepsilon \right] |\nabla h|^2 \geq 0 . \tag{2.21}$$

Taking $\varepsilon = 2n^2(n+4)/[3(n+2)^2]$ in (2.21), we get

$$\int_M \left(-\frac{1}{2} S + \frac{1}{2} n \right) |\nabla h|^2 \geq 0 .$$

Since $S \geq n$, we see that $S \equiv n$ so that M is a Clifford torus. This completes the proof of the Theorem.

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Department of Mathematics
Tokyo Metropolitan University
Hachioji, Tokyo, 192–03
Japan