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JULIA SET OF THE FUNCTION $z\exp(z+\mu)$ II

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Abstract. We are concerned with bifurcation of Julia sets for the one-parameter family of functions in the title with the real parameter μ. In particular, the distribution of values of μ, for which the Julia sets of the functions coincide with the complex plane, is discussed.

Introduction. Let f_μ be an entire transcendental function $z\mapsto z\exp(z+\mu)$, where μ is a complex parameter. Put $f_\mu^n=f_\mu\circ f_\mu^{n-1}$ for a positive integer n, where f_μ^0 means the identity mapping of the complex plane $\boldsymbol{C}$. The Julia set J_μ of f_μ is defined as the set of all points on $\boldsymbol{C}$, in any neighbourhood of every point of which the sequence $\{f_\mu^n\}_{n=0}^\infty$ does not form a normal family.

Baker [1] proved the following theorem.

THEOREM. *There exists a real value of the parameter μ such that the Julia set J_μ of f_μ coincides with $\boldsymbol{C}$.*

Jang [3] proved the following result by studying Baker's argument in detail: There are infinitely many positive real values of μ with the property $J_\mu=\boldsymbol{C}$.

In this article, we study the distribution of values of μ stated in the above result of Jang. Noting another result $J_\mu\neq\boldsymbol{C}\,(-\infty<\mu<2)$ of Jang [3], we restrict the parameter μ to the real value not less than 1.

1. Values μ_n and $\mu^{(n)}$ of the parameter μ. Obviously the set of singular values of $f: z\mapsto z\exp(z+\mu)$ consists of two values $z=0$ and $z=f_\mu(-1)$. The point $z=0$ is the only one finite transcendental singularity of the inverse function f_μ^{-1} of f_μ and this is fixed by f_μ. The point $z=f_\mu(-1)$ is the only one finite algebraic singularity of f_μ^{-1}.

For a fixed value μ of the parameter, we put

$$s_0(\mu)=-1 \quad \text{and} \quad s_n(\mu)=f_\mu(s_{n-1}(\mu))\,, \qquad n\geqq 1\,.$$

The sequence $\{s_n(\mu)\}_{n=1}^\infty$ is the so-called orbit of the critical value $z=f_\mu(-1)$ of f_μ under the iteration of f_μ. The behaviour of this orbit plays a very important role in the study of the bifurcation of Julia sets J_μ. So, first we state some properties of $s_n(\mu)$.

Since the parameter μ is real, every $s_n(\mu)$ is negative and we have

$$s_n(\mu)=s_k(\mu)\exp\psi_{k,n-k}(\mu)\,, \qquad 0\leqq k\leqq n-1\,, \tag{1}$$

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where

(2) $$\psi_{k,l}(\mu)=\sum_{j=k}^{k+l-1}(s_j(\mu)+\mu)\,,\qquad l\geqq 1\,.$$

For an arbitrary real constant α, we see

(3) $$\lim_{\mu\to\infty}(s_1(\mu)+\alpha\mu)=-\infty\,.$$

As Jang [3] showed, (3) implies

(4) $$\lim_{\mu\to\infty}s_n(\mu)=0\,,\qquad n\geqq 2\,.$$

Evidently we see

(5) $$\mu\leqq -s_1(\mu)=\exp(-1+\mu)\,,$$

where the equality holds only for $\mu=1$. In other words, the equation $s_1(\mu)+\mu=0$ in the unknown μ has the only one root $\mu_1=1$. We see also that the equation $s_1(\mu)+1=0$ has the only one root $\mu^{(1)}=1$. A simple calculation shows that $s_2(\mu)+\mu=0$ has the only one root $\mu_2=1$ in the interval $1\leqq\mu<\infty$ and that $s_2(\mu)+\mu$ is positive for $\mu>\mu_2$. It is also easy to see that the equation $\psi_{0,2}(\mu)=-1+s_1(\mu)+2\mu=0$ has two roots $\mu=1$ and $\mu=\mu^{(2)}$ (>1) and $\psi_{0,2}(\mu)$ is positive in the interval $1<\mu<\mu^{(2)}$ and is negative in the intervals $0<\mu<1$ and $\mu^{(2)}<\mu<\infty$. Since we see

$$\psi_{0,2}(1+\log 3)=-4+2(1+\log 3)>0\,,$$

the equation $s_2(\mu)+1=-\exp\psi_{0,2}(\mu)+1=0$ has the greatest root $\mu^{(2)}$ greater than $1+\log 3$.

For completeness of our discussion, we recall Jang's argument in [3] under a slight improvement. Since $s_2(\mu^{(2)})+1=0$, (5) implies

$$s_3(\mu^{(2)})+\mu^{(2)}=s_1(\mu^{(2)})+\mu^{(2)}<0\,.$$

Hence (4) gives us the existence of the greatest root $\mu=\mu_3$ $(>\mu^{(2)})$ of the equation $s_3(\mu)+\mu=0$. Clearly $s_3(\mu)+\mu$ is positive for $\mu>\mu_3$. Since $s_3(\mu_3)=-\mu_3<-\mu^{(2)}<-(1+\log 3)$, the equality (4) shows the existence of the greatest root $\mu^{(3)}$ $(>\mu_3)$ of the equation $s_3(\mu)+1=0$. Obviously $s_3(\mu)+1$ is positive for $\mu>\mu^{(3)}$.

We use $\mu^{(3)}$ instead of $\mu^{(2)}$ in the above observation and see the existence of the greatest root μ_4 $(>\mu^{(3)})$ of the equation $s_4(\mu)+\mu=0$ and the existence of the greatest root $\mu^{(4)}$ $(>\mu_4)$ of the equation $s_4(\mu)+1=0$. It is easy to check that $s_4(\mu)+\mu$ is positive for $\mu>\mu_4$ and $s_4(\mu)+1$ is also positive for $\mu>\mu^{(4)}$.

Repeating the above procedure, we have a sequence of infinitely many values μ_n and $\mu^{(n)}$ of the parameter μ such that

(6) $1=\mu_1=\mu^{(1)}=\mu_2<1+\log 3<\mu^{(2)}<\mu_3<\mu^{(3)}<\cdots<\mu_n<\mu^{(n)}<\mu_{n+1}<\mu^{(n+1)}<\cdots$,

where

$$\begin{cases} s_n(\mu_n)+\mu_n=0\,, & n\geqq 1\,, \\ s_n(\mu)+\mu>0 \quad \text{for} \quad \mu>\mu_n\,, & n\geqq 2 \end{cases} \tag{7}$$

and

$$\begin{cases} s_n(\mu^{(n)})+1=0\,, & n\geqq 1\,, \\ s_n(\mu)+1>0 \quad \text{for} \quad \mu>\mu^{(n)}\,, & n\geqq 2\,. \end{cases} \tag{8}$$

REMARK. Jang [3] states only that, for $n\geqq 3$, the equation $s_n(\mu)+\mu=0$ has a root μ_n ($>\mu^{(n-1)}$) (not necessarily the greatest) and that the equation $s_n(\mu)+1=0$ has a root $\mu^{(n)}$ ($>\mu_n$) (not necessarily the greatest).

2. Distribution of the sequence $\{\mu_n\}_{n=1}^{\infty}$. First we prove the following proposition.

PROPOSITION 1. *For values $\mu^{(n)}$ ($n\geqq 2$) of the parameter μ, the n points $s_k(\mu^{(n)})$, $0\leqq k\leqq n-1$, are mutually distinct and are super-attractive n-th periodic points of $f_{\mu^{(n)}}$. Therefore, the Julia set of $f_{\mu^{(n)}}$ does not coincide with* $\boldsymbol{C}$.

PROOF. Suppose that there are integers k and l ($0\leqq k<l\leqq n-1$) with the property $s_k(\mu^{(n)})=s_l(\mu^{(n)})$. Clearly $s_k(\mu^{(n)})=s_{k+q(l-k)}(\mu^{(n)})$ for any non-negative integer q. There is a positive integer p satisfying $k+p(l-k)\leqq n<k+(p+1)(l-k)$. The sequence $\{s_j(\mu^{(n)})\}_{j=k+p(l-k)}^{k+(p+1)(l-k)}$ containing $s_n(\mu^{(n)})$ coincides with the sequence $\{s_j(\mu^{(n)})\}_{j=k}^{l}$ and this shows the existence of such a j ($k\leqq j<l$) that $s_j(\mu^{(n)})=s_n(\mu^{(n)})$. This contradicts (8). Thus n points $s_k(\mu^{(n)})$ ($0\leqq k\leqq n-1$) are mutually distinct. Since $f'_{\mu^{(n)}}(-1)=0$, it is easy to see that these n points are super-attractive n-th periodic points of $f_{\mu^{(n)}}$.

On the value μ_n ($n\geqq 3$) of the parameter μ, we can see that the point $s_n(\mu_n)$ is a repulsive fixed point of $f=f_{\mu_n}$. To see this, we note (7) and (6) and have

$$f(s_n(\mu_n))=f(-\mu_n)=-\mu_n$$

and

$$f'(s_n(\mu_n))=f'(-\mu_n)=-\mu_n+1<-\log 3\,.$$

Thus $s_n(\mu_n)$ is a repulsive fixed point of f. Hence, as Jang stated in [3], Baker's argument in [1], which was used to prove the theorem stated in the introduction of this article, leads us to the following result of Jang stated also in the introduction: The Julia set of f_{μ_n} ($n\geqq 3$) coincides with $\boldsymbol{C}$. This is also proved in the following way. By Eremenko-Lyubich's theorem [2], the function f_{μ_n} has no wandering domain and no Baker domain. Hence Sullivan's argument [4] implies $J_{\mu_n}=\boldsymbol{C}$.

Now we prove the following theorem.

THEOREM 2.

$$\lim_{n\to\infty} \mu^{(n)} = \lim_{n\to\infty} \mu_n = \infty .$$

PROOF. By (6), it suffices to show $\lim_{n\to\infty} \mu^{(n)} = \infty$. Since the sequence $\{\mu^{(n)}\}_{n=1}^{\infty}$ is increasing, we see the existence of $\mu^{(\infty)} = \lim_{n\to\infty} \mu^{(n)} \leqq \infty$. Assume $\mu^{(\infty)} < \infty$. Clearly we have $1+\log 3 < \mu^{(\infty)}$ by (6) and $-1 < s_n(\mu^{(\infty)}) < 0$ $(n \geqq 2)$ by (8). Hence we have

$$s_{n+1}(\mu^{(\infty)})/s_n(\mu^{(\infty)}) = \exp(s_n(\mu^{(\infty)}) + \mu^{(\infty)}) > \exp(-1+\mu^{(\infty)}) > 3$$

for every n $(\geqq 2)$, which implies

$$-1 < s_{n+1}(\mu^{(\infty)}) < 3^{n-1} s_2(\mu^{(\infty)}) .$$

The right hand side of this tends to $-\infty$, as n tends to infinity. This is a contradiction. Hence $\mu^{(\infty)}$ must be infinity.

The above theorem can also be deduced from the following proposition.

PROPOSITION 3. $\mu^{(n)} > 1+\log(n+1)$ *for* $n \geqq 2$.

PROOF. In the case $n=2$, we have seen $1+\log 3 < \mu^{(2)}$ in (6). Hereafter, we consider the case $n \geqq 3$.

Put $y_1 = y_1(\mu) = -s_1(\mu)$, $y_2 = y_2(\mu) = \psi_{0,n}(\mu) - s_1(\mu)$ and $y_3 = y_3(\mu) = -(n-1) + n\mu$. We see easily that the equation $y_1 = y_3$ has two roots $\mu = 1$ and $\mu = \mu_*$ (>1) and that $y_1 < y_3$ if and only if μ is in the open interval $1 < \mu < \mu_*$.

In the case $\mu_* \leqq \mu^{(n-1)}$, (6) implies $\mu_* < \mu^{(n)}$.

Consider the contrary case $\mu^{(n-1)} < \mu_*$. In this case, (6) and (8) give us $s_k(\mu)+1 > 0$ in $\mu > \mu^{(n-1)}$ for $2 \leqq k \leqq n-1$. Hence we have

$$y_2 - y_3 = \sum_{j=0}^{n-1} (s_j(\mu)+\mu) - s_1(\mu) + (n-1) - n\mu > 0$$

for $\mu > \mu^{(n-1)}$. As was seen already, we have $y_1 < y_3$ in the interval $\mu^{(n-1)} < \mu < \mu_*$. Hence we see $y_1 < y_2$ in this interval. On the other hand, (3) and (4) imply

$$\lim_{\mu\to\infty} (y_2 - y_1) = \lim_{\mu\to\infty} \psi_{0,n}(\mu) = -\infty .$$

Since $y_2(\mu_*) - y_1(\mu_*) = y_2(\mu_*) - y_3(\mu_*)$ is positive, the equation $y_1 - y_2 = 0$ has a root greater than μ_*. As $\mu^{(n)}$ is the greatest root of $s_n(\mu)+1=0$ and of $\psi_{0,n}(\mu) = y_2 - y_1 = 0$, we see $\mu_* < \mu^{(n)}$.

Thus we have always $\mu_* < \mu^{(n)}$. On the other hand, we have

$$\begin{aligned} y_1(1+\log(n+1)) &= n+1 < 1 + n\log(n+1) \\ &= y_3(1+\log(n+1)) , \end{aligned}$$

which implies $1+\log(n+1) < \mu_*$. Therefore, we have

$$1+\log(n+1)<\mu^{(n)}$$

for $n \geqq 3$. This is the required.

REMARK. By more careful observation, we can see

$$\mu^{(n)}>\begin{cases} 1+\log(2n+1) & n\geqq 4\,, \\ 1+\log(3n+1)\,, & n\geqq 9\,, \\ 1+\log(4n+1)\,, & n\geqq 20 \end{cases}$$

and so on. The proofs of these may be omitted here.

We have also the following proposition.

PROPOSITION 4. $\mu^{(3)}>3$.

PROOF. A direct calculation gives us

$$-74/10<s_1(3)=-\exp 2<-7\,.$$

Hence we see

$$s_2(3)=-\exp(5+s_1(3))>-\exp(-2)>-1/7$$

and

$$\begin{aligned} s_3(3) &= -\exp(8+s_1(3)+s_2(3)) \\ &< -\exp(8-74/10-1/7)<-1\,. \end{aligned}$$

Since the value $\mu^{(3)}$ is the greatest root of $s_3(\mu)+1=0$, we have $\mu^{(3)}>3$ by (4).

REMARK. According to Sagawa, $\mu^{(3)}$ lies between 31/10 and 32/10.

3. Repulsive periodic points of f_μ for some values of μ. In the preceding section, we were concerned with the values μ_n of the parameter μ, each of which is the greatest root of the equation $\psi_{n,1}(\mu)=s_n(\mu)+\mu=0$. In this section, we are concerned with the greatest root of the equation $\psi_{n,k}(\mu)=0$ for $n\geqq 3$ and $k\geqq 2$. We see easily by (1) that, for this greatest root μ of $\psi_{n,k}(\mu)=0$, $s_{n+k}(\mu)$ is equal to $s_n(\mu)$ so that $s_n(\mu)$ is a periodic point of f_μ.

Under the conditions $n\geqq 3$ and $k\geqq 2$, we see $\mu^{(n+k-2)}\geqq\mu^{(3)}$ by (6). If μ is not less than $\mu^{(n+k-2)}$, we see $s_{n+k-2}(\mu)+1\geqq 0$ and $-1<s_j(\mu)<0$ for $2\leqq j\leqq n+k-3$. Those are conclusions from (8). Hence we have

$$\begin{aligned} s_{n+k-3}(\mu) &= s_{n+k-2}(\mu)\exp(-s_{n+k-3}(\mu)-\mu) \\ &> s_{n+k-2}(\mu)\exp(1-\mu)>-\exp(1-\mu) \end{aligned}$$

for $\mu\geqq\mu^{(n+k-2)}$. Similarly, for $2\leqq j\leqq n+k-4$, we have

$$\begin{aligned} s_j(\mu) &> s_{j+1}(\mu)\exp(1-\mu) \\ &> s_{n+k-3}(\mu)\exp((n+k-3-j)(1-\mu)) \\ &> -\exp((n+k-2-j)(1-\mu)) \end{aligned}$$

for $\mu \geqq \mu^{(n+k-2)}$. Therefore, for $2 \leqq p \leqq n+k-3$ and for $\mu \geqq \mu^{(n+k-2)}$, we have

$$\begin{aligned} \sum_{j=p}^{n+k-3} s_j(\mu) &> -\sum_{j=p}^{n+k-3} \exp((n+k-2-j)(1-\mu)) \\ &> -1/(\exp(\mu-1)-1)\,. \end{aligned}$$

Proposition 4 and (6) imply

$$\sum_{j=p}^{n+k-3} s_j(\mu) > -1/((\exp 2)-1) > -1/6$$

for $2 \leqq p \leqq n+k-3$ and $\mu \geqq \mu^{(n+k-2)}$. Hence we see

$$\begin{aligned} &\psi_{0,n+k-2}(\mu^{(n+k-2)}) - \sum_{j=0}^{1} (s_j(\mu^{(n+k-2)}) + \mu^{(n+k-2)}) - (k-2)\mu^{(n+k-2)} \\ &= \sum_{j-2}^{n+k-3} s_j(\mu^{(n+k-2)}) + (n-2)\mu^{(n+k-2)} > 0\,. \end{aligned}$$

Here we recall $\mu^{(n+k-2)}$ is a root of $s_{n+k-2}(\mu)+1=0$, that is, a root of $\psi_{0,n+k-2}(\mu)=0$. Hence the above inequality shows

$$\sum_{j=0}^{1} s_j(\mu^{(n+k-2)}) + k\mu^{(n+k-2)} < 0\,. \tag{9}$$

Now we can prove the following proposition.

PROPOSITION 5. *For $n \geqq 3$ and $k \geqq 2$, the equation $\psi_{n,k}(\mu)=0$ has the greatest root $\mu=\mu_{n,k}$, and $\psi_{n,k}(\mu)$ is positive for $\mu>\mu_{n,k}$. In addition, the inequalities $\mu^{(n+k-2)}<\mu_{n,k}<\mu^{(n+k-1)}$ hold.*

PROOF. The inequality (9) shows

$$\begin{aligned} \psi_{n,k}(\mu^{(n+k-2)}) &= \sum_{j=n}^{n+k-1} (s_j(\mu^{(n+k-2)}) + \mu^{(n+k-2)}) \\ &< s_{n+k-2}(\mu^{(n+k-2)}) + s_{n+k-1}(\mu^{(n+k-2)}) + k\mu^{(n+k-2)} \\ &= s_0(\mu^{(n+k-2)}) + s_1(\mu^{(n+k-2)}) + k\mu^{(n+k-2)} < 0 \end{aligned}$$

by virtue of $s_j(\mu)<0$ and of $s_{n+k-2}(\mu^{(n+k-2)}) = -1 = s_0(\mu^{(n+k-2)})$. On the other hand, for $\mu \geqq \mu^{(n+k-1)}$, we see

$$\psi_{n,k}(\mu) = \sum_{j=n}^{n+k-1} (s_j(\mu)+\mu) > -k+k\mu > 0 \tag{10}$$

by (8) and (6). Hence there is the greatest root $\mu_{n,k}$ of the equation $\psi_{n,k}(\mu)=0$ such that $\mu^{(n+k-2)}<\mu_{n,k}<\mu^{(n+k-1)}$. Thus we have our proposition.

Using this proposition, we prove the following proposition.

PROPOSITION 6. *For $n\geqq 3$ and $k\geqq 2$, the points $s_j(\mu_{n,k})$ $(n\leqq j\leqq n+k-1)$ are mutually distinct k-th periodic points of $f_{\mu_{n,k}}$.*

PROOF. For simplicity, put $\mu=\mu_{n,k}$ and $f=f_\mu$. As was stated at the beginning of this section, $s_n(\mu)$ is equal to $s_{n+k}(\mu)$. So, it suffices to prove $s_{n+j}(\mu)\neq s_{n+l}(\mu)$ for $0\leqq j<l\leqq k-1$.

Assume $s_{n+j}(\mu)=s_{n+l}(\mu)$ for $0\leqq j<l\leqq k-1$. Then we see

$$s_{n+j}(\mu)=s_{n+l}(\mu)=f^{l-j}(s_{n+j}(\mu))=s_{n+j}(\mu)\exp\psi_{n+j,l-j}(\mu)\,,$$

which shows $\psi_{n+j,l-j}(\mu)=0$. Proposition 5 shows that the greatest root of the equation $\psi_{n+j,l-j}(\mu)=0$ lies between $\mu^{(n+l-2)}$ and $\mu^{(n+l-1)}$. So we have $\mu<\mu^{(n+l-1)}\leqq\mu^{(n+k-2)}$. Since $\mu=\mu_{n,k}$ is greater than $\mu^{(n+k-2)}$ by Proposition 5, we have a contradiction. Therefore, we see $s_{n+j}(\mu)\neq s_{n+l}(\mu)$ for $0\leqq j<l\leqq k-1$ and we have our proposition.

PROPOSITION 7. *For $n\geqq 3$ and $k\geqq 2$, the values $\mu_{n,k}$ in Proposition 5 satisfy the following:*

$$\mu^{(n+k-2)}<\mu_{3,n+k-3}<\mu_{4,n+k-4}<\cdots<\mu_{n+k-2,2}<\mu_{n+k-1}<\mu^{(n+k-1)}\,.$$

PROOF. First, as was stated in Proposition 5, we have

$$\psi_{n,k}(\mu_{n,k})=\sum_{j=n}^{n+k-1}(s_j(\mu_{n,k})+\mu_{n,k})=0\,.$$

Hence we see

$$\begin{aligned}\psi_{n+1,k-1}(\mu_{n,k})&=\sum_{j=n+1}^{n+k-1}(s_j(\mu_{n,k})+\mu_{n,k})\\&=\psi_{n,k}(\mu_{n,k})-s_n(\mu_{n,k})-\mu_{n,k}\\&=-s_n(\mu_{n,k})-\mu_{n,k}\,.\end{aligned}$$

By Proposition 5 and (6), we see $\mu^{(n)}\leqq\mu^{(n+k-2)}<\mu_{n,k}$, which shows $s_n(\mu_{n,k})+1>0$. Hence (6) leads us to

$$\psi_{n+1,k-1}(\mu_{n,k})=-s_n(\mu_{n,k})-\mu_{n,k}<1-\mu_{n,k}<0\,.$$

Therefore, we see by Proposition 5 that the greatest root $\mu_{n+1,k-1}$ of the equation $\psi_{n+1,k-1}(\mu)=0$ is greater than $\mu_{n,k}$. From this observation, we have

$$\mu^{(n+k-2)}<\mu_{3,n+k-3}<\mu_{4,n+k-4}<\cdots<\mu_{n+k-2,2}<\mu^{(n+k-1)}\,.$$

Furthermore, since μ_{n+k-1} is the greatest root of the equation $\psi_{n+k-1,1}(\mu)=s_{n+k-1}(\mu)+$

$\mu=0$, we may put $\mu_{n+k-1}=\mu_{n+k-1,1}$ in the notation used in Proposition 5. So, similarly to the above, we see easily $\mu_{n+k-2,2}<\mu_{n+k-1}<\mu^{(n+k-1)}$. Thus we have our proposition.

Now we prove the following theorem.

THEOREM 8. *Assume $n\geqq 3$ and $k\geqq 2$. Then, for the values $\mu_{n,k}$ of the parameter μ obtained in Proposition 5, the Julia set of $f_{\mu_{n,k}}$ coincides with $\boldsymbol{C}$.*

PROOF. Proposition 6 shows that k-th periodic points $s_j(\mu_{n,k})$ $(n\leqq j\leqq n+k-1)$ of $f=f_{\mu_{n,k}}$ are mutually distinct. Suppose that there is a j $(n\leqq j\leqq n+k-1)$ with the property $s_j(\mu_{n,k})=-1$. This means that the point -1 is a k-th periodic point of f and we have $s_k(\mu_{n.k})=f^k(-1)=-1$. This and (8) imply $\mu_{n,k}\leqq\mu^{(k)}$. Proposition 5 leads us to a contradiction. Hence every point $s_j(\mu_{n,k})$ $(n\leqq j\leqq n+k-1)$ is different from -1. The equation $z\exp(z+\mu)=s_1(\mu)=-\exp(-1+\mu)$ has the only one real root $z=-1$ and hence the sequence $\{s_j(\mu_{n,k})\}_{j=n}^{n+k-1}$ does not contain $s_1(\mu_{n,k})$. Therefore, the critical point $s_1(\mu_{n,k})$ of f is a preperiodic point of f. In the same way as was stated after Proposition 1, Eremenko-Lyubich's theorem [2] and Sullivan's argument [4] give us the desired.

REFERENCES

[1] I. N. BAKER, Limit functions and sets of non-normality iteration theory, Ann. Acad. Sci. Fenn., A. I. Math. 467 (1970), 1–9.

[2] A. E. EREMENKO AND M. LYUBICH, Dynamical properties of some classes of entire functions, Ann. Inst. Fourier, Grenoble 42 (1992), 989–1020.

[3] C. M. JANG, Julia set of the function $z\exp(z+\mu)$, Tôhoku Math. J. 44 (1992), 271–277.

[4] D. SULLIVAN, Conformal dynamical systems, in Geometric Dynamics, Lecture Notes in Math. 1007, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983, 725–752.

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Added in proof (Received October 6, 1997). After this paper was submitted, the authors learned through Professor S. Nakane that N. Fagella also discussed the same problem as ours in the following paper from another view point: Limiting dynamics for the complex standard family, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 5 (1995), 673–699.