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AN INTEGRAL REPRESENTATION OF EIGENFUNCTIONS FOR MACDONALD'S q-DIFFERENCE OPERATORS

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Abstract. We give the eigenfunctions for Macdonald's q-difference operators in terms of q-Selberg type integrals. Our result can be applied not only to the case of Macdonald symmetric polynomials but also to the cases of rational and meromorphic solutions.

1. Introduction. The purpose of this paper is to study the eigenvalue problem for Macdonald's commuting family of q-difference operators from the viewpoint of integrals. We will show in particular that Macdonald's q-difference equations arise naturally from integrals associated with certain (many-valued) meromorphic functions. It implies also that various solutions to Macdonald's q-difference equations, other than the Macdonald polynomials, can be obtained by taking different cycles.

The argument of this paper is carried out in the sense of q-de Rham cohomology as in Aomoto [1], although we will not formulate it precisely below. If one knows that a system of differential or difference equations arises from some de Rham cohomology, one could provide the fundamental system of its solutions by determining the corresponding homology cycles. In spite of its importance, such an application of de Rham cohomology to q-difference equations has not been thoroughly developed yet. We intend this paper to be the first step of an approach in this direction to q-difference equations of Macdonald type.

Macdonald [2] introduced a commuting family of q-difference operators defined by

$$D_y^r = t^{r(r-1)/2} \sum_{i_1<\cdots<i_r} \left[\prod_{\substack{s=1,\dots,r \\ j\notin\{i_1,\dots,i_r\}}} \frac{ty_{i_s}-y_j}{y_{i_s}-y_j} \right] T_{q,y_{i_1}} \cdots T_{q,y_{i_r}} \tag{1.1}$$

for $y=(y_1,\dots,y_n)$ and $r=1,\dots,n$. Our aim is to study the simultaneous eigenfunctions of these operators in terms of integrals. Here q is a real number satisfying $0<q<1$, and t is a nonzero real number. The shift operator T_{q,y_i} is defined by $(T_{q,y_i}f)(y_1,\dots,y_n)=f(\dots,qy_i,\dots)$.

We will make use of the generating function

$$\begin{aligned} D_y(u) &= \sum_{r=0}^{n} (-u)^r D_y^r \\ &= \frac{1}{\Delta(y)} \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) \prod_{j=1}^{n} (y_j^{n-\sigma(j)}(1-ut^{n-\sigma(j)}T_{q,y_j})) \end{aligned} \tag{1.2}$$

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of this commuting family, where $\Delta(y)$ denotes the Vandermonde determinant

$$\Delta(y)=\prod_{1\le i<j\le n}(y_i-y_j)=\det(y_j^{n-i})_{1\le i,j\le n}\,.$$

We define the function $\Psi^{(s)}=\Psi^{(s)}(y;x)$ of the variables $y=(y_1,y_2,\ldots,y_n)$ and $x=(x_1,x_2,\ldots,x_m)$ as

$$\Psi^{(s)}=\Psi^{(s)}(y;x)=\prod_{j=1}^{n}y_j^s\prod_{\substack{1\le r\le n\\1\le j\le m}}\frac{(ty_r/x_j)_\infty}{(y_r/x_j)_\infty}\prod_{\substack{1\le i,j\le m\\i\ne j}}\frac{(x_i/x_j)_\infty}{(tx_i/x_j)_\infty}\tag{1.3}$$

for $s\in\boldsymbol{C}$, where $(a)_\infty=(a;q)_\infty=\prod_{i\ge0}(1-aq^i)$. More generally, we define the function $\Psi^{(s_0,s_1,\ldots,s_{l-1})}$ of the set of variables

$$x^{(i)}=(x_1^{(i)},x_2^{(i)},\ldots,x_{k^{(i)}}^{(i)})\qquad(i=0,1,2,\ldots,l)$$

as

$$\begin{aligned}\Psi^{(s_0,s_1,\ldots,s_{l-1})}&=\Psi^{(s_0,s_1,\ldots,s_{l-1})}(x^{(0)};x^{(1)};\ldots;x^{(l)})\\&=\prod_{i=1}^{l}\Psi^{(s_{i-1})}(x^{(i-1)};x^{(i)})\end{aligned}\tag{1.4}$$

for $s_0,s_1,\ldots,s_{l-1}\in\boldsymbol{C}$.

In this paper we study integrals on a cycle C whose integration variables are

$$x^{(i)}=(x_1^{(i)},x_2^{(i)},\ldots,x_{k^{(i)}}^{(i)})\qquad(i=1,2,\ldots,l)\,,$$

and whose measure $d\xi=d\xi(x^{(1)},\ldots,x^{(l)})$ is invariant with respect to the shift operators $T_{q,x_j^{(i)}}$ $(1\le j\le k^{(i)},\ i=1,\ldots,l)$. Here C is an arbitrary cycle, but is fixed in what follows.

Our main result is the following:

THEOREM 1. *For the variables* $x^{(i)}=(x_1^{(i)},x_2^{(i)},\ldots,x_{k^{(i)}}^{(i)})$ $(0\le i\le l)$, *if a function* $\varphi(x^{(l)})$ *satisfies the q-difference equation*

$$D_{x^{(l)}}(u)\varphi(x^{(l)})=c_{x^{(l)}}(u)\varphi(x^{(l)})\,,$$

where $c_{x^{(l)}}(u)$ *is a generating function of the eigenvalues not depending on* $x^{(l)}$, *then the function*

$$\psi(x^{(0)})=\int_C\Psi^{(s_0,s_1,\ldots s_{l-1})}(x^{(0)};x^{(1)};\ldots;x^{(l)})\varphi(x^{(l)})d\xi(x^{(l)};x^{(2)};\ldots;x^{(l)})$$

satisfies

$$D_{x^{(0)}}(u)\psi(x^{(0)})=c_{x^{(0)}}(u)\psi(x^{(0)})$$

with

$$c_{x^{(0)}}(u)=c_{x^{(l)}}(uq^{s_0+\cdots+s_{l-1}}t^{k^{(0)}-k^{(l)}})\prod_{i=0}^{l-1}(uq^{s_0+\cdots+s_i}t^{k^{(0)}-k^{(i)}};t)_{k^{(i)}-k^{(i+1)}},$$

where $(u;t)_k=(u;t)_\infty/(ut^k;t)_\infty$.

When $\varphi(x^{(l)})=(\prod_{1\le j\le k^{(l)}}x_j^{(l)})^{s_l}$, it follows from (1.2) that

$$c_{x^{(l)}}(u)=(uq^{s_l};t)_{k^{(l)}}.$$

Hence we have

$$c_{x^{(0)}}(u)=\prod_{i=0}^{l}(uq^{s_0+\cdots+s_i}t^{k^{(0)}-k^{(i)}};t)_{k^{(i)}-k^{(i+1)}},$$

where $k^{(l+1)}$ is defined to be zero. Thus, also setting $l=n-1$, $k^{(i)}=n-i$ $(0\le i\le n-1)$, $s_0=\lambda_n$, $s_i=\lambda_{n-i}-\lambda_{n-i+1}$ $(1\le i\le n-1)$, and $x^{(0)}=y=(y_1,y_2,\ldots,y_n)$, we obtain the following corollary, which gives an affirmative answer to Conjecture 6.2 in [3].

COROLLARY 2. *Set*

$$\Phi=\Psi^{(\lambda_n,\lambda_{n-1}-\lambda_n,\ldots,\lambda_2-\lambda_3)}(x^{(0)};x^{(1)};\ldots;x^{(n-1)})\cdot(x^{(n-1)})^{\lambda_1-\lambda_2}.$$

Then we have

$$D_y^r\int_C\Phi d\xi=c_\lambda^r\int_C\Phi d\xi\qquad(r=1,\ldots,n)$$

for an arbitrary cycle C, where

$$d\xi=d\xi(x^{(1)};x^{(2)};\ldots;x^{(n-1)}),$$

and

$$c_\lambda^r=\sum_{i_1<\cdots<i_r}\prod_{1\le s\le r}q^{\lambda_{i_s}}t^{(n-i_s)}.$$

When the λ_j are integers such that $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n\ge0$, the function

$$\int_{T^{n(n-1)/2}}\Phi d\xi$$

expresses the Macdonald symmetric polynomial $P_\lambda(y;q,t)$ up to a constant factor, as is noted in Theorem 7.1 of [3]. Here T^r denotes the r-dimensional torus. This is a motivation for the present work (also see [4]).

Here is a remark concerning the measure $d\xi$ and the cycle C. According to the situation, we could take either the usual invariant measure

$$d\xi(x)=\frac{dx_1}{x_1}\cdots\frac{dx_k}{x_k}$$

for $x=(x_1, x_2, \dots, x_k)$, or the discrete measure given by Jackson integral. In this paper, however, we do not discuss such an explicit form of the measure and the cycle.

In Section 2, we give a proof of Theorem 1. In Section 3, we give a variant of Theorem 1 for the eigenfunctions of the single q-difference operator D_y^1, which would also be useful in applications.

2. Proof of Theorem 1. For the sake of brevity, we frequently express the equality

$$\int_C \varphi_1(x)d\xi(x) = \int_C \varphi_2(x)d\xi(x)$$

simply by

$$\varphi_1(x) \equiv \varphi_2(x) .$$

In this section, we set

$$F(u\,|\,x\,;y) = \sum_{I\subset\{1,\dots,m\}} (-u)^{|I|} t^{|I|(|I|-1)/2} \prod_{\substack{i\in I\\ j\notin I}} \frac{1-tx_i/x_j}{1-x_i/x_j} \prod_{\substack{i\in I\\ 1\le j\le n}} \frac{1-y_jx_i}{1-ty_jx_i} \tag{2.1}$$

for $x=(x_1, x_2, \dots, x_m)$ and $y=(y_1, y_2, \dots, y_n)$. Then the action of the operators $D_y(u)$ and $D_x^*(u)$ on the function $\Psi^{(s)}(y\,;x)$, defined in (1.3), are given by

$$D_y(u)\Psi^{(s)}(y\,;x) = \Psi^{(s)}(y\,;x)F(uq^s\,|\,y\,;x^{-1}) ,$$

$$D_x^*(u)\Psi^{(s)}(y\,;x) = \Psi^{(s)}(y\,;x)F(u\,|\,x^{-1}\,;y) ,$$

where x^{-1} represents $(x_1^{-1}, x_2^{-1}, \dots, x_m^{-1})$, and

$$D_x^*(u) = \sum_{I\subset\{1,\dots,m\}} (-u)^{|I|}(T_{q,x}^I)^{-1}A_I(x)$$

is a formal adjoint operator of

$$D_x(u) = \sum_{I\subset\{1,\dots,m\}} (-u)^{|I|}A_I(x)T_{q,x}^I$$

with

$$A_I(x) = t^{|I|(|I|-1)/2} \prod_{\substack{i\in I\\ j\notin I}} \frac{1-tx_i/x_j}{1-x_i/x_j} ,$$

and

$$T_{q,x}^I = T_{q,x_{i_1}} \cdots T_{q,x_{i_r}}$$

for $I=\{i_1, \dots, i_r\}$.

To proceed, we use the equality

$$F(u \mid y; x) = (u; t)_{n-m} F(ut^{n-m} \mid x; y)\,, \tag{2.2}$$

which will be proved later in Lemma 3. By (2.2), we have

$$\begin{aligned} D_y(u)\Psi^{(s)}(y; x)\varphi(x) &= \Psi^{(s)}(y; x) F(uq^s \mid y; x^{-1})\varphi(x) \\ &= \Psi^{(s)}(y; x) F(uq^s t^{n-m} \mid x^{-1}; y)(uq^s; t)_{n-m}\varphi(x) \\ &= (D_x^*(uq^s t^{n-m})\Psi^{(s)}(y; x))(uq^s; t)_{n-m}\varphi(x) \\ &\equiv \Psi^{(s)}(y; x)(uq^s; t)_{n-m}(D_x(uq^s t^{n-m})\varphi(x))\,. \end{aligned}$$

More generally, repeating such a process, we have

$$\begin{aligned} &D_{x^{(0)}}(u)\Psi^{(s_0, s_1, \ldots, s_{l-1})}(x^{(0)}; x^{(1)}; \ldots; x^{(l)})\varphi(x^{(l)}) \\ &\quad \equiv \Psi^{(s_0)}(x^{(0)}; x^{(1)})(uq^{s_0}; t)_{k^{(0)}-k^{(1)}} \\ &\qquad \times D_{x^{(1)}}(uq^{s_0} t^{k^{(0)}-k^{(1)}})\Psi^{(s_1, s_2, \ldots, s_{l-1})}(x^{(1)}; \ldots; x^{(l)})\varphi(x^{(l)}) \\ &\quad \equiv \cdots \\ &\quad \equiv \Psi^{(s_0, \ldots, s_{l-1})}(x^{(0)}; \ldots; x^{(l)}) \prod_{i=0}^{l-1} (uq^{s_0+\cdots+s_i} t^{k^{(0)}-k^{(i)}}; t)_{k^{(i)}-k^{(i+1)}} \\ &\qquad \times D_{x^{(l)}}(uq^{s_0+s_1+\cdots+s_{l-1}} t^{k^{(0)}-k^{(l)}})\varphi(x^{(l)})\,. \end{aligned}$$

To complete the proof of Theorem 1, it is enough to show the following:

LEMMA 3. *We have*

$$F(u \mid y; x) = (u; t)_{n-m} F(ut^{n-m} \mid x; y) \tag{2.3}$$

for $x = (x_1, x_2, \ldots, x_m)$ *and* $y = (y_1, y_2, \ldots, y_n)$.

PROOF. We rewrite (2.1) as

$$\begin{aligned} F(u \mid x; y) &= \sum_{I \subset \{1, \ldots, m\}} (-u)^{|I|} \frac{T^I_{t,x}\{\Delta(x)\prod_{i,j}(1 - y_i x_j)^{-1}\}}{\Delta(x)\prod_{i,j}(1 - y_i x_j)^{-1}} \\ &= \left\{\Delta(x)\prod_{i,j}(1 - y_i x_j)^{-1}\right\}^{-1} \\ &\qquad \times (1 - uT_{t,x_1})\cdots(1 - uT_{t,x_m})\left\{\Delta(x)\prod_{i,j}(1 - y_i x_j)^{-1}\right\}. \end{aligned} \tag{2.4}$$

Similarly,

$$\begin{aligned} F(u \mid y; x) &= \sum_{K \subset \{1, \ldots, n\}} (-u)^{|K|} t^{|K|(|K|-1)/2} \prod_{\substack{i \in K \\ j \notin K}} \frac{1 - ty_i/y_j}{1 - y_i/y_j} \prod_{\substack{i \in K \\ 1 \le j \le m}} \frac{1 - x_j y_i}{1 - tx_j y_i} \\ &= \left\{\Delta(y)\prod_{i,j}(1 - y_i x_j)^{-1}\right\}^{-1} \end{aligned} \tag{2.5}$$

$$\times(1-uT_{t,y_1})\cdots(1-uT_{t,y_n})\left\{\Delta(y)\prod_{i,j}(1-y_ix_j)^{-1}\right\}.$$

Recall the formula

$$\prod(1-y_ix_j)^{-1}=\sum_{l(\mu)\le\min\{m,n\}}s_\mu(x)s_\mu(y) \tag{2.6}$$

in terms of the Schur function

$$s_\mu(x)=\frac{1}{\Delta(x)}\det(x_j^{\mu_i+m-i})_{1\le i,j\le m}$$

for the partition $\mu=(\mu_1,\mu_2,\dots)$. Here $l(\mu)$ denotes the length of μ.

Moreover, we obtain

$$\begin{aligned}(2.7)\qquad &\Delta(x)^{-1}(1-uT_{t,x_1})\cdots(1-uT_{t,x_m})\{\Delta(x)s_\mu(x)\}\\ &=\Delta(x)^{-1}(1-uT_{t,x_1})\cdots(1-uT_{t,x_m})\det(x_j^{\mu_i+m-i})_{1\le i,j\le m}\\ &=\Delta(x)^{-1}\det((1-ut^{\mu_i+m-i})x_j^{\mu_i+m-i})\\ &=\Delta(x)^{-1}(1-ut^{\mu_1+m-1})\cdots(1-ut^{\mu_m})\det(x_j^{\mu_i+m-i})_{1\le i,j\le m}\\ &=(1-ut^{\mu_1+m-1})\cdots(1-ut^{\mu_m})s_\mu(x)\,.\end{aligned}$$

Similarly,

$$\Delta(y)^{-1}(1-uT_{t,y_1})\cdots(1-uT_{t,y_n})\{\Delta(y)s_\mu(y)\}=(1-ut^{\mu_1+n-1})\cdots(1-ut^{\mu_n})s_\mu(y)\,. \tag{2.8}$$

Combining (2.4)–(2.8) and noting that $\mu_i=0$ for $\min\{m,n\}+1\le i\le\max\{m,n\}$, we obtain the desired result.

This completes the proof of Theorem 1.

3. A variant for the eigenfunctions of D_y^1. So far we have discussed simultaneous eigenfunctions of the commuting family of q-difference operators $D_y^1,\dots,D_y^n$ of Macdonald. In applications, however, it is sometimes more convenient and even necessary to deal with the eigenfunctions of the single operator D_y^1. In this section we reformulate our main theorem in this form, confining ourselves to the case of $l=1$, and give a direct proof based on the method of partial fractions.

PROPOSITION 4. *For $x=(x_1,\dots,x_m)$ and $y=(y_1,y_2,\dots,y_n)$, if*

$$D_x^1\varphi(x)=c_x^1\varphi(x)\,,$$

then we have

$$D_y^1\psi(y)=q^s\left(c_x^1t^{n-m}+\frac{1-t^{n-m}}{1-t}\right)\psi(y)\,,$$

with

$$\psi(y)=\int_C \Psi^{(s)}(y;x)\varphi(x)d\xi(x)\,.$$

PROOF. First, we set, for $x=(x_1,\dots,x_m)$ and $y=(y_1,y_2,\dots,y_n)$,

$$F(x;y)=\sum_{i=1}^{m}\prod_{\substack{1\le k\le m\\ k\ne i}}\frac{1-tx_i/x_k}{1-x_i/x_k}\prod_{1\le r\le n}\frac{1-y_rx_i}{1-ty_rx_i}\,. \tag{3.1}$$

This corresponds to the coefficient of u in $F(u\,|\,x;y)$ defined in (2.1). Then we have

$$D_y^1(\Psi^{(s)}(y;x))\varphi(x)=q^s\Psi^{(s)}(y;x)\varphi(x)F(y;x^{-1})\,,$$

and

$$\begin{aligned}\Psi^{(s)}(y;x)D_x^1\varphi(x)&=\sum_{k=1}^{m}\Psi^{(s)}(y;x)A_k(x)T_{q,x_k}\varphi(x)\\ &\equiv\sum_{k=1}^{m}(T_{q,x_k})^{-1}\{\Psi^{(s)}(y;x)A_k(x)T_{q,x_k}\varphi(x)\}\\ &=\sum_{k=1}^{m}(T_{q,x_k})^{-1}\{\Psi^{(s)}(y;x)A_k(x)\}\varphi(x)\\ &=\Psi^{(s)}(y;x)F(x^{-1};y)\varphi(x)\,,\end{aligned}$$

where x^{-1} represents $(x_1^{-1},x_2^{-1},\dots,x_m^{-1})$. Thus our task is reduced to showing the equality

$$F(y;x)=t^{n-m}F(x;y)+\frac{1-t^{n-m}}{1-t}\,. \tag{3.2}$$

Regarding

$$F(y;x)=\sum_{i=1}^{n}\prod_{\substack{1\le k\le n\\ k\ne i}}\frac{1-ty_i/y_k}{1-y_i/y_k}\prod_{1\le r\le m}\frac{1-x_ry_i}{1-tx_ry_i}$$

as a function of one variable y_1, we expand $F(y;x)$ into partial fractions

$$F(y;x)=\sum_{j=2}^{n}\frac{a_j}{y_1-y_j}+\sum_{j=1}^{m}b_j\frac{1-x_jy_1}{1-tx_jy_1}+c\,. \tag{3.3}$$

The residue at $y_1=(tx_j)^{-1}$ for each $j=1,\dots,m$ gives

$$b_j=t^{n-m}\prod_{2\le k\le n}\frac{1-y_kx_j}{1-ty_kx_j}\prod_{\substack{1\le r\le m\\ r\ne j}}\frac{1-tx_j/x_r}{1-x_j/x_r}\,.$$

For each $j=2,\dots,n$, the residue at $y_1=y_j$ gives $a_j=0$. Hence the equality (3.3) is reduced to

$$F(y;x)=t^{n-m}F(x;y)+c_n\,. \tag{3.4}$$

To obtain (3.2), it is enough to show

$$c_n=\frac{1-t^{n-m}}{1-t}$$

for $n\in\boldsymbol{N}$.

In the case $n=1$, considering the case $y_1=0$, we have

$$F(0;x)=1$$

and

$$F(x;0)=\sum_{i=1}^{m}\prod_{\substack{1\le k\le m\\ k\neq i}}\frac{1-tx_i/x_k}{1-x_i/x_k}=\frac{1-t^m}{1-t}$$

by definition. Therefore, the equality (3.4) implies

$$c_1=\frac{1-t^{1-m}}{1-t}\,.$$

Next, substituting the equalities

$$F(y_1,\dots,y_{n-1},0;x)=1+tF(y_1,\dots,y_{n-1};x)$$

and

$$F(x;y_1,\dots,y_{n-1},0)=F(x;y_1,\dots,y_{n-1})$$

into (3.4), we obtain

$$1+tF(y_1,\dots,y_{n-1};x)=t^{n-m}F(x;y_1,\dots,y_{n-1})+c_n\,. \tag{3.5}$$

The case $n-1$ of (3.4) with (3.5) gives

$$1+tc_{n-1}=c_n\,.$$

This completes the derivation of (3.2).

By repeating the argument of this proof, it is not difficult to generalize Proposition 4 to the case where $l>1$ as in Theorem 1.

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