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NORMAL CONTACT STRUCTURES ON 3-MANIFOLDS

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Abstract. We give a classification of the closed 3-manifolds that admit normal contact forms or normal almost contact structures.

1. Introduction. Let ω be a *contact form* on a closed 3-manifold M, that is $\omega \wedge d\omega$ is a volume form. The *Reeb vector field* ξ of ω is defined by $d\omega(\xi, \cdot) \equiv 0$ and $\omega(\xi) \equiv 1$. On the *contact structure* $\mathscr{D} = \ker \omega$ (or, more generally, on a 2-plane distribution $\mathscr{D} = \ker \eta$ transverse to ξ) one can find an endomorphism $J : \mathscr{D} \to \mathscr{D}$ compatible with $d\omega$ in the sense that $d\omega(JX, JY) = d\omega(X, Y)$ for all vector fields $X, Y \in \mathscr{D}$ and $d\omega(X, JX) > 0$ for $X \neq 0$. This J is uniquely defined up to homotopy. The triple (J, ξ, η), in other words, a reduction of the structure group of M to $U(1) \times 1$, is called an *almost contact structure compatible with* ω.

On $M \times \boldsymbol{R}$ one can now define an almost complex structure (still denoted by J) which extends $J : \mathscr{D} \to \mathscr{D}$ and satisfies $J\xi = \partial_t$, where t denotes the $\boldsymbol{R}$-coordinate. If J is integrable, then the almost contact structure (J, ξ, η) and the contact form ω are called *normal*.

In [8] Sato proved that if M admits a normal almost contact structure, then $\pi_2(M) = 0$ or M is homotopy equivalent to $S^1 \times S^2$. In the present paper we complete the investigation begun by Sato. For the reader familiar with the geometries of 3-manifolds (in the sense of Thurston) we can now state our main results; all the notation will be explained below.

THEOREM 1. *A closed* 3*-manifold admits a normal contact form if and only if it is diffeomorphic to one of the following manifolds*:

(a) $\Gamma \backslash S^3$ *with* $\Gamma \subset SO(4) = \mathrm{Isom}_0(S^3)$,
(b) $\Gamma \backslash \widetilde{SL}_2$ *with* $\Gamma \subset \mathrm{Isom}_0(\widetilde{SL}_2)$,
(c) $\Gamma \backslash Nil^3$ *with* $\Gamma \subset \mathrm{Isom}_0(Nil^3)$.

REMARK. The manifolds in this theorem are precisely the Seifert fibred 3-manifolds with non-zero Euler number over orientable base orbifolds (without reflectors).

THEOREM 2. *A closed* 3*-manifold admits a normal almost contact structure if and only if it is diffeomorphic to one of the manifolds listed in Theorem* 1 *or one of the following manifolds*:

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(a) $\Gamma\backslash(H^2 \times E^1)$ *with* $\Gamma \subset \mathrm{Isom}_0(H^2 \times E^1)$,
(b) T^2*-bundles over* S^1 *with periodic monodromy*,
(c) $S^2 \times S^1$.

Our notation here is that of [9], to which we refer for all basic facts about geometric 3-manifolds. $\widetilde{SL}_2$ denotes the universal cover of $PSL_2\boldsymbol{R}$, Nil^3 the Heisenberg group of upper triangular (3×3)-matrices and H^2 the hyperbolic plane. $\mathrm{Isom}_0(X)$ stand for the identity component of the isometry group of a Riemannian manifold X. In each case Γ denotes any discrete subgroup of $\mathrm{Isom}_0(X)$ acting freely on X.

We note in passing that the manifolds in (b) of Theorem 2 can also be described as left-quotients of the universal cover of the Euclidean group, that is, the group of orientation preserving isometries of the Euclidean plane E^2. This class comprises exactly five manifolds.

As a consequence of Theorems 1 and 2 there are 3-manifolds that admit normal almost contact structures not induced from a contact form.

2. Compact complex surfaces. If a closed 3-manifold M admits a normal contact form, then $M \times \boldsymbol{R}$ admits an $\boldsymbol{R}$-invariant integrable almost complex structure, hence $M \times S^1$ is a compact complex surface with a smooth S^1-action by holomorphic automorphisms. Therefore the starting point for our proof of Theorem 1 is the following theorem from [3]. Part of the argument employed there to prove this theorem is parallel to that of Sato. We should like to point out to the reader that Section 4 of [3], which contains Theorem 3 and its proof as well as other arguments relevant to the discussion here, can be read independently of the preceding sections of that paper.

THEOREM 3. *A compact complex surface W is diffeomorphic to a complex surface of the form* $M \times S^1$ *on which the obvious smooth* S^1*-action is by holomorphic automorphisms, if and only if W is one of the following.*

(a) *A Hopf surface that is (topologically) of the form* $(\Gamma\backslash S^3) \times S^1$ *with* $\Gamma \subset U(2)$.

(b) *A properly elliptic surface of the form* $(\Gamma\backslash(H^2 \times E^1)) \times S^1$ *or* $(\Gamma\backslash\widetilde{SL}_2) \times S^1$ *with* $\Gamma \subset \mathrm{Isom}_0(H^2 \times E^1)$ *or* $\Gamma \subset \mathrm{Isom}_0(\widetilde{SL}_2)$, *respectively.*

(c) *One of the hyperelliptic surfaces (which are topologically* T^2*-bundles over* T^2 *with monodromy* A, I, *where* $A \in SL_2\boldsymbol{Z}$ *is periodic and* I *the identity matrix,* $A \neq I$) *with Euler class* $(0, 0)$. *Up to diffeomorphism, there are four such surfaces.*

(d) *A complex torus, diffeomorphic to* T^4.

(e) *A primary or secondary Kodaira surface of the form* $(\Gamma\backslash Nil^3) \times S^1$ *with* $\Gamma \subset \mathrm{Isom}_0(Nil^3)$.

(f) *A ruled surface of genus* 1 *diffeomorphic to* $S^2 \times T^2$.

In the following sections we deal with each of the geometries in turn.

3. Spherical geometry. Suppose (M, ω) is a 3-manifold with normal contact form ω such that $W = M \times S^1$ with the induced complex structure is a Hopf surface.

If $\pi_1(W)$ is non-abelian, then W is elliptic. Furthermore, the elliptic structure is unique and has no singular fibres by (the proof of) Lemma 7.2 of [13]. By [7, Theorem 27], cf. [11], the base orbifold B of the elliptic fibration $W \to B$ is a sphere with three cone points and positive orbifold characteristic. We now have the following argument from [3]: Since the general fibre of the elliptic fibration represents a homology class with self-intersection zero, positivity of intersections implies that the holomorphic S^1-action generated by ∂_t sends general fibres to general fibres and fixes the exceptional (i.e. multiple) fibres. So the action descends to an S^1-action on B with fixed points in the three cone points, which must be the trivial action. Hence ∂_t is tangent to the fibres.

Thus ξ is also tangent to the elliptic fibres, and the flow of ξ induces a Seifert fibration on the quotient $M = W/\langle \partial_t \rangle$. (In $W = M \times S^1$ an orbit of ξ is tangent to the M-factor and cuts an orbit of ∂_t at most once. Since both the orbits of ξ and of ∂_t are along the fibres of $W \to B$, it follows that the orbits of ξ are closed. By a fundamental result of Epstein [1], this implies that M is Seifert fibred.) Hence M is a geometric manifold (cf. [9]), and the geometric type can only be S^3 since the geometric type (if any) of a Hopf surface can only be $S^3 \times E^1$ [13, Theorem 10.1].

If $\pi_1(W)$ is abelian, then $\pi_1(M) = \mathbf{Z} \oplus \mathbf{Z}_m$ (including the case $m = 1$ of primary Hopf surfaces, that is, $\pi_1(W) = \mathbf{Z}$) by [6], and $\pi_1(W)$, considered as the deck transformation group on $\mathbf{C}^2 \setminus \{(0, 0)\}$, is generated by a contraction T and, if $m > 1$, a torsion generator U. There are two possible cases:

Case (1):

$$T(z_1, z_2) = (\alpha z_1, \beta z_2), \quad 0 < |\alpha| \leq |\beta| < 1,$$
$$U(z_1, z_2) = (\varepsilon_1 z_1, \varepsilon_2 z_2), \quad \varepsilon_1^m = \varepsilon_2^m = 1.$$

Case (2):

$$T(z_1, z_2) = (\beta^n z_1 + \gamma z_2^n, \beta z_2), \quad 0 < |\beta| < 1, \quad \gamma \neq 0,$$
$$U(z_1, z_2) = (\varepsilon_2^n z_1, \varepsilon_2 z_2), \quad \varepsilon_2^m = 1, \quad (m, n) = 1.$$

Now let φ_t denote the flow of ∂_t on $W = M \times S^1$. Lift φ_t to a one-parameter group of holomorphic automorphisms of $\mathbf{C}^2 \setminus \{(0, 0)\}$ (still denoted by φ_t). Since φ_t commutes with T, the arguments on pp. 230–231 of [6] show that φ_t has to be of the form

$$\varphi_t(z_1, z_2) = (\tilde{a}(t) z_1, \tilde{b}(t) z_2)$$

in case (1) and

$$\varphi_t(z_1, z_2) = (\tilde{b}^n(t) z_1 + \tilde{c}(t) z_2^n, \tilde{b}(t) z_2)$$

in case (2). The condition $\varphi_{t+t_0} = \varphi_t \circ \varphi_{t_0}$ implies

$$\varphi_t(z_1, z_2) = (e^{at} z_1, e^{bt} z_2)$$

or

$$\varphi_t(z_1, z_2) = (e^{nbt}z_1 + cte^{nbt}z_2^n, e^{bt}z_2),$$

respectively, where the real parts of a and b are non-zero and have the same sign (since there is a time t_0 such that φ_{t_0} is equal to T up to sign and up to a power of U, so φ_{t_0} is contracting or expanding).

Now consider the set

$$S = \{(z_1, z_2) \in \boldsymbol{C}^2 : |z_1|^2 + K|z_2|^2 = 1\}$$

with K a positive real constant. This set is U-invariant and diffeomorphic to S^3, and it is easy to check that the flow of φ_t is transverse to S for K sufficiently large. This implies that M is diffeomorphic to a lens space $S/\langle U \rangle = L(m, n)$.

Conversely, suppose M is of the form $\Gamma \backslash S^3$ with Γ a discrete subgroup of $U(2)$ (or $SO(4)$, which amounts to the same thing; see the remark below) acting freely on S^3. Identifying S^3 with the unit sphere in $\boldsymbol{C}^2$, we have a natural multiplication of S^3 on itself given by

$$(z_1, z_2)(w_1, w_2) = (z_1 w_1 - z_2 \bar{w}_2, z_2 \bar{w}_1 + z_1 w_2).$$

(This formula is simply quaternionic multiplication when identifying (z_1, z_2) with $q = z_1 + z_2 j$.) There is a natural epimorphism

$$\Phi : S^3 \times S^3 \to SO(4)$$

given by

$$\Phi(q_1, q_2)(x) = q_1 x q_2^{-1},$$

where q_1, q_2, x are unit quaternions, see [9]. Moreover, it is shown there that any finite subgroup of $SO(4)$ acting freely on S^3 is conjugate in $O(4)$ to a subgroup of $\Phi(S^3 \times S^1)$, where S^1 is identified with $\{(z_1, 0) \in \boldsymbol{C}^2 : |z_1| = 1\}$. Hence, up to diffeomorphism, we may assume that $M = \Gamma \backslash S^3$ with $\Gamma \subset \Phi(S^3 \times S^1)$.

REMARK. Alternatively, one may conjugate Γ in $O(4)$ to a subgroup of $\Phi(S^1 \times S^3)$. It is easy to seen that $\Phi(S^1 \times S^3) = U(2)$, with $U(2)$ acting on $\boldsymbol{C}^2$ by matrix multiplication from the left. This implies the statement above about the equivalence of considering subgroups $\Gamma \subset U(2)$ or $\Gamma \subset SO(4)$.

We leave the proof of the following lemma to the reader (cf. Lemma 6 below).

LEMMA 4. *The Lie algebra (of left-invariant vector fields) $su(2)$ of $S^3 = SU(2)$ admits a basis e_1, e_2, e_3 with $[e_i, e_j] = e_k$ for any cyclic permutation (i, j, k) of $(1, 2, 3)$ such that the right action of S^1 is given by the flow of e_1. Hence, this action preserves e_1 and rotates the (e_2, e_3)-plane uniformly.*

It follows from Lemma 4 that the dual 1-form $\omega = e_1^*$ is a contact form on S^3

invariant under the action of $\Phi(S^3 \times S^1)$. The fact that ω is a contact form follows directly from the non-integrability condition $[e_2, e_3] \notin \mathscr{D}$ of the 2-plane field $\mathscr{D} = \ker \omega$, indeed

$$\omega \wedge d\omega(e_1, e_2, e_3) = d\omega(e_2, e_3) = -\omega([e_2, e_3]) \neq 0 .$$

It remains to show that ω is a normal contact form. Here (and in the following sections) we use the following result of Fröhlicher [2], cf. [12].

LEMMA 5. *Let J be a left-invariant almost complex structure on the real Lie group G. Then J is integrable if and only if the $(+i)$-eigenspace $(T_eG \otimes \boldsymbol{C})^{(1,0)}$ of J on the complexified Lie algebra $T_eG \otimes \boldsymbol{C}$ is a Lie subalgebra.*

In other words, the integrability condition is that the bracket of two left-invariant vector fields of type $(1, 0)$ is again of type $(1, 0)$; by writing two arbitrary vector fields of type $(1, 0)$ as linear combinations (with coefficients in $C^\infty(G, \boldsymbol{R})$) of left-invariant vector fields of that type it is easy to see that their bracket is also of type $(1, 0)$.

The almost complex structure J on $M \times S^1$ induced by ω is given by $Je_1 = \partial_t$ and $Je_3 = e_2$ on $S^3 \times E^1$. Observe that J is invariant under the action of $S^3 \times S^1$ on S^3. Hence a basis of $(T_eG \otimes \boldsymbol{C})^{(1,0)}$ (where $G = S^3 \times E^1$) is given by

$$\partial_t - iJ\partial_t = \partial_t + ie_1 \quad \text{and} \quad e_2 - iJe_2 = e_2 + ie_3 .$$

We compute

$$[\partial_t + ie_1, e_2 + ie_3] = e_2 + ie_3 ,$$

which proves that J is integrable.

4. $\widetilde{SL}_2$-geometry. Now let (M, ω) be a 3-manifold with normal contact form ω such that $W = M \times S^1$ with the induced complex structure is a properly elliptic surface of the form $W = (\Gamma \backslash \widetilde{SL}_2) \times S^1$ with $\Gamma \subset \mathrm{Isom}_0(\widetilde{SL}_2)$. By arguments completely analogous to those used in Section 3 (cf. [3]) we deduce that $M = \Gamma \backslash \widetilde{SL}_2$.

For the converse we have the following analogue of Lemma 4, which is proved in [4].

LEMMA 6. *The Lie algebra sl_2 of $\widetilde{SL}_2$ has a basis e_1, e_2, e_3 with*

$$[e_1, e_2] = e_3 , \quad [e_2, e_3] = -e_1 , \quad [e_3, e_1] = e_2 ,$$

such that the $\boldsymbol{R}$-factor of $\mathrm{Isom}_0(\widetilde{SL}_2) = \widetilde{SL}_2 \times_{\boldsymbol{Z}} \boldsymbol{R}$ *is given by the flow of e_1.*

As before we define $\omega = e_1^*$. This is invariant under the action of $\mathrm{Isom}_0(\widetilde{SL}_2)$, and the induced almost complex structure is given by $Je_1 = \partial_t$, $Je_2 = e_3$. The same computation as in Section 3 shows that J is integrable.

5. The Heisenberg group. The Heisenberg group Nil^3 is the group of upper triangular (3×3)-matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

$\mathrm{Isom}_0(Nil^3)$ is generated by left multiplication,

$$(x_0, y_0, z_0)(x, y, z) = (x_0 + x, y_0 + y, z_0 + z + x_0 y),$$

and the action of the isotropy group S^1 (cf. [9]),

$$(x, y, z) \mapsto (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta, z + (\sin\theta(y^2\cos\theta - x^2\cos\theta - 2xy\sin\theta)/2)).$$

Set $\omega = dz - x\,dy$. A straightforward computation shows that ω is a contact form on Nil^3 invariant under the action of $\mathrm{Isom}_0(Nil^3)$. The induced invariant almost complex structure on $Nil^3 \times E^1$ is given by

$$J\partial_x = -(\partial_y + x\partial_z), \quad J\partial_z = \partial_t,$$

and J is integrable since

$$[\partial_x + i(\partial_y + x\partial_z), \partial_z - i\partial_t] = 0.$$

It remains to show that if $W = M \times S^1$ is a Kodaira surface with complex structure induced from a normal contact form on M, then M is modelled on Nil^3. The fact that $M \times S^1$ is diffeomorphic to $(\Gamma \backslash Nil^3) \times S^1$ implies that M is homotopy equivalent to $\Gamma \backslash Nil^3$. By a result of Scott [10], M and $\Gamma \backslash Nil^3$ are homeomorphic and hence diffeomorphic by the uniqueness of differentiable structures on topological 3-manifolds, cf. [5].

Alternatively, one can again observe that the holomorphic S^1-action on W sends fibres to fibres and that the flow of ∂_t actually has to be along the fibres of the (unique) elliptic fibration, so the quotient M under this S^1-action is a Seifert fibred manifold by the argument used in Section 3. Hence M is a geometric manifold, and the geometric type can only be Nil^3 because of the uniqueness of the geometric type of W.

6. The remaining geometries. Let W be a complex surface diffeomorphic to $(\Gamma \backslash (H^2 \times E^1)) \times S^1$ with holomorphic S^1-action. Such a surface has an elliptic fibration with Euler number zero, and the S^1-action has to go along the fibres, because the base orbifold is of negative orbifold characteristic and does not admit any non-trivial S^1-actions. If the complex structure on $W = M \times S^1$ were induced from a normal contact form on M, this argument would show that M is a Seifert manifold of the form $M = \Gamma \backslash (H^2 \times E^1)$ with Seifert fibration given by the flow of the Reeb vector field ξ. Since the Euler number of this Seifert fibration is zero, ω would lift to a contact form on $\Sigma_g \times S^1$, where Σ_g denotes a surface of genus $g > 1$, with Reeb vector field ξ tangent to the S^1-factor. But then $d\omega$ would induce an exact area form on the transversal Σ_g, which is absurd.

The argument in the remaining cases of Theorem 3 is analogous. If W were a complex torus, then $\partial_t - iJ\partial_t$ would lift to a holomorphic vector field on C^2 with fourfold periodic coefficient functions and thus would have to be constant; hence so would $\xi = -J\partial_t$. Then M would have to be a 3-torus and ξ a constant slope vector field on T^3, which would admit a transverse 2-torus.

If W were a ruled surface of genus 1 of the form $W = S^2 \times T^2$, the flow of ∂_t would have to be transverse to the S^2-factor by the hairy ball theorem (cf. [3, Section 5.1]). Hence we would have $M = S^2 \times S^1$ with ξ transverse to the S^2-factor (see Section 7 below), which leads to a contradiction as before.

This concludes the proof of Theorem 1.

7. Normal almost contact structures. By Theorem 3 and the arguments used in the preceding sections, the only manifolds other than those from Theorem 1 that might admit a normal almost contact structure are manifolds that are homotopy equivalent to one of those listed in Theorem 2. In case (a) the arguments from Section 3 are actually strong enough to prove that M is diffeomorphic to a manifold of the form $\Gamma \backslash (H^2 \times E^1)$. In cases (a) and (b) homotopy equivalence implies diffeomorphism by the result of Scott [10] mentioned above.

In case (c) we argue as follows. Suppose M is a 3-manifold with normal almost contact structure such that $W = M \times S^1$ is a ruled surface diffeomorphic to $S^2 \times T^2$. Let $\pi : W \to T^2$ denote the ruling. By the same arguments as in the elliptic case the S^1-action has to send fibres to fibres and induces an S^1-action on the base T^2 of the ruling (which can be seen to be given by a constant slope vector field because of the holomorphicity of $\pi_*(\partial_t - iJ\partial_t)$). We claim that the period of this S^1-action equals the period of the S^1-action on W. Assuming this claim, we see that a fibre of W descends to an embedded sphere in $M = W/\langle \partial_t \rangle$, and from [5, Lemma 3.13] or by a direct geometric argument it follows that M is diffeomorphic to $S^2 \times S^1$ (with ξ transverse to the S^2-factor, thus proving the assertion at the end of Section 6). To prove the claim we argue by contradiction. If the period of the S^1-action on W were k times the period of the S^1-action on $T^2 (k > 1)$, we would get an induced holomorphic Z_k-action on a fixed fibre S^2. However, such an action would necessarily have a fixed point, contradicting the fact that the S^1-action on W is principal.

Thus, to complete the proof of Theorem 2 it only remains to exhibit a normal almost contact structure on each of the manifolds listed there. In each case there is an obvious complex structure on $M \times S^1$ compatible with the geometry, see [13]. Denote the S^1-coordinate by t and set $\xi = -J\partial_t$. Furthermore, one finds an S^1-invariant complex line complementary to $\{\partial_t, J\partial_t\}$ and tangent to the M-factor. Let η be the 1-form on M defining this complex line; then $(J|\ker \eta, \xi, \eta)$ defines a normal almost contact structure on M.

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