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ON SYMMETRIES OF CONSTANT MEAN CURVATURE SURFACES, PART I: GENERAL THEORY

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Abstract. We start the investigation of immersions of a simply connected domain into three dimensional Euclidean space which have constant mean curvature (CMC-immersions), and allow for a group of automorphisms of the domain which leave the image invariant. This leads to a detailed description of symmetric CMC-surfaces and the associated symmetry groups.

1. Introduction. This is the first of two parts of a note in which we start the investigation of conformal CMC-immersions $\Psi: \mathscr{D} \to \boldsymbol{R}^3$, $\mathscr{D}$ an open, simply connected subset of $\boldsymbol{C}$, which allow for groups of spatial symmetries

$$\operatorname{Aut} \Psi(\mathscr{D}) = \{\tilde{T} \text{ proper Euclidean motion of } \boldsymbol{R}^3 \mid \tilde{T}\Psi(\mathscr{D}) = \Psi(\mathscr{D})\} .$$

More precisely (see the definition in Section 2), we consider a Riemann surface M with universal covering $\pi: \mathscr{D} \to M$, and a conformal CMC-immersion $\Phi: M \to \boldsymbol{R}^3$ with nonzero mean curvature, such that $\Phi \circ \pi = \Psi$. Then we consider the groups

$$\operatorname{Aut} \mathscr{D} = \{g: \mathscr{D} \to \mathscr{D} \text{ biholomorphic}\} ,$$

$$\operatorname{Aut} M = \{g: M \to M \text{ biholomorphic}\} ,$$

$$\operatorname{Aut}_\pi \mathscr{D} = \{g \in \operatorname{Aut} \mathscr{D} \mid \text{there exists } \hat{g} \in \operatorname{Aut} M: \pi \circ g = \hat{g} \circ \pi\} ,$$

$$\operatorname{Aut}_\Phi M = \{\hat{g} \in \operatorname{Aut} M \mid \text{there exists } \tilde{T} \in \operatorname{Aut} \Psi(\mathscr{D}): \Phi \circ \hat{g} = \tilde{T} \circ \Phi\} ,$$

and

$$\operatorname{Aut}_\Psi \mathscr{D} = \{g \in \operatorname{Aut} \mathscr{D} \mid \text{there exists } \tilde{T} \in \operatorname{Aut} \Psi(\mathscr{D}): \Psi \circ g = \tilde{T} \circ \Psi\} .$$

There are many well-known examples of CMC-surfaces with large spatial symmetry groups. The classic Delaunay surfaces (see [2]) have a nondiscrete group $\operatorname{Aut} \Psi(\mathscr{D})$ containing the group of all rotations around their generating axis. Other examples are the Smyth surface [9], which were visualized by D. Lerner, I. Sterling, C. Gunn and U. Pinkall. These surfaces have an $(m+2)$-fold rotational symmetry in $\boldsymbol{R}^3$, where the axis of rotation passes through the single umbilic of order m. More recent is the large class of examples provided by Große-Brauckmann and Polthier (see e.g. [4], [5]) of

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singly, doubly and triply periodic CMC-surfaces.

Other interesting classes of surfaces are the ones with a large group $\mathrm{Aut}_\Psi \mathscr{D}$. Examples for this are the compact CMC-surfaces, whose Fuchsian or elementary group is contained in $\mathrm{Aut}_\Psi \mathscr{D}$.

Yet another class of surfaces (M, Φ), for which $\mathrm{Aut}_\Psi \mathscr{D}$ is interesting are the surfaces with branch points. If one deletes the set $B \subset M$ of those points in M which are mapped by Φ to be branch points, then such a surface can be constructed as an immersion $\hat{\Phi}$ of the non-simply connected Riemann surface $M \setminus B$ into $\boldsymbol{R}^3$. To get an immersion Ψ of a simply connected domain $\mathscr{D}$ into $\boldsymbol{R}^3$, which covers $(M \setminus B, \hat{\Phi})$, one can also apply the discussion of this paper as will be shown in Section 4.5 of the second part.

It is our goal to describe properties of the group $\mathrm{Aut}\, \Psi(\mathscr{D})$ in terms of biholomorphic automorphisms of the Riemann surface M or the simply connected cover $\mathscr{D}$, i.e., in terms of $\mathrm{Aut}_\Phi M$ or $\mathrm{Aut}_\Psi \mathscr{D}$. To this end we investigate the relation between these groups. This is done in Chapter 2. After defining in Section 2.1, what we mean by a CMC-immersions (M, Φ), we start in Section 2.2 by listing some well known properties of the groups $\mathrm{Aut}\, M$, $\mathrm{Aut}\, \mathscr{D}$, $\mathrm{Aut}_\pi \mathscr{D}$. These follows entirely from the underlying Riemannian structure of M and $\mathscr{D}$. In Section 2.3 we derive the transformation properties of the metric and the Hopf differential under an automorphism in $\mathrm{Aut}_\Psi \mathscr{D}$. This will lead in Section 2.4 to some general restrictions on $\mathrm{Aut}_\Psi \mathscr{D}$ in the case $\mathscr{D} = \boldsymbol{C}$. In Section 2.5–2.7 we will introduce group homomorphisms $\bar{\pi}: \mathrm{Aut}_\pi \mathscr{D} \to \mathrm{Aut}\, M$, $\phi: \mathrm{Aut}_\Phi M \to \mathrm{Aut}\, \Psi(\mathscr{D})$ and $\psi: \mathrm{Aut}_\Psi \mathscr{D} \to \mathrm{Aut}\, \Psi(\mathscr{D})$. We will also prove, that, in case M with the metric induced by Ψ is complete, ψ is surjective (Corollary 2.7). In Section 2.8 it will be shown that we furthermore can restrict our investigation to those CMC-immersions $\Phi: M \to \boldsymbol{R}^3$, for which ϕ is an isomorphism of Lie groups. In Sections 2.9 and 2.10 we will investigate, for which complete CMC-surfaces the group $\mathrm{Aut}_\Psi \mathscr{D}$ is nondiscrete. To this end we give a simple condition on the image $\Psi(\mathscr{D})$ of Ψ under which the group $\mathrm{Aut}\, \Psi(\mathscr{D})$ is a closed Lie subgroup of $\mathrm{OAff}(\boldsymbol{R}^3)$. Here we denote by $\mathrm{OAff}(\boldsymbol{R}^3)$ the group of proper (i.e., orientation preserving) Euclidean motions of $\boldsymbol{R}^3$. Using these results and the work of Smyth [9], we will prove in the second part (Section II.2.4), that $\mathrm{Aut}\, \Psi(\mathscr{D})$ is nondiscrete, if and only if the surface $\Psi(\mathscr{D})$ is isometric to a CMC-surface of revolution, i.e., a Delaunay surface. In Section 2.11 we illustrate the discussion in Chapter 2, using the examples of Delaunay and Smyth. The groups $\mathrm{Aut}_\Phi M$, $\mathrm{Aut}_\Psi \mathscr{D}$ and $\mathrm{Aut}\, \Psi(\mathscr{D})$ are explicitly given for these examples.

In the second part of these notes we will analyze symmetry conditions in terms of the Weierstraß-type representation of conformal CMC-immersions [1]. Using results of this paper, we will also obtain a description of CMC-immersions from arbitrary Riemann surfaces M into $\boldsymbol{R}^3$ in terms of the Weierstraß-type data.

Equations and sections in the second part of the note will be referenced by adding the roman numeral 'II' in front of the equation and section number, respectively.

2. Automorphisms of CMC-surfaces.

2.1. Before we can start the investigation of CMC-immersions (M, Φ) we have to define, what we mean by a CMC-immersion, if M is not a domain in $\boldsymbol{R}^2$. We will restrict our investigations to the case of nonzero mean curvature, i.e. we will exclude the special case of minimal surfaces.

DEFINITION. Let M be a connected C^2-manifold and let $\Phi: M \to \boldsymbol{R}^3$ be an immersion of type C^2. Φ is called a CMC-immersion, if there exists an atlas of M, s.t. every chart (U, φ) in this atlas defines a C^2-surface $\Phi \circ \varphi^{-1}: \varphi(U) \to \boldsymbol{R}^3$ with nonzero constant mean curvature.

This definition makes sense due to the well known fact that CMC surfaces are orientable and thus can be equipped with a complex structure compatible with the induced conformal structure.

In this paper we will therefore restrict ourselves to conformal CMC-immersions $\Phi: M \to \boldsymbol{R}^3$, where M is a Riemann surface. I.e., if we write "(M, Φ) *is a* CMC-*immersion*", we always mean "*M is a Riemann surface and Φ is a conformal* CMC-*immersion*".

2.2. In this section we strip the CMC-surface M of its metric and leave only the complex structure. We will recollect some well known facts a about Riemann surfaces (see e.g. [3]).

Up to conformal equivalence the only Riemann surfaces, which are simply connected are the sphere $\boldsymbol{CP}_1 \cong \boldsymbol{C} \cup \{\infty\}$, the complex plane $\boldsymbol{C}$ and the upper half plane $\Delta = \{z \in \boldsymbol{C} \mid \mathrm{Im}(z) > 0\}$, which is conformally equivalent to the open unit disk. Each of these surfaces is equipped with its standard complex structure. Every Riemann surface M can be represented as the quotient of one of these three Riemann surfaces by a freely acting Fuchsian group Γ of biholomorphic automorphisms, i.e., we may write $M = \Gamma \diagdown \mathscr{D}$, where $\mathscr{D}$ is the simply connected cover of M.

If $\pi: \mathscr{D} \to M$ is the covering map, then Γ is also the covering group of π. Then by [3, V.4.5]

$$\mathrm{Aut}\, M = N(\Gamma)/\Gamma\,, \tag{2.2.1}$$

where $N(\Gamma)$ is the normalizer of Γ in $\mathrm{Aut}\,\mathscr{D}$. Since $g \in \mathrm{Aut}_\pi \mathscr{D}$ is equivalent to

$$\pi \circ g \circ \gamma = \pi \circ g \tag{2.2.2}$$

for all $\gamma \in \Gamma$, we have $g \circ \gamma = \gamma_1 \circ g$ for some $\gamma_1 \in \Gamma$, and thus

$$\mathrm{Aut}_\pi \mathscr{D} = N(\Gamma)\,. \tag{2.2.3}$$

Therefore,

$$\mathrm{Aut}\, M \cong \mathrm{Aut}_\pi \mathscr{D}/\Gamma\,, \tag{2.2.4}$$

where $\mathrm{Aut}_\pi \mathscr{D}$ is a closed subgroup of $\mathrm{Aut}\,\mathscr{D}$, and Γ is a normal subgroup of $\mathrm{Aut}_\pi \mathscr{D}$.

The following is also well known (see e.g. [3, IV.5, IV.6, V.4]):

LEMMA. (a) *The group Γ is discrete, i.e., either finite or countable, and consists of conformal (biholomorphic) automorphisms of $\mathscr{D}$, which act fixed point free on $\mathscr{D}$.*

(b) *If $\mathscr{D}=\boldsymbol{C}P_1$, then Γ is trivial and M is the sphere.*

(c) *If $\mathscr{D}=\boldsymbol{C}$, then Γ is abelian and M is either the plane, the cylinder or a torus.*

(d) *For a Riemann surface $M=\Gamma\backslash\mathscr{D}$ the following are equivalent:*

- *The Fuchsian group is abelian.*
- *The Lie group* $\mathrm{Aut}\,M$ *of conformal automorphisms of M is nondiscrete.*

Surfaces of this kind are called exceptional surfaces.

2.3. In the following we will need two standard results from the theory of CMC surfaces which can be found e.g. in [6]:

1. $Edz^2=\langle \Psi_{zz}, N\rangle dz^2$, defines a holomorphic quadratic form, the Hopf differential, on M.
2. CMC surfaces come in S^1-families, the so called associated families. In other words, two CMC surfaces are isometric if and only if they are in the same associated family.

We define $\mathrm{OAff}(\boldsymbol{R}^3)$ to be the group of proper Euclidean motions in $\boldsymbol{R}^3$. It will be convenient at times to decompose an element of $\tilde{T}\in\mathrm{OAff}(\boldsymbol{R}^3)$ into a rotational and a translational part:

$$\tilde{T}v=R_{\tilde{T}}v+t_{\tilde{T}}\,, \qquad v\in\boldsymbol{R}^3\,. \tag{2.3.1}$$

We will also write $\tilde{T}=(R_{\tilde{T}}, t_{\tilde{T}})$.

DEFINITION. As already mentioned in the introduction we define

$$\mathrm{Aut}_\pi \mathscr{D}=\{g\in\mathrm{Aut}\,\mathscr{D}\mid \text{there exists } \hat{g}\in\mathrm{Aut}\,M: \pi\circ g=\hat{g}\circ\pi\}\,, \tag{2.3.2}$$

$$\mathrm{Aut}_\Phi M=\{\hat{g}\in\mathrm{Aut}\,M\mid \text{there exists } \tilde{T}\in\mathrm{Aut}\,\Psi(\mathscr{D}): \Phi\circ\hat{g}=\tilde{T}\circ\Phi\}\,, \tag{2.3.3}$$

$$\mathrm{Aut}_\Psi \mathscr{D}=\{g\in\mathrm{Aut}\,\mathscr{D}\mid \text{there exists } \tilde{T}\in\mathrm{Aut}\,\Psi(\mathscr{D}): \Psi\circ g=\tilde{T}\circ\Psi\}\,, \tag{2.3.4}$$

and

$$\mathrm{Aut}\,\Psi(\mathscr{D})=\{\tilde{T}\in\mathrm{OAff}(\boldsymbol{R}^3)\mid \tilde{T}\Psi(\mathscr{D})=\Psi(\mathscr{D})\}\,. \tag{2.3.5}$$

LEMMA. *Let $\Psi:\mathscr{D}\to\boldsymbol{R}^3$ be a conformal* CMC-*immersion with metric*

$$ds^2=\frac{1}{2}e^u dzd\bar{z} \tag{2.3.6}$$

where $u=u(z,\bar{z}):\mathscr{D}\to\boldsymbol{R}$, and Hopf differential Edz^2. Let $g\in\mathrm{Aut}\,\mathscr{D}$. Then the following are equivalent:

1. *The automorphism g is in* $\mathrm{Aut}_\Psi \mathscr{D}$.

2. *The functions u and E transform under g as*

$$e^{(u\circ g)(z,\bar z)}|g'(z)|^2=e^{u(z,\bar z)}, \tag{2.3.7}$$

$$(E\circ g)(z)(g'(z))^2=E(z). \tag{2.3.8}$$

PROOF. Let us define the immersion $\Psi_1=\Psi\circ g$. Then $\Psi_1:\mathscr{D}\to\boldsymbol{R}^3$ is also a CMC-immersion. By the definition of u we have

$$e^{u_1(z,\bar z)}=e^{(u\circ g)(z,\bar z)}|g'(z)|^2. \tag{2.3.9}$$

Since the Hopf differential is a holomorphic quadratic form we get

$$E_1(z)=(E\circ g)(z)(g'(z))^2. \tag{2.3.10}$$

We have $g\in\mathrm{Aut}_\Psi\mathscr{D}$ if and only if Ψ_1 and Ψ give the same surface up to a proper Euclidean motion. By the fundamental theorem of surface theory this is the case if and only if both surfaces have the same first and second fundamental form, which by Eqs. (2.3.6) and the well-known expression

$$\mathrm{II}=\frac{1}{2}\begin{pmatrix}(E+\bar E)+He^u & i(E-\bar E)\\ i(E-\bar E) & -(E+\bar E)+He^u\end{pmatrix}, \tag{2.3.11}$$

for the second fundamental form is equivalent to $E_1=E$ and $u_1=u$. This, together with Eq. (2.3.9) and Eq. (2.3.10), proves the lemma. □

COROLLARY. *The elements of* $\mathrm{Aut}_\Psi\mathscr{D}$ *act as self-isometries of* $(\mathscr{D},\Psi)$, *i.e.* $\mathrm{Aut}_\Psi\mathscr{D}\subset\mathrm{Iso}_\Psi\mathscr{D}$.

PROOF. By Eq. (2.3.7) and the definition (2.3.6) of u, the metric ds^2 is invariant under $g\in\mathrm{Aut}_\Psi\mathscr{D}$.

2.4. We will draw some further conclusions from the automorphicity of the Hopf differential, Eq. (2.3.8). We recall that CMC-surfaces with Hopf differential identically zero are part of a round sphere. Such surfaces will be called *spherical.*

PROPOSITION. *Let* (M,Φ) *be a* CMC-*surface with simply connected cover* $(\mathscr{D},\Psi)$, $\mathscr{D}=\boldsymbol{C}$. *Then either* $E\equiv0$ *or the group* $\mathrm{Iso}_\Psi\mathscr{D}\subset\mathrm{Aut}\,\boldsymbol{C}$ *of self-isometries of* $(\mathscr{D},\Psi)$ *consists only of rigid motions of the plane, i.e., every* $g\in\mathrm{Iso}_\Psi\mathscr{D}$ *can be written as* $g:z\mapsto az+b$, *with* $a,b\in\boldsymbol{C}$ *and* $|a|=1$.

PROOF. Let us assume, that there exists an automorphism g in $\mathrm{Iso}_\Psi\mathscr{D}\subset\mathrm{Aut}\,\boldsymbol{C}$, which is of the form

$$g(z)=az+b \tag{2.4.1}$$

with a, b being complex constants and $|a|\neq1$.

Case I: If $b\neq0$ then we can, by a biholomorphic change of coordinates

$$z \mapsto \tilde{z} = \frac{a-1}{b} z + 1 , \tag{2.4.2}$$

turn g into a scaling with rotation $g(\tilde{z}) = a\tilde{z}$. Let us also define $\tilde{E}: \mathscr{D} \to \boldsymbol{C}$ by $\tilde{E} d\tilde{z}^2 = E dz^2$, then

$$\tilde{E}(\tilde{z}) = \frac{b^2}{(a-1)^2} E\left(\frac{b}{a-1}(z-1)\right). \tag{2.4.3}$$

The Hopf differential $\tilde{E}$ transforms under $\tilde{g}$ according to

$$|(E \circ g)(z)| \cdot |g'(z)|^2 = |E(z)| . \tag{2.4.4}$$

We therefore get for all $n \in \boldsymbol{Z}$:

$$|\tilde{E}(a^n \tilde{z})| \cdot |a^{2n}| = |\tilde{E}(\tilde{z})| . \tag{2.4.5}$$

For $|a| > 1$ this implies that the absolute value of $\tilde{E}$ is decreasing from the fixed point $\tilde{z} = 0$ of g in all directions in the $\tilde{z}$-plane to zero. Since $\tilde{E}$ is holomorphic in $\tilde{z}$, this gives $\tilde{E} \equiv 0$ and therefore $E \equiv 0$.

If $|a| < 1$, consider the inverse $g^{-1} \in \mathrm{Iso}_{\Psi} \mathscr{D}$:

$$g^{-1}(z) = \frac{1}{a} z - \frac{b}{a} . \tag{2.4.6}$$

Since $|1/a| > 1$ we can use the first part of the proof again.

Case II: If $b = 0$ then Eq. (2.4.4) gives directly

$$|E(a^n z)| = |a^{-2n}| \, |E(z)| . \tag{2.4.7}$$

We can therefore argue in the same way as in the first case. □

From the results of this section we can draw the following conclusion for $\mathrm{Iso}_{\Psi}(\mathscr{D})$, if $\mathscr{D} = \boldsymbol{C}$:

THEOREM. *Let (M, Φ) be a complete, nonspherical* CMC-*surface with universal covering immersion $(\mathscr{D}, \Psi)$ and $\mathscr{D} = \boldsymbol{C}$. Let $\mathrm{Iso}_{\Psi} \mathscr{D} \subset \mathrm{Aut}\, \mathscr{D}$ be the group of self-isometries of $(\mathscr{D}, \Psi)$. Let $E dz^2$ be the Hopf differential of $(\boldsymbol{C}, \Psi)$.*

1. *If $\mathrm{Iso}_{\Psi} \boldsymbol{C}$ contains the group of all rotations around a fixed point, then, up to a biholomorphic change of coordinates, we have $E = d(z - z_0)^m$, where $d \in \boldsymbol{C} \setminus \{0\}$ and $m = 0, 1, 2, \ldots$ is an integer.*
2. *If $\mathrm{Iso}_{\Psi} \boldsymbol{C}$ contains a one-parameter group $\mathscr{T}$ of translations, then $E = a e^{bz}$ with complex constants a, b, $a \neq 0$.*

PROOF. 1. Without loss of generality we can choose $z_0 = 0$. Let $\mathscr{R} = \{g_\varphi \in \mathrm{Aut}\, \boldsymbol{C} \mid g_\varphi(z) = e^{i\varphi} z, \varphi \in [0, 2\pi)\}$ be the one-parameter group of rotations around the origin. Clearly, $|E|$ is invariant under all automorphisms in $\mathscr{R} \subset \mathrm{Iso}_{\Psi} \mathscr{D}$. Therefore, for each $\varphi \in \boldsymbol{R}$, we get $E(g_\varphi(z)) = e^{i\theta} E(z)$, where $\theta = \theta(\varphi) \in \boldsymbol{R}$ depends linearly on φ. It follows, since E is holomorphic, that $\theta=m\phi$, m a nonnegative integer, and, up to a biholomorphic change of coordinates, $E=dz^m$, $d\in\boldsymbol{C}$. Since by assumption $E\not\equiv 0$, we have $d\neq 0$.

2. $|E|$ and therefore also the set of zeroes of E is invariant under the group $\mathscr{T}$. Since the set of zeroes of a holomorphic function is discrete, it follows, that the holomorphic function $E(z)$ has no zeroes. Thus, $\ln(E)$ and its derivative are entire functions. Let us write the group $\mathscr{T}$ as

$$\mathscr{T}=\{g_r \mid g_r(z)=z+rv\,,\ r\in\boldsymbol{R}\,,\ v\in\boldsymbol{C}\setminus\{0\}\}\,. \tag{2.4.8}$$

Then for all $r\in\boldsymbol{R}$ and some $v\in\boldsymbol{C}\setminus\{0\}$ we have $E(g_r(z))=e^{ir\phi}E(z)$, with some $\phi\in[0,2\pi)$. Therefore, the logarithmic derivative of E is constant along any orbit of $\mathscr{T}$, whence it is a globally constant function. It follows, that $E(z)=e^{bz+c}$ for some complex constants b, c. Setting $a=e^c\neq 0$ gives the desired statement. □

COROLLARY. *Let* (M,Φ), (C,Ψ) *and* $\mathrm{Iso}_\Psi\mathscr{D}$ *be defined as in Theorem* 2.4.

1. *If* $\mathrm{Iso}_\Psi\mathscr{D}$ *contains the one-parameter group* $\mathscr{R}$ *of rotations around a fixed point* $z_0\in\boldsymbol{C}$, *then either* $\mathrm{Iso}_\Psi\mathscr{D}=\mathscr{R}$ *or* $\Phi(M)$ *is a cylinder.*
2. *If* $\mathrm{Iso}_\Psi\mathscr{D}$ *contains a one-parameter group* $\mathscr{T}$ *of translations, then either* $\Phi(M)$ *is a cylinder, or* $\mathrm{Iso}_\Psi\mathscr{D}=\mathscr{T}\times Q$, *or* $\mathrm{Iso}_\Psi\mathscr{D}=\mathscr{T}\times Q\times R$, *where* $\times$ *denotes the product of sets,* Q *is a, possibly trivial, discrete group of translations, not contained in* $\mathscr{T}$, *and* R *is the group generated by the* 180°*-rotation around a fixed point,* $z\to 2z_0-z$, $z_0\in\boldsymbol{C}$.

PROOF. The first part follows immediately from Theorem 2.4 and the fact that, due to (2.3.8), isometries leaves the zeroes of E fixed. To prove the second part it is enough to remark that, by (2.3.7), the given isometry groups are the only possibilities for a nonconstant function u, i.e. for a noncylindrical surface. □

REMARK. The immersions considered in the theorem and in the corollary will be investigated in more detail in Section 2.12.

2.5. In the next sections we will investigate some properties of the groups defined in Definition 2.3. We begin with the following.

LEMMA. (a) *Let* $g\in\mathrm{Aut}_\pi\mathscr{D}$ *and* $\hat{g}\in\mathrm{Aut}\,M$ *be as in* (2.3.2), *then* $\hat{g}$ *is uniquely defined.*

(b) *Let* $\hat{g}\in\mathrm{Aut}_\Phi M$ *and* $\tilde{T}\in\mathrm{Aut}\,\Psi(\mathscr{D})$ *be as in* (2.3.3), *then* $\tilde{T}$ *is uniquely defined.*

(c) *Let* $g\in\mathrm{Aut}_\Psi\mathscr{D}$ *and* $\tilde{T}\in\mathrm{Aut}\,\Psi(\mathscr{D})$ *be as in* (2.3.4), *then* $\tilde{T}$ *is uniquely defined.*

PROOF. (a) Assume $\hat{g}$ and $\hat{g}'$ both satisfy (2.3.2), then $\hat{g}(\pi(z))=\hat{g}'(\pi(z))$ for all $z\in\mathscr{D}$. This implies $\hat{g}=\hat{g}'$, since π is surjective.

(b) A proper Euclidean motion in $\boldsymbol{R}^3$ is determined uniquely by its restriction to an affine two dimensional subspace. If we choose a point $z\in M$, then for each point p of the affine tangent plane $\Phi(z)+d\Phi(T_zM)$, $p=\Phi(z)+d\Phi(v)$, we have

$$\tilde{T}(p)=\Phi(\hat{g}(z))+(\hat{g}_*d\Phi)(v)\,. \tag{2.5.1}$$

Therefore $\tilde{T}$ is uniquely determined.

(c) Similarly. $\square$

REMARK. It actually follows from the proof, the $\tilde{T}$ is already determined by the restriction of $\hat{g}$ to an arbitrary open subset of M.

2.6. Using Lemma 2.5 we define the following maps:

DEFINITION.

$$\bar{\pi}: \mathrm{Aut}_\pi \mathscr{D} \to \mathrm{Aut}\, M, \qquad \bar{\pi}: g \mapsto \hat{g}, \tag{2.6.1}$$

where g and $\hat{g}$ are as in (2.3.2),

$$\phi: \mathrm{Aut}_\Phi M \to \mathrm{OAff}(\boldsymbol{R}^2), \qquad \phi: \hat{g} \mapsto \tilde{T}, \tag{2.6.2}$$

where $\hat{g}$ and $\tilde{T}$ are as in (2.3.3), and

$$\psi: \mathrm{Aut}_\Psi \mathscr{D} \to \mathrm{OAff}(\boldsymbol{R}^3), \qquad \psi: g \mapsto \tilde{T}, \tag{2.6.3}$$

where g and $\tilde{T}$ are as in (2.3.4).

By Lemma 2.5, ϕ and ψ are group homomorphisms. Also note that the images of ϕ and ψ are contained in $\mathrm{Aut}\, \Psi(\mathscr{D})$.

THEOREM. (a) *The groups* $\mathrm{Aut}_\Psi \mathscr{D}$ *and* $\mathrm{Aut}_\Phi M$ *are closed Lie subgroups of* $\mathrm{Aut}\, \mathscr{D}$ *and* $\mathrm{Aut}\, M$, *respectively.*

(b) *The maps* $\bar{\pi}$, ϕ *and* ψ *are analytic homomorphisms of Lie groups.*

PROOF. (a) Let $g_n \in \mathrm{Aut}_\Psi \mathscr{D}$ be a sequence which converges to $g \in \mathrm{Aut}\, \mathscr{D}$. Then g_n converges uniformly on each compact subset of $\mathscr{D}$. In particular, $\Psi \circ g_n = \tilde{T}_n \circ \Psi$ converges uniformly to $\Psi \circ g$ on each sufficiently small closed ball around any point $z \in \mathscr{D}$. Therefore, also the differentials converge, whence $(\tilde{T}_n)_* d_z \Psi = R_{\tilde{T}_n} d_z \Psi$ converges, where we have written $\tilde{T}_n = (R_{\tilde{T}_n}, t_{\tilde{T}_n})$ as in (2.3.1). This implies that $R_{\tilde{T}_n}$ converges to a rotation R in $\boldsymbol{R}^3$. Since also $\tilde{T}_n \circ \Psi = t_{\tilde{T}_n} + R_{\tilde{T}_n} \circ \Psi$ converges, $t_{\tilde{T}_n} \to t$ for some $t \in \boldsymbol{R}^3$. Altogether, this shows $\tilde{T}_n \to \tilde{T} = (R, t)$. But now $\Psi \circ g_n \to \Psi \circ g = \tilde{T} \circ \Psi$. This shows that $\tilde{T} \in \mathrm{Aut}\, \Psi(\mathscr{D})$ and $g \in \mathrm{Aut}_\Psi \mathscr{D}$. The argument for $\mathrm{Aut}_\Phi M$ is similar.

(b) We know that $\mathrm{Aut}\, M$ and $\mathrm{Aut}\, \mathscr{D}$ are Lie groups and, by the argument above, we know that $\mathrm{Aut}_\Psi \mathscr{D}$ and $\mathrm{Aut}_\Phi M$ are closed subgroups of $\mathrm{Aut}\, \mathscr{D}$ and $\mathrm{Aut}\, M$, respectively. Therefore, with the induced topology $\mathrm{Aut}_\Psi \mathscr{D}$ and $\mathrm{Aut}_\Phi M$ are Lie groups. We show that in this topology the maps ϕ and ψ are continuous, from which analyticity follows [7, Th. II.2.6].

Assume $g_n \to g$ in $\mathrm{Aut}_\Psi \mathscr{D}$. Then g_n converges to g uniformly on each compact subset of $\mathscr{D}$. In particular, $\Psi \circ g_n = \tilde{T}_n \circ \Psi$ converges to $\Psi \circ g = \tilde{T} \circ \Psi$ on each sufficiently small closed ball around any point $z \in \mathscr{D}$. By the proof of Lemma 2.5, $\tilde{T}$ and $\tilde{T}_n$ are uniquely determined by the restriction of g and g_n to an arbitrary open subset of $\mathscr{D}$. Therefore, $\tilde{T}_n$ converges to $\tilde{T}$. This shows that ψ is continuous. For ϕ we proceed analogously.

For $\bar{\pi}$ the claim is trivial. $\square$

2.7. We will need the following.

THEOREM. *We retain the notation of Section* 2.3. *If M with the metric induced by Φ is complete, then for every Euclidean motion $\tilde{T} \in \operatorname{Aut} \Psi(\mathscr{D})$ there exists a $g \in \operatorname{Aut}_\Psi \mathscr{D}$, such that $\Psi \circ g = \tilde{T} \circ \Psi$, i.e., ψ maps $\operatorname{Aut}_\Psi \mathscr{D}$ onto $\operatorname{Aut} \Psi(\mathscr{D})$. The automorphism g is unique up to multiplication with an element of* $\operatorname{Ker} \psi$.

PROOF. Let $\tilde{T}$ be a Euclidean motion, which leaves the image $\Psi(\mathscr{D})$ invariant. Let us choose two arbitrary points z_0 and z_1 in $\mathscr{D}$, such that $\Psi(z_1) = \tilde{T}\Psi(z_0)$. Then, since Ψ is locally injective and conformal, for each such pair (z_0, z_1) there exists an open neighbourhood U_0 of z_0 and an open neighbourhood U_1 of z_1, such that $\tilde{T} \circ \Psi(z) = \Psi(h(z))$, $z \in U_0$, defines an orientation preserving isometry $h: U_0 \to U_1$.

Since M is complete, also $\mathscr{D}$ is complete [7, Prop. I.10.6]. Thus, $\mathscr{D}$ is an analytic, complete, simply connected manifold. Therefore, by [7, Sect. I.11], the local isometry h can be extended to a unique, global, orientation preserving self-isometry $g \in \operatorname{Aut} \mathscr{D}$ of $\mathscr{D}$, such that $g|_{U_0} = h$. By the definition of h we have $\Psi(g(z)) = \tilde{T}\Psi(z)$ on U_0. Since all occuring maps are analytic, we get $\tilde{T} \circ \Psi = \Psi \circ g$ on $\mathscr{D}$. Uniqueness of g up to an element of $\operatorname{Ker} \psi$ is trivial. $\square$

The only group still to be discussed is $\operatorname{Aut} \Psi(\mathscr{D})$. Unfortunately, in general $\operatorname{Aut} \Psi(\mathscr{D})$ doesn't seem to be closed in $\operatorname{OAff}(\boldsymbol{R}^3)$. In Section 2.10 we will give a simple condition on (M, Φ), under which $\operatorname{Aut} \Psi(\mathscr{D})$ can be shown to be closed.

However, here we are able to conclude:

COROLLARY. *If (M, Φ) is complete, then $\psi: \operatorname{Aut}_\Psi \mathscr{D} \to \operatorname{Aut} \Psi(\mathscr{D})$ is surjective. In particular, $\operatorname{Aut} \Psi(\mathscr{D}) \cong \operatorname{Aut}_\Psi \mathscr{D} / \operatorname{Ker} \psi$ is a Lie group.*

Theorem 2.7 and Corollary 2.7 have well known equivalents for the map π. The arguments leading to Eq. (2.2.4) prove the following

PROPOSITION. *Let M be a Riemann surface with simply connected cover $\pi: \mathscr{D} \to M$. With the notation as above we have:*

(a) *For every $\hat{g} \in \operatorname{Aut} M$ there exists a $g \in \operatorname{Aut}_\pi \mathscr{D}$, such that $\pi \circ g = \hat{g} \circ \pi$.*

(b) *The map $\bar{\pi}: \operatorname{Aut}_\pi \mathscr{D} \to \operatorname{Aut} M$ is surjective.*

2.8. We want to investigate, how the groups defined in Section 2.3 are related to each other by the maps $\bar{\pi}$, ϕ and ψ.

For every $g \in \bar{\pi}^{-1}(\operatorname{Aut}_\Phi M)$ we have

(2.8.1) $$\Psi \circ g = \Phi \circ \pi \circ g = \Phi \circ \bar{\pi}(g) \circ \pi = \phi(\bar{\pi}(g)) \circ \Psi,$$

hence

(2.8.2) $$\bar{\pi}^{-1}(\operatorname{Aut}_\Phi M) \subset \operatorname{Aut}_\Psi \mathscr{D}$$

and $\psi = \phi \circ \bar{\pi}$ on $\bar{\pi}^{-1}(\mathrm{Aut}_{\Phi} M)$.

For $g \in \operatorname{Ker} \bar{\pi}$ we have $\Psi \circ g = \Phi \circ \pi \circ g = \Phi \circ \pi = \Psi$. Therefore,

$$\operatorname{Ker} \bar{\pi} \subset \operatorname{Ker} \psi . \tag{2.8.3}$$

We also recall that $\operatorname{Ker} \bar{\pi} = \Gamma$, the Fuchsian group of M.

LEMMA. *Let (M, Φ) be a* CMC-*immersion with* $\operatorname{Ker} \psi = \operatorname{Ker} \bar{\pi}$. *Then the following holds*:

(a) $(\bar{\pi})^{-1}(\mathrm{Aut}_{\Phi} M) = \mathrm{Aut}_{\Psi} \mathcal{D}$.

(b) $\phi : \mathrm{Aut}_{\Phi} M \rightarrow \mathrm{Aut}\, \Psi(\mathcal{D})$ *is an injective group homomorphism.*

If, in addition, (M, Φ) is complete, then:

(c) *The action of $\tilde{T} \in \mathrm{Aut}\, \Psi(\mathcal{D})$ can be lifted to an action on M, i.e., ϕ is surjective.*

(d) $\phi : \mathrm{Aut}_{\Phi} M \rightarrow \mathrm{Aut}\, \Psi(\mathcal{D})$ *is a group isomorphism.*

PROOF. (a) Since $\operatorname{Ker} \psi = \operatorname{Ker} \bar{\pi}$, we have that $\mathrm{Aut}_{\Psi} \mathcal{D}$ is in the normalizer of $\operatorname{Ker} \bar{\pi} = \Gamma$, whence $\mathrm{Aut}_{\Psi} \mathcal{D} \subset \mathrm{Aut}_{\pi} \mathcal{D}$, by (2.2.3). Therefore, for $g \in \mathrm{Aut}_{\Psi} \mathcal{D}$ we have

$$\Phi \circ \bar{\pi}(g) \circ \pi = \Phi \circ \pi \circ g = \Psi \circ g = \psi(g) \circ \Psi = \psi(g) \circ \Phi \circ \pi , \tag{2.8.4}$$

and $\bar{\pi}(g) \in \mathrm{Aut}_{\Phi} M$ follows, i.e., $\mathrm{Aut}_{\Psi} \mathcal{D} \subset \bar{\pi}^{-1}(\mathrm{Aut}_{\Phi} M)$. Now (a) follows from Eq. (2.8.2).

(b) From Eq. (2.8.1) and (a) it follows that $\psi = \phi \circ \bar{\pi}$ on $\mathrm{Aut}_{\Psi} \mathcal{D}$, which implies $\operatorname{Ker} \phi = \{\mathrm{id}\}$ and therefore (b).

(c) Let $\tilde{T} \in \mathrm{Aut}\, \Psi(\mathcal{D})$. Since (M, Φ) is complete, there exists, by Theorem 2.7, $g \in \mathrm{Aut}_{\Psi} \mathcal{D}$, such that $\tilde{T} = \psi(g)$. From (a) it follows that there exists $\hat{g} \in \mathrm{Aut}_{\Phi} M$, such that $\hat{g} = \bar{\pi}(g)$, whence $\tilde{T} = \phi(\hat{g})$. The map ϕ is therefore surjective.

(d) follows from (b) and (c). □

PROPOSITION. (a) $\operatorname{Ker} \psi$ *is a discrete subgroup of* $\mathrm{Aut}_{\Psi} \mathcal{D}$ *and acts freely and discontinuously on* $\mathcal{D}$.

(b) $M' = \operatorname{Ker} \psi \backslash \mathcal{D}$ *is a Riemann surface.*

(c) *Let $\pi' : \mathcal{D} \rightarrow M'$ denote the natural projection. Then there exists an immersion Φ' of M' into $\boldsymbol{R}^3$, such that the following diagram commutes*:

$$\begin{tikzcd}
& M = \operatorname{Ker} \bar{\pi} \backslash \mathcal{D} \arrow[dd] \arrow[dr, "\Phi"] & \\
\mathcal{D} \arrow[ur, "\pi"] \arrow[rr, "\Psi"] \arrow[dr, "\pi'"'] & & \boldsymbol{R}^3 \\
& M' = \operatorname{Ker} \psi \backslash \mathcal{D} \arrow[ur, "\Phi'"'] &
\end{tikzcd} \tag{2.8.5}$$

(d) *For the* CMC-*immersion (M', Φ') as above we define $\bar{\pi}'$ and ϕ' as in Section* 2.6. *Then*

$$\operatorname{Ker} \bar{\pi}' = \operatorname{Ker} \psi . \tag{2.8.6}$$

PROOF. (a) Since $\psi : \mathrm{Aut}_{\Psi} \mathcal{D} \rightarrow \mathrm{OAff}(\boldsymbol{R}^3)$ is a continuous homomorphism of Lie groups, $\operatorname{Ker}\psi$ is, with the induced topology, a Lie subgroup of $\operatorname{Aut}_\Psi \mathscr{D}$. Therefore, if $\operatorname{Ker}\psi$ were nondiscrete, it would contain a one-parameter subgroup $\gamma(t)$. Hence $\Psi(\gamma(t).z)=\Psi(z)$ for all $z\in\mathscr{D}$ and all $t\in\boldsymbol{R}$. This implies $\gamma(t).z=z$ for all $z\in\mathscr{D}$ and all $t\in\boldsymbol{R}$, whence $\gamma(t)=I$ for all t, a contradiction.

Now let us assume that $g\in\operatorname{Ker}\psi$ has a fixed point $z_0\in\mathscr{D}$. Then

$$\Psi(g(z))=\Psi(z) \qquad \text{for all} \quad z\in\mathscr{D}\,, \tag{2.8.7}$$

$$g(z_0)=z_0\,. \tag{2.8.8}$$

Taking into account the injectivity of the derivative of Ψ one gets by differentiating Eq. (2.8.7) at $z=z_0$,

$$g'(z_0)=1\,. \tag{2.8.9}$$

In the case of $\mathscr{D}=\boldsymbol{C}$ one has $g(z)=az+b$, with $a, b\in\boldsymbol{C}$. It follows from Eqs. (2.8.9) and (2.8.8) that $g=\mathrm{id}$.

In the case of $\mathscr{D}$ being the unit circle we can view g as an isometry w.r.t. the Bergmann metric on $\mathscr{D}$. This together with [7, Lemma I.11.2] implies again $g=\mathrm{id}$.

It remains to be proved that $\operatorname{Ker}\psi$ acts discontinuously, i.e. that there is a point $z_0\in\mathscr{D}$, such that the orbit of $\operatorname{Ker}\psi$ through z_0 is discrete. For the unit circle this follows from the discreteness of $\operatorname{Ker}\psi$ and [3, Theorem IV.5.4]. For $\mathscr{D}=\boldsymbol{C}$ it is trivial, since then, by the arguments above, $\operatorname{Ker}\psi$ is a discrete group of translations.

(b) Since by (a), $\operatorname{Ker}\psi$ is a Fuchsian or elementary group, it follows that $M=\operatorname{Ker}\psi\diagdown\mathscr{D}$ is a Riemann surface (see e.g. [3, Section IV.5]).

(c) From the definition of $\operatorname{Ker}\psi$ and M' it is clear that Ψ factors through M'. This defines an immersion $\Phi'\colon M'\to\boldsymbol{R}^3$. If $\pi'\colon\mathscr{D}\to M'$ is the natural projection, then $\Phi'\circ\pi'=\Psi$ and (2.8.5) follows.

(d) is clear from the definition of M'. □

The last lemma shows that for our purposes it is actually enough to restrict our attention to surfaces with

$$\operatorname{Ker}\psi=\operatorname{Ker}\bar{\pi}\,. \tag{2.8.10}$$

For these surfaces the conclusions of Lemma 2.8 hold.

2.9. The following proposition shows what it means for the symmetry group $\operatorname{Aut}_\Psi\mathscr{D}$ that $\operatorname{Aut}\Psi(\mathscr{D})$ is not discrete.

Proposition. *Let (M, Φ) be a* CMC*-immersion with simply connected cover $\mathscr{D}$, which is complete w.r.t. the induced metric and admits a one parameter group of Euclidean motions $P\subset\operatorname{Aut}\Psi(\mathscr{D})$. Then $\operatorname{Aut}_\Psi\mathscr{D}$ also contains a one parameter group.*

Proof. Let $P=\{\tilde{T}_x, x\in\boldsymbol{R}\}$ be a one parameter subgroup of $\operatorname{Aut}\Psi(\mathscr{D})$, where $\operatorname{Aut}\Psi(\mathscr{D})$ carries the Lie group structure stated in Corollary 2.7. Let $A\subset\mathscr{D}$ be an open subset such that Ψ is injective on A. Let $a \in A$ be arbitrary. Since $\tilde{T}_0 \Psi(a) = \Psi(a) \in \Psi(A)$, there exists some $\varepsilon > 0$ and an open subset $A_\varepsilon \subset A$, such that $\tilde{T}_x \Psi(A_\varepsilon) \subset \Psi(A)$ for all $|x| < \varepsilon$. Therefore, by Theorem 2.7, there exists an automorphism $g_x \in \operatorname{Aut} \mathscr{D}$, $|x| < \varepsilon$, satisfying $\Psi \circ g_x = \tilde{T}_x \circ \Psi$, which is unique up to multiplication with an element in $\operatorname{Ker} \psi$. In addition, it follows from the proof of Theorem 2.7 that we can choose g_x such that

$$g_x(A_\varepsilon) \subset A . \tag{2.9.1}$$

Since $\operatorname{Ker} \psi$ is discrete, the condition (2.9.1) determines g_x uniquely, if A is small enough. This shows that $g_{x+y} = g_x g_y$ for all sufficiently small $x, y \in \boldsymbol{R}$. For $x \to 0$ we have $\tilde{T}_x \to \tilde{T}_0 = I$, therefore $\Psi \circ g_x \to \Psi$ uniformly on $\mathscr{D}$. This shows that g_x converges to an element of $\operatorname{Ker} \psi$. Since $\operatorname{Ker} \psi$ acts freely on $\mathscr{D}$, Eq. (2.9.1) implies $g_x \to g_0 = I$ for $x \to 0$. This shows that $\omega : x \to g_x$ is a continuous and thus analytic (see [7, Th. II.2.6]) homomorphism from some interval $(-\tilde{\varepsilon}, \tilde{\varepsilon})$, $\tilde{\varepsilon} > 0$, into $\operatorname{Aut}_\Psi \mathscr{D}$. For an arbitrary $x \in \boldsymbol{R}$ we write $x = m\tilde{\varepsilon}/2 + r$, where $r \in [0, \tilde{\varepsilon}/2)$ and $m \in \boldsymbol{Z}$ are uniquely determined. The definition

$$g_x = g_r (g_{\tilde{\varepsilon}/2})^m \in \operatorname{Aut}_\Psi \mathscr{D} , \tag{2.9.2}$$

extends ω to a one-parameter subgroup of $\operatorname{Aut}_\Psi \mathscr{D}$, which finishes the proof. □

2.10. It remains to be investigated under which circumstances the existence of a cluster point of $\operatorname{Aut} \Psi(\mathscr{D})$ implies $\dim \operatorname{Aut} \Psi(\mathscr{D}) \geq 1$. This is certainly the case, if $\operatorname{Aut} \Psi(\mathscr{D})$ is closed in $\operatorname{OAff}(\boldsymbol{R}^3)$.

To this end we introduce the notion of an admissible immersion.

Definition. Let (M, Φ) be an immersed manifold in $\boldsymbol{R}^3$. A point $p \in \Phi(M)$ is called *admissible*, if there is an open neighbourhood U of p in $\boldsymbol{R}^3$, such that the intersection $\Phi(M) \cap U$ is closed in U. The immersion (M, Φ) is called admissible, if $\Phi(M)$ contains at least one admissible point.

We think it is fair to say that, basically, every surface of interest is admissible. Most surfaces studied actually belong to the smaller class of locally closed surfaces (see [8, II.2]), for which each point of the image is admissible. Among the locally closed surfaces are e.g. the immersed surfaces with closed image $\Phi(M)$ in $\boldsymbol{R}^3$, especially compact submanifolds of $\boldsymbol{R}^3$, and immersed surfaces (M, Φ), for which Φ is proper (see e.g. [10, I.2.30]).

Also note that, geometrically speaking, a surface has to return infinitely often to each neighbourhood of each of its nonadmissible points. A nonadmissible surface is therefore in a sense a two-dimensional analog of a Peano curve.

Remark. It is important to note that admissibility is a property of the image $\Phi(M)$ of the immersion Φ. We don't claim that it is preserved under isometries. In particular, for an admissible surface it may well be, that not all members of the associated family are admissible.

The definition of admissible surfaces allows us to describe a large class of surfaces, for which $\operatorname{Aut}\Psi(\mathscr{D})$ is closed.

THEOREM. *If* (M, Φ) *is a complete, admissible surface in* $\boldsymbol{R}^2$ *and* $(\mathscr{D}, \Psi)$ *is the simply connected cover of* M *with the covering immersion* $\Psi=\Phi\circ\pi$, *then the group* $\operatorname{Aut}\Psi(\mathscr{D})$ *is closed in* $\operatorname{OAff}(\boldsymbol{R}^3)$.

PROOF. Let $\tilde{T}_n\in\operatorname{Aut}\Psi(\mathscr{D})$ be a sequence of symmetry transformations of $\Psi(\mathscr{D})$ which converges to $\tilde{T}\in\operatorname{OAff}(\boldsymbol{R}^3)$. Therefore, also the sequence $\tilde{T}_n^{-1}$ converges in $\operatorname{OAff}(\boldsymbol{R}^3)$.

Since (M, Φ) is admissible, there exists an admissible point $p\in\Psi(\mathscr{D})$, together with an open ball $B(p, \varepsilon)$ of radius $\varepsilon<1$ around p in $\boldsymbol{R}^3$, such that $B(p, \varepsilon)\cap\Psi(\mathscr{D})$ is closed in $B(p, \varepsilon)$.

Without loss of generality we can assume that p and the whole bounded sequence $\{\tilde{T}_n^{-1}(p)\}$ lies in $B(0, 1/2)$. Otherwise we first apply a scaling transformation of $\boldsymbol{R}^3$, which changes neither the admissibility of (M, Φ) nor the group structure of $\operatorname{Aut}\Psi(\mathscr{D})$.

We take $N\in\boldsymbol{N}$ such that $\|\tilde{T}-\tilde{T}_n\|<\varepsilon/3$ for $n\geq N$, where $\|\cdot\|$ denotes the operator norm.

We choose $p'=\tilde{T}_N^{-1}(p)$ and $z'\in\mathscr{D}$. such that $p'=\Psi(z')$. Since $\tilde{T}_N$ is a Euclidean motion we have that $\tilde{T}_N(B(p', \varepsilon))=B(p, \varepsilon)$.

For all $q\in B(p', \varepsilon/3)\cap\Psi(\mathscr{D})$ we get $|q|\leq|q-p'|+|p'|<\varepsilon/3+1/2<1$, therefore, if $n\geq N$,

$$|\tilde{T}_n(q)-p|\leq|\tilde{T}_n(q)-\tilde{T}(q)|+|\tilde{T}(q)-\tilde{T}(p')|+|\tilde{T}(p')-\tilde{T}_N(p')|<5\varepsilon/6\,. \tag{2.10.1}$$

Thus we have $\tilde{T}_n(q)\in B(p, 5\varepsilon/6)\cap\Phi(\mathscr{D})$ for all $n\geq N$ and $\overline{\{\tilde{T}_n(q), n\geq N\}}\subset B(p, \varepsilon)$. Now we know that $B(p, \varepsilon)\cap\Psi(\mathscr{D})$ is closed in $B(p, \varepsilon)$, and that $\tilde{T}_n(q)$ converges by assumption to $\tilde{T}(q)\in\boldsymbol{R}^3$. Therefore, the limit $\tilde{T}(q)$ is in $\Psi(\mathscr{D})$. Since Ψ is an immersion, there exists an open neighbourhood A of z' which is mapped into $B(p', \varepsilon/3)$ by Ψ. Therefore, if we choose $z\in\mathscr{D}$, such that $p=\Psi(z)$, *then* $\tilde{T}$ induces an isometry g of A onto an open neighbourhood $B\subset\mathscr{D}$ of z, such that $\tilde{T}\circ\Psi=\Psi\circ g$ on A. By [7, Sect. I.11], this isometry can be extended globally to a unique automorphism $g\in\operatorname{Aut}\mathscr{D}$. Since all maps are analytic and globally defined, the relation $\tilde{T}\circ\Psi=\Psi\circ g$ holds on the whole of $\mathscr{D}$. It follows that $g\in\operatorname{Aut}_\Psi\mathscr{D}$ and $\tilde{T}\in\operatorname{Aut}\Psi(\mathscr{D})$. □

COROLLARY. *Let* (M, Φ) *and* $\mathscr{D}$ *be as in Theorem* 2.10. *If* $\operatorname{Aut}\Psi(\mathscr{D})$ *is nondiscrete, then also* $\operatorname{Aut}_\Psi\mathscr{D}$ *is nondiscrete.*

PROOF. By the assumptions, $\operatorname{Aut}\Psi(\mathscr{D})$ is closed in $\operatorname{OAff}(\boldsymbol{R}^3)$ and nondiscrete. It therefore contains a one parameter group and the corollary follows from Proposition 2.9 above. □

Finally, we state the following result for the translational parts of the elements of $\operatorname{Aut}\Psi(\mathscr{D})$:

PROPOSITION. *If* (M, Φ) *is admissible and complete, and* Aut $\Psi(\mathscr{D})$ *is discrete, then the set* $\mathscr{L}=\{t \mid \tilde{T}=(R, t) \in \operatorname{Aut} \Psi(\mathscr{D})\}$ *of translations is discrete.*

PROOF. Assume, $\mathscr{L}$ is not discrete. Then there exists a $t \in \mathscr{L}$ and a sequence $\{t_n\} \subset \mathscr{L}$, $t_n \neq t$, such that t_n converges to t. Since the set of rotations is compact, we obtain a subsequence $\tilde{T}_n=(R_n, t_n) \in \operatorname{Aut} \Psi(\mathscr{D})$, which converges in $\operatorname{OAff}(\boldsymbol{R}^3)$. Since by Theorem 2.10, Aut $\Psi(\mathscr{D})$ is closed, this sequence converges in Aut $\Psi(\mathscr{D})$ to some $\tilde{T}$. But then $\tilde{T}_n=\tilde{T}$ for sufficiently large n, since Aut $\Psi(\mathscr{D})$ is discrete. This shows that $t_n=t$ for sufficiently large n, a contradiction. □

2.11. As examples for the discussion in this chapter, let us investigate two well known classes of CMC-surfaces, the Delaunay and the Smyth surfaces.

We recall that a Delaunay surface is defined as a complete, immersed surface of constant mean curvature which is generated in $\boldsymbol{R}^3$ by rotating a curve around a given axis. We will restrict the definition without loss of generality to an arbitrary but fixed mean curvature $H \neq 0$ and exclude the degenerate case of the sphere.

Let us translate two well known facts about Delaunay surfaces into our language:

PROPOSITION. 1. *Let* (M, Φ) *be a noncylindrical* CMC-*immersion with universal covering immersion* $(\mathscr{D}, \Psi)$, *such that* $\Phi(M)$ *is a Delaunay surface. Then* $\Phi(M)$ *is generated by rotating the roulette of an ellipse (unduloid) or a hyperbola (nodoid) along the line on which the conic rolled.*

2. *Let* $S \subset \boldsymbol{R}^3$ *be a Delaunay surface. Then there exists a* CMC-*immersion* (M, Φ) *with universal covering immersion* $(\mathscr{D}, \Psi)$, $\mathscr{D}=\boldsymbol{C}$, *such that*

- $S=\Phi(M)$,
- $\operatorname{Aut}_\Psi \mathscr{D}$ *contains a one parameter group* $\mathscr{T}$ *of translations, which is mapped by the surjective homomorphism* $\psi: \operatorname{Aut}_\Psi \mathscr{D} \rightarrow \operatorname{Aut} \Psi(\mathscr{D})$ *to the group of rotations around the axis of revolution of the Delaunay surface.*

PROOF. 1. Well known, see e.g. [2].
2. Such immersions (with $\Phi=\Psi$, $M=\mathscr{D}=\boldsymbol{C}$) are explicitly constructed in [9]. □

REMARK. Besides of being periodic, the roulette of an ellipse or hyperbola has another important property: In each of its periods there is a unique point of maximal distance from the line A on which the conic rolls. The roulette is symmetric w.r.t. the reflection at any line perpendicular to A, which passes through such a point of maximal distance from A. The Delaunay surface is therefore invariant under a 180°-rotation around any axis which is perpendicular to the axis of revolution A and passes through a parallel of maximal radius.

Let us recall again that in the discussion of Delaunay surfaces we exclude the catenoid, the roulette of the parabola, which is a minimal surface. Since we also exclude the sphere, we get the following result, the proof of which is an exercise in elementary geometry:

LEMMA. *Each Delaunay surface determines its axis of revolution uniquely.*

A simple argument using Proposition 2.11 gives the following.

COROLLARY. *Let* (M, Φ) *be as in Proposition* 2.11. *Let* A *be the generating axis of the Delaunay surface* $\Phi(M)$. *Then, as a set,* $\operatorname{Aut}\Psi(\mathscr{D})$ *can be written as*

$$\operatorname{Aut}\Psi(\mathscr{D})=\mathscr{R}\times\tilde{Q}\times\{I, \tilde{R}\}\,, \tag{2.11.1}$$

where $\mathscr{R}$ *is the one-parameter group of rotations around* A, $\tilde{Q}$ *is a nontrivial discrete group of translations along* A, *and* $\tilde{R}$ *is a* $180°$*-rotation around an axis which is perpendicular to* A.

Smyth [9] introduced for every integer $m \geq 0$ a one-parameter family of conformal immersions

$$\Psi_c^m: \boldsymbol{C} \to \boldsymbol{R}^3\,, \qquad c \in \boldsymbol{C}\setminus\{0\}\,, \tag{2.11.2}$$

with constant mean curvature, such that the induced metric is complete and invariant under the one-parameter group of rotations around $z=0$ in $\boldsymbol{C}$. We will call these surfaces Smyth surfaces. The Hopf differential of $(\boldsymbol{C}, \Psi_c^m)$ is $cz^m dz^2$, and therefore each $(\boldsymbol{C}, \Psi_c^m)$ has an umbilic of order m at the origin. For $m=0$ the family Ψ_c^m contains the cylinder. We will call a Smyth surface *nondegenerate*, if its image in $\boldsymbol{R}^3$ is not a cylinder.

Two surfaces in $\boldsymbol{R}^3$ will be called congruent if they are related by a proper Euclidean motion of $\boldsymbol{R}^3$. Recall also the definition of the associated family.

The following results were proved by Smyth [9]:

THEOREM. *Let* (M, Φ), *with covering immersion* $(\mathscr{D}, \Psi)$, *be a complete, immersed surface of constant mean curvature, admitting a one-parameter group of self-isometries. Then the following holds*:

1. *The simply connected cover of the Riemann surface* M *is* $\mathscr{D}=\boldsymbol{C}$.
2. *The associated family of* $(\mathscr{D}, \Psi)$ *contains either a Delaunay or a Smyth surface, i.e.,* $(\mathscr{D}, \Psi)$ *is isometric to the simply connected cover of a Delaunay or a Smyth surface.*
3. *The surface* $(\mathscr{D}, \Psi)$ *admits a one-parameter group* P *of self-isometries which is*
   (a) *a one-parameter group of translations* ($P \cong \boldsymbol{R}$) *in case of the Delaunay surfaces,*
   (b) *a one-parameter group of rotations around a fixed point in* $\boldsymbol{C}$ ($P \cong S^1$) *in case of the Smyth surfaces.*

2.12. With the results in the previous section we can also easily derive the uniformization of Delaunay and Smyth surfaces.

PROPOSITION. 1. *Each Delaunay surface is conformally equivalent to the cylinder,* *i.e., its simply connected cover is* $\boldsymbol{C}$ *and the Fuchsian group is a one-parameter group of translations.*

2. *Each nondegenerate Smyth surface is conformally equivalent to* $\boldsymbol{C}$.

3. *For a nondegenerate Smyth surface we have*

$$\operatorname{Ker} \bar{\pi} = \Gamma = \{\mathrm{id}\}\,. \tag{2.12.1}$$

PROOF. We already know from Theorem 2.11 that both, Delaunay and Smyth surfaces, have, as Riemann surfaces, the simply connected cover $\boldsymbol{C}$. Therefore, by Lemma 2.2, they are biholomorphically equivalent to the plane, the cylinder or a torus and the Fuchsian group is either trivial or it is a discrete group of translations.

1. is clear.

2. Let $\Psi : \boldsymbol{C} \to \boldsymbol{R}^3$ be an immersion such that $\Psi(\mathscr{D}) = \Psi_c^m(\mathscr{D})$ is a Smyth surface for some parameters $m \in \boldsymbol{N}$, $c \in \boldsymbol{C}$. Assume that the Fuchsian group of the surface contains a nontrivial translation. Then, by Theorem 2.11, the group $\mathrm{Aut}_\Psi \mathscr{D}$ satisfies the assumptions of the first part of Lemma 2.4. Therefore, $\Psi(\mathscr{D})$ is a cylinder. For a nondegenerate Smyth surface this gives $M = \boldsymbol{C} = \mathscr{D}$, i.e., the Smyth surface is conformally equivalent to the complex plane.

3. From 2, it follows that $\pi = \mathrm{id}$ and $\mathrm{Aut}_\pi \mathscr{D} = \mathrm{Aut}\, \mathscr{D} = \mathrm{Aut}\, M$, therefore $\operatorname{Ker} \bar{\pi} = \Gamma = \{\mathrm{id}\}$. □

We continue with the following.

LEMMA. 1. *For each noncylindrical Delaunay surface* $S \subset \boldsymbol{R}^3$, *there exists a* CMC-*immersion* (M, Φ) *with* $\Phi(M) = S$, *such that*

$$\operatorname{Ker} \psi = \operatorname{Ker} \bar{\pi}\,, \tag{2.12.2}$$

$$\mathrm{Aut}_\Phi M \cong \mathrm{Aut}\, \Psi(\mathscr{D})\,, \tag{2.12.3}$$

and, as a product of sets, we have

$$\mathrm{Aut}_\Psi D = \mathrm{Iso}_\Psi D = \mathscr{T} \times Q \times R\,, \tag{2.12.4}$$

where $\mathscr{T} \subset \mathrm{Aut}\, \boldsymbol{C}$ *is a one-parameter group of translations,* $Q \subset \mathrm{Aut}\, \boldsymbol{C}$ *is a discrete group of translations with one generator, and* $R = \{I, R_\pi\} \subset \mathrm{Aut}\, \boldsymbol{C}$ *is the group generated by the inversion* $R_\pi : z \mapsto -z$.

2. *If* $(\boldsymbol{C}, \Psi = \Psi_c^m)$ *is a nondegenerate Smyth surface, then*

$$\operatorname{Ker} \psi = \operatorname{Ker} \bar{\pi} = \{\mathrm{id}\} \tag{2.12.5}$$

and

$$\mathrm{Aut}\, \Psi(\mathscr{D}) \cong \mathrm{Aut}_\Psi \mathscr{D} = \mathscr{R}\,, \tag{2.12.6}$$

where $\mathscr{R}$ *is a finite group of rotations around* $z = 0$ *in* $\boldsymbol{C}$.

PROOF. 1. The equations (2.12.2) and (2.12.3) follow easily from Proposition

2.11, Proposition 2.8, and Corollary 2.11. Furthermore, with the definitions in Corollary 2.11 we can describe the group $\mathrm{Aut}_{\Phi} M$, and, by the identification of M with a subset of $\boldsymbol{C}$, also $\mathrm{Aut}_{\Psi} \mathscr{D}$, explicitly:

- The group $\mathscr{R} \subset \mathrm{Aut}\, \Psi(\mathscr{D})$ is identified in $\mathrm{Aut}\, M$ with the set of translations parallel to the imaginary axis in $\boldsymbol{C}$. We therefore have $\mathscr{T} = \psi^{-1}(\mathscr{R})$.
- The group $\tilde{Q} \subset \mathrm{Aut}\, \Psi(\mathscr{D})$ is identified in $\mathrm{Aut}\, M$ with the group Q of translations in $\boldsymbol{C}$ which leave $\mathscr{L}$ invariant. We therefore have $\psi^{-1}(\tilde{Q}) = Q$.
- The rotation $\tilde{R}$ is identified in $\mathrm{Aut}\, M$ with a $180°$-rotation around an arbitrary fixed point $z \in \mathscr{L}$. We choose $z=0$. Then $\tilde{R}$ is identified with $R_\pi : z \to -z$ up to an automorphism in $\Gamma \subset \mathscr{T}$. We therefore have $\psi^{-1}(\mathscr{R} \times \{I, \tilde{R}\}) = \mathscr{T} \times \{I, R_\pi\}$.

Since ψ is a surjective homomorphism and since $\mathrm{Aut}\, \Psi(\mathscr{D}) = \mathscr{R} \times \tilde{Q} \times \{I, \tilde{R}\}$, we have

$$\mathrm{Aut}_{\Psi} \mathscr{D} = \psi^{-1}(\mathrm{Aut}\, \Psi(\mathscr{D})) = \mathscr{T} \times Q \times \{I, R_\pi\} . \tag{2.12.7}$$

Since with Theorem 2.4,

$$\mathscr{T} \times Q \times \{I, R_\pi\} \subset \mathrm{Aut}_{\Psi} \mathscr{D} \subset \mathrm{Iso}_{\Psi} \mathscr{D} \subset \mathscr{T} \times Q \times \{I, R_\pi\} , \tag{2.12.8}$$

we get $\mathrm{Iso}_{\Psi} \mathscr{D} = \mathscr{T} \times Q \times \{I, R_\pi\}$, and therefore Eq. (2.12.4).

2. For the immersions Ψ_c^m, the metric $1/2 e^u dz d\bar{z}$ and, by the well-known equation

$$K = H^2 - 4 |E|^2 e^{-2u} \tag{2.12.9}$$

for the Gauß curvature K also $|E|$ is invariant under the one-parameter group of rotations around $z=0$. Therefore, by Corollary 2.3 and Theorem 2.4, we have that either $\mathrm{Aut}_{\Psi} \mathscr{D}$ is contained in the group of rotations around a fixed point, or $\Psi(\mathscr{D})$ is a cylinder. Therefore, for a nondegenerate Smyth surface, $\mathrm{Ker}\, \psi \subset \mathrm{Aut}_{\Psi} \mathscr{D}$ consists only of rotations. But since, by Proposition 2.8, $\mathrm{Ker}\, \psi$ acts freely on $M = \boldsymbol{C}$, this implies that $\mathrm{Ker}\, \psi = \{\mathrm{id}\} = \mathrm{Ker}\, \bar{\pi}$ and, with Corollary 2.7, $\psi : \mathrm{Aut}_{\Psi} \mathscr{D} \to \mathrm{Aut}\, \Psi(\mathscr{D})$ is an isomorphism of Lie groups, i.e.,

$$\mathrm{Aut}\, \Psi(\mathscr{D}) \cong \mathrm{Aut}_{\Psi} \mathscr{D} . \tag{2.12.10}$$

Let $g \in \mathrm{Aut}_{\Psi} \mathscr{D}$. Then g is a rotation around $z=0$, i.e., $g(z) = e^{i\theta} z$ for some $\theta \in [0, 2\pi)$. By Eq. (2.3.8) and Corollary 2.4, we get that $e^{i(m+2)\theta} = 1$ for some integer $m \geq 0$. This shows that $\mathrm{Aut}_{\Psi} \mathscr{D}$ is a discrete group of rotations around $z=0$ in $\boldsymbol{C}$, finishing the proof. □

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