Tohoku Math. J.
51 (1999), 227–236

# CODIMENSION-ONE FOLIATIONS AND ORIENTED GRAPHS

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(Received January 29, 1998, revised June 30, 1998)

**Abstract.** In this paper, an oriented graph $G(M, F)$ is assigned to each codimension-one foliation $(M, F)$, and topological relations between $(M, F)$ and $G(M, F)$ are studied. A strong relation between admissible functions of $(M, F)$ and $G(M, F)$ is given.

**1. Introduction.** Let $(M, F)$ be a transversely oriented codimension-one foliation $F$ of a closed oriented manifold $M$. On the set of all leaves of $F$, Novikov [6] introduced a partial order to define a so-called Novikov component. On the other hand, it is well-known that a partially ordered set is described as an oriented graph. In this paper, we assign to each $(M, F)$, in a unique way, an oriented graph $G(M, F)$ by a similar way to Novikov's method, and show that for any oriented graph $G$, there is a codimension-one foliation $(M, F)$ with $G=G(M, F)$. These are done in §3. We also show that there is a 'nice' embedding $\phi : G(M, F) \to M$, and in §4 we prove that the induced homomorphism $\phi_* : \pi_1(G(M, F)) \to \pi_1(M)$ is injective. Walczak [15] introduced the notion of admissible functions of $(M, F)$ and the present author defined the notion of admissible functions of oriented graphs in [10]. As an application of the viewpoint obtained above, we show that these two notions of admissible functions are essentially same. This is done in §5. Finally, in §6, we give a brief discussion on Riemannian labels of oriented graphs, whose definition comes naturally from our viewpoint, and on the Laplacians on graphs.

**2. Preliminaries.** We begin this section with some definitions on graphs. For the definition of cellular complexes, see Spanier [13], and for generalities on graph theory, see Bollobas [2].

$G$ is called a graph if $G$ is a finite one-dimensional cellular complex. We set $V=V(G)=\{v_i\}=\{\text{all 0-cells of } G\}$ and $E=E(G)=\{e_a\}=\{\text{all 1-cells of } G\}$. We call each $v\in V(G)$ a vertex, and $e\in E(G)$ an edge. For $e\in E(G)$, we also set $V(e)=\mathrm{Cl}(e)-e=\{\text{endpoints of } e\}$, where the closure $\mathrm{Cl}(e)$ of $e$ is taken in $G$.

REMARK. (a) $V(e)$ may consist of only one point $\{v\}$. In this case, we call $e$ a loop at $v$.

(b) $V(e_a)=V(e_b)$ may occur even if $e_a\neq e_b$. In this case, $G$ is called a multigraph (see Bollobas [2]).

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* 1991 *Mathematics Subject Classification*. Primary 57R30; Secondary 05C20.

A path $P=(V(P), E(P))$ is a pair of ordered elements of $V(G)$ and $E(G)$ of the form

$$V(P)=\langle v_0, v_1, \dots, v_l\rangle\,, \quad E(P)=\langle e_1, e_2, \dots, e_l\rangle \quad \text{with} \quad V(e_i)=\{v_{i-1}, v_i\}\,.$$

The length $l(P)$ of $P$ is defined to be the cardinality of the set $E(P)$, that is, $l$. In case $v_l=v_0$, we call $P$ a closed path.

Let $G=(V, E)$ be a graph. Since each edge $e$ is homeomorphic to $(0, 1)$, $e$ has a natural orientation induced, via a fixed homeomorphism, from that of $(0, 1)$. In this case, we say that $e$ is oriented. With this in mind, we give the following definition.

$G$ is called an oriented graph if $G$ is a graph and each edge is oriented. For each edge $e\in E(G)$, we call $I(e)=H(0)$ the initial vertex, and $T(e)=H(1)$ the terminal vertex. Here $H:[0, 1]\to G$ is the extended map of the given homeomorphism $h:(0, 1)\to e\subset G$. In case $I(e)=v$ and $T(e)=w$, we occasionally denote $e$ by $[v, w]$. Note that if $e$ is a loop at the vertex $v$, then $v=I(e)=T(e)$.

Now, let $(M, F)$ be a transversely oriented codimension-one foliation $F$ of a closed oriented manifold $M$. For generalities on foliations, see Hector and Hirsh [3]. In the following, we shall work in the $C^\infty$-category.

A compact saturated domain $D$ of $M$ is said to be a foliated trivial $I$-bundle if $D$ is the total space of a trivial $I$-bundle over a compact leaf $L$ of $F$ and if the induced foliation on $D$ from $F$ is everywhere transverse to the fibers $I$. Note that the boundary $\partial D$ consists of two copies of the compact leaf $L$. A compact saturated domain $D$ of $M$ is said to be a $(+)$-fcd (resp. $(-)$-fcd) if $N$ is outward (resp. inward) everywhere on the boundary $\partial D$ of $D$, where $N$ is a non-vanishing vector field on $M$ transverse to $F$ so that the direction of $N$ coincides with the transverse orientation of $F$.

It is well-known that if $F$ has an infinite number of compact leaves, then all but a finite number of them are contained in some foliated trivial $I$-bundles (cf. Hector and Hirsh [3]).

Let $L\subset \operatorname{Int} M$ be a compact leaf of a foliated manifold $(M, F)$ with a boundary which is a union of compact leaves of $F$. Construct a new foliated manifold $(M_0, F_0)$ as follows: Delete the subset $L$ from $M$ and add two copies of $L$ to $M-L$ by the natural identification so that the resulting manifold $M_0$ to be compact with $\partial M_0=\partial M\cup\{$two copies of $L\}$ and $F_0=(F-L)\cup\{$two copies of $L\}$. We say that $(M_0, F_0)$ is obtained from $(M, F)$ by cutting $M$ along $L$.

Let $(M, F)$ be as above and $(G, V, E)$ be an oriented graph. We say that a mapping $\phi: G\to M$ is a transverse embedding if $\phi$ is a continuous injection and the restriction $\phi|_{\mathrm{Cl}(e)}$ of $\phi$ to each $\mathrm{Cl}(e)$, $e\in E$, can be extended to a smooth transverse embedding of some open interval containing $[0, 1]$, the domain of the extended map $H:[0, 1]\to G$ of $e$. Furthermore, if the image of $\phi$ intersects all leaves of $F$ and the induced orientation on $e$ from the transverse orientation of $F$ coincides with the original one of $e$, we call $\phi$ to be nice.

**3. Construction of graphs and foliations.** Let $(M, F)$ be a transversely oriented codimension-one foliation $F$ of a closed oriented manifold $M$ with $\dim M \geq 3$. In this section, we shall construct, in a unique way, an oriented graph $G(M, F)$ from $(M, F)$. Furthermore, from an arbitrarily given oriented graph $G$, we shall construct a transversely oriented codimension-one foliation of a closed oriented manifold $(M, F)$ so that $G = G(M, F)$.

First, assume that $F$ has no compact leaves. In this case, it is well-known that there is a closed transversal intersecting all leaves of $F$, where a closed transversal means an embedding $\phi : S^1 \to M$ which is transverse to the leaves of $F$. Then take a point $v$ on $S^1$ and regard $S^1$ as an oriented graph $G(M, F)$ with one vertex $\{v\}$ and one loop $\{S^1\}$ at $v$ with the orientation induced from the transverse orientation of $F$.

Second, assume that $F$ has at least one compact leaf, say $L$, and by cutting along which the foliated manifold obtained from $M$ is a foliated trivial $I$-bundle. In this case, it is also well-known that there is a closed transversal $S^1$ intersecting all leaves of $F$. By the same way as in the first case, take a point $v$ on $S^1$ and regard $S^1$ as an oriented graph $G(M, F)$ with one vertex $\{v\}$ and one loop $\{S^1\}$ at $v$ with the orientation induced from the transverse orientation of $F$.

Finally, we assume that $F$ has compact leaves, but none of them have the property in the second case. In this case, take all (set-theoretical) maximal foliated trivial $I$-bundles $D_1, D_2, \ldots, D_s$, and set $M_1 = M - \bigcup_{i=1}^{s} \operatorname{Int}(D_i)$. By assumption, $M_1$ is not empty. Then take minimal $(\pm)$-fcd's $D_{s+1}, D_{s+2}, \ldots, D_t$, and set $M_2 = M_1 - \bigcup_{i=s+1}^{t} \operatorname{Int}(D_i)$. If $M_2$ is not empty, cut $M_2$ along all compact leaves in the interior of $M_2$, and list all connected components as $D_{t+1}, D_{t+2}, \ldots, D_u$. Note that the number of compact leaves in $M_2$ is finite from the fact stated in Section 2.

Now we construct an oriented graph $G(M, F)$ from $(M, F)$. Take $v_i \in \operatorname{Int}(D_i)$ $(i = 1, 2, \ldots, u)$ and set $V(G) = \{v_1, v_2, \ldots, v_u\}$. In case $M_2 = \varnothing$, the argument below is valid by simply replacing $u$ with $t$. For each compact leaf $L_{i_j} \subset \partial D_i$, take a point $p_{i_j} \in L_{i_j}$. If $L_{i_j} = L_{k_l} \subset \partial D_i \cap \partial D_k$, then choose $p_{i_j}$'s so that $p_{i_j} = p_{k_l}$. On each $D_i$ it is easy to construct smooth arcs $\{c_{i_j}\}$ satisfying the following conditions: $c_{i_j}$ is a smooth arc between $v_i$ and $p_{i_j}$, $c_{i_j} \cap c_{i_l} = \{v_i\}$ if $j \neq l$, each $c_{i_j}$ is properly contained in a smooth transverse curve, and the set $\bigcup_j c_{i_j}$ intersects all leaves of $F|D_i$. For each compact leaf $L = L_{i_j} = L_{k_l} \subset \partial D_i \cap \partial D_k$, take a union $c_{i_j} \cup c_{k_l}$ and deform it slightly near $L$ so that the resulting curve is again a smooth transverse curve between $v_i$ and $v_k$. We denote this curve by $e_L$ and give $e_L$ an orientation induced from the transverse orientation of $F$. Note that this definition makes sense even in the case $i = k$. In this case, $e_L$ is a loop. Set $E(G) = \{e_L\}$. It is easy to see that $G = (V(G), E(G))$ is an oriented graph. We define $G(M, F) = (V(G), E(G))$.

By the construction above, we get the following result.

THEOREM 1. *Let $(M, F)$ be as above. For each $(M, F)$ there exist an oriented graph $G(M, F)$ and a nice transverse embedding $\phi : G(M, F) \to (M, F)$. Furthermore, for each* *edge* $e \in E(G)$, $\phi(\mathrm{Int}(e))$ *intersects each compact leaf of* $F$ *at most once.*

Conversely, we have the following

THEOREM 2. *Let* $G$ *be an oriented graph. Then there is a foliated manifold* $(M, F)$ *so that* $G = G(M, F)$.

PROOF. Let $G=(V, E)$ be an arbitrarily given oriented graph. We shall construct a codimension-one foliation $(M^3, F)$ on a 3-dimensional manifold $M^3$ so that $G(M^3, F)=G$.

The idea is the following: For each vertex $v \in V$ adjacent $k$ edges, construct a 3-dimensional manifold $M_v^3$ with $k$ tori $T^2$'s as boundary components and a codimension-one foliation $F_v$ with $\partial M_v^3 \subset F_v$. If $v$ is adjacent to $w$, then glue suitable $T^2 \subset \partial M_v$ and $T^2 \subset \partial M_w$. After glue all $T^2$'s, we get the desired $(M^3, F)$.

Let $v \in V$ be a vertex adjacent $k_v$ outward edges and $l_v$ inward edges, that is, $v$ is a initial point of $k_v$ edges and is a terminal point of $l_v$ edges. Take a 2-dimensional sphere $S^2$, delete $k_v + l_v$ small open discs from $S^2$, and denote by $D_v$ the resulting disc with $k_v + l_v$ boundary components. Set $M_v = D_v \times S^1$, and list all boundary components of $\partial M_v$ as $C_1^+, \dots, C_{k_v}^+, C_1^-, \dots, C_{l_v}^-$ so that each $C_i^+$ corresponds to a vertex adjacent to $v$ with an oriented edge outward at $v$ and that each $C_j^-$ corresponds to a vertex adjacent to $v$ with an oriented edge inward at $v$. Now construct a transversely oriented codimension-one foliation $F_v$ with $\partial M_v = \partial D_v \times S^1 \subset F_v$ as follows: Give $S^1 = \boldsymbol{R}/\boldsymbol{Z}$ the canonical orientation induced from the one of $\boldsymbol{R}$. Wind $(\mathrm{Int}\, D_v) \times \{t\}$, $t \in S^1$, along $S^1$ in the negative direction near $C_i^+$'s and in the positive direction near $C_j^-$'s (cf. turbulization in [3] or [7]). Then we get a foliation $F_v$ consisting of these leaves and compact leaves $(\partial D_v) \times S^1$. Note that the transverse orientation along $C_i^+ \times S^1$ is outward and is inward along $C_l^- \times S^1$. The resulting foliated manifold $(M_v, F_v)$ is the desired one. In case $k_v = l_v = 1$, this construction simply gives a foliated trivial $I$-bundle over $T^2$ and we need to deform it. To do this, the simplest way is to use the $*$-operation defined by Lawson [5]. Let $(M_v', F_v')$ be the foliated manifold obtained by the above construction. Define $(M_v, f_v) = (M_v', F_v') * (T^3, F_a)$, where $(T^3, F_a)$ is the codimension-one foliation of $T^3$ with irrational $a$ 'slant', that is, $F_a$ is defined by a closed 1-form and all leaves are dense in $T^3$. $*$-operation is an identification of foliations along closed transversals, and produces no new compact leaves.

If $v$ and $w$ are adjacent, then $w$ corresponds to one of $C_i^\pm \times S^1$'s $\subset M_v$, say, $C_i^+ \times S^1$, and $v$ to one of $C_j^\mp \times S^1$'s $\subset M_w$, say, $C_j^- \times S^1$. Identify $M_v$ and $M_w$ along $C_i^+ \times S^1$ and $C_j^- \times S^1$ naturally. In this way, identifying all $C_i^\pm \times S^1$'s in $M_v$ for $v \in V(G)$, we get the desired codimension-one foliation

$$(M^3, F) = \bigcup_{v \in V(G)} (M_v, F_v)/\{\text{identification given above}\}\,.$$

It is easy to see that $G(M, F) = G$. This completes the proof of Theorem 2.

**4. A topological relation.** In this section, we show the following topological relation between $(M, F)$ and $G(M, F)$ constructed in Section 3.

THEOREM 3. *Let $(M, F)$, $G(M, F)$ and $\phi$ be as in Theorem 1. If $F$ has a compact leaf, then the induced map $\phi_* : \pi_1(G) \to \pi_1(M)$ is injective.*

PROOF. We shall identify the oriented graph $G=G(M, F)$ and $\phi(G(M, F)) \subset M$. We assume that $\operatorname{Ker} \phi_* \neq \{1\}$ and derive a contradiction. Let $\alpha$ be a non-trivial element in $\operatorname{Ker} \phi_*$. Represent $\alpha$ by a closed path of the smallest length, say, $\alpha = \langle e_1 e_2 \cdots e_k \rangle$. Note that, as $F$ has a compact leaf, each edge $e_i$ intersects at least one compact leaf and distinct edges do not intersect the same compact leaf except at their vertices. Let $f\colon D^2 \to M$ be a continuous map with $f(\partial D) = e_1 e_2 \cdots e_k$. We deform $f$ so that except near vertices of $e_i$'s $f$ is smooth and is in general position with respect to $F$. If $L_1$ is a compact leaf intersecting $\operatorname{Int}(e_1)$, then $f^{-1}(L_1 \cap f(D^2))$ is a set of circles and arcs on $D^2$, and one of the arcs connects a point in $\operatorname{Int}(e_1)$ to a point in $\operatorname{Int}(e_j)$ for some $j$. If $e_1 = [v, w]$, then, by considering the orientations, it is easy to see that $e_j = [w, v]$. If $L_2$ is a compact leaf intersecting $\operatorname{Int}(e_2)$, then $f^{-1}(L_2 \cap f(D^2))$ is a set of circles and arcs on $D^2$, and one of the arcs connects a point in $\operatorname{Int}(e_2)$ to a point in $\operatorname{Int}(e_l)$ for some $l$. As the compact leaves $L_1$ and $L_2$ does not intersect, the arc between $\operatorname{Int}(e_2)$ and $\operatorname{Int}(e_l)$ does not intersect the arc between $\operatorname{Int}(e_1)$ and $\operatorname{Int}(e_j)$. This implies $l<j$, and if $e_2 = [x, y]$, then $e_l = [y, x]$. We can repeat this process until we find $i$ so that $e_i = [u, z]$ and $e_{i+1} = [z, u]$. Therefore, $\alpha = \langle e_1 \cdots e_{i-1} [u, z][z, u] e_{i+2} \cdots e_k \rangle = \langle e_1 \cdots e_{i-1} e_{i+2} \cdots e_k \rangle$, which contradicts the minimality of the length of $e_1 e_2 \cdots e_k$ representing $\alpha$. This completes the proof of Theorem 3.

REMARK. It is still an open problem whether any smooth codimension-one foliation on a simply connected closed manifold always admit compact leaves or not (cf. Langevin [4]).

By the well-known Novikov's compact leaf theorem (see Novikov [6]), any smooth codimension-one foliation on $S^3$ has a compact leaf. Thus, combining this with Theorem 3, we have the following.

COROLLARY 1. *For any $(S^3, F)$, the graph $G(S^3, F)$ is a tree. Here, the orientation of edges of $G(S^3, F)$ are negrected.*

**5. Admissible functions.** First, we give further definitions on graphs. Let $G = (V(G), E(G))$ be a graph. A graph $K$ is called a full subgraph of $G$ if

(i) $K$ is a non-empty subcomplex of $G$, and

(ii) any $e \in E(G)$ with $V(e) \subset K$ implies $e \in E(K)$.

A proper full subgraph $K$ of an oriented graph $G$ is called a $(+)$-subgraph (resp. $(-)$-subgraph) if $e \in E(G)$ with $V(e) \cap V(K) \neq \emptyset$ and $V(e) \cap (V(G) - V(K)) \neq \emptyset$ implies $I(e) \in V(K)$ (resp. $T(e) \in V(K)$).

Recall the definition of admissible functions of an oriented graph $G$ (see Oshikiri [10]). We call a function $f\colon V(G)\to \boldsymbol{R}$ admissible if every minimal $(+)$-subgraph contains a vertex $v$ with $f(v)>0$, and every minimal $(-)$-subgraph contains a vertex $w$ with $f(w)<0$. Here "minimal" means the usual set theoretical sense, that is, being non-empty and containing no non-empty proper $(+)$-subgraphs (resp. $(-)$-subgraphs). In case $G$ has no $(+)$-subgraphs, any function $f$ with $f(v)>0$ and $f(w)<0$ for some $v, w\in V(G)$ or $f\equiv 0$ is called admissible.

Next we recall some definitions on foliations. Let $F$ be a transversely oriented codimension-one foliation of a closed connected oriented manifold $M$. Let $g$ be a Riemannian metric on $M$. Then there is a unique unit vector field $N$ orthogonal to $F$ whose direction coincides with the given transverse orientation of $F$. We give an orientation to $F$ as follows: Let $\{E_1, \dots, E_n\}$ be an oriented local orthonormal frame for the tangent bundle $TF$ of $F$. The orientation of $M$ given by $\{N, E_1, \dots, E_n\}$ then coincides with the given one of $M$. We denote the mean curvature of a leaf $L$ at $x$ with respect to $N$ by $H(x)$, that is,

$$H=\sum_{i=1}^{n}\langle \nabla_{E_i}E_i, N\rangle\,,$$

where $\langle\ ,\ \rangle$ means $g(\ ,\ )$, $\nabla$ is the Riemannian connection of $(M, g)$, and $\{E_i\}$ is a local orthonormal frame for $TF$ with $\dim F=n$. We call $H(x)$ the mean curvature function of $F$ with respect to $g$. We also define an $n$-form $\chi_F$ on $M$ by

$$\chi_F(V_1, \dots, V_n)=\det(\langle E_i, V_j\rangle)_{i,j=1,\cdots,n} \qquad \text{for} \quad V_j\in TM\,,$$

where $\{E_1, \dots, E_n\}$ is an oriented local orthonormal frame for $TF$. Note that the restriction $\chi_F|_L$ is the volume element of $(L, g|_L)$ for $L\in F$. Then we have the following formula.

Proposition R (Rummler [12]). *$d\chi_F=-H\,dV(M, g)=\mathrm{div}_g(N)\,dV(M, g)$, where $dV(M, g)$ is the volume element of $(M, g)$ and $\mathrm{div}_g(N)$ is the divergence of $N$ with respect to $g$, that is,*

$$\mathrm{div}_g(N)=\sum_{i=1}^{n}\langle \nabla_{E_i}N, E_i\rangle\,.$$

Let $f$ be a smooth function on $M$. We call $f$ admissible if there is a Riemannian metric $g$ on $M$ so that $-f$ coincides with the mean curvature function of $F$ with respect to $g$ (see Walczak [15] or Oshikiri [8], [9]). A characterization of admissible functions, which is conjectured by Walczak (see Langevin [4]) and proved affirmatively by the author (see Oshikiri [11]), is the following

Theorem O. *Let $F$ be a transversely oriented codimension-one foliation of a closed connected oriented manifold $M$. Assume that $F$ contains at least one $(+)$-fcd. Then $f$ is admissible if and only if $f(x)>0$ somewhere in any minimal $(+)$-fcd and $f(y)<0$*

*somewhere in any minimal* $(-)$*-fcd. In case* $F$ *contains no* $(+)$*-fcd's, any smooth function* $f$ *with* $f(x)>0$ *and* $f(y)<0$ *for some* $x, y\in M$ *or* $f\equiv 0$ *is admissible.*

Now we shall discuss a relation between these two definitions of admissible functions. Let $(M, F)$ and $G=G(M, F)$ be as in Section 3. Let $v\in V(G)$ and $e\in E(G)$ correspond to a foliated compact domain $D_v\subset M$ and to a compact leaf $L_e\in F$, respectively. For a saturated compact domain $D$ of $(M, F)$ we denote by $G(D)$ the subgraph of $G$ consisting of all vertices $v\in V(G)$ with $\operatorname{Int} D\cap D_v\neq\varnothing$ and all edges $e\in E(G)$ with $L_e\subset\operatorname{Int} D$. It is easy to see the following.

LEMMA. *If* $D$ *is a* $(+)$*-fcd (resp.* $(-)$*-fcd), then* $G(D)$ *is a* $(+)$*-subgraph (resp.* $(-)$*-subgraph). Furthermore,* $D$ *is a minimal* $(+)$*-fcd (resp.* $(-)$*-fcd) if and only if* $G(D)$ *is a minimal* $(+)$*-subgraph (resp.* $(-)$*-subgraph).*

Let $f$ be a smooth function on $M$ and $dV$ a volume element on $M$. Define a function $G_{dV}(f)\colon V(G)\to \boldsymbol{R}$ by

$$G_{dV}(f)(v)=\int_{D_v} f\,dV \qquad \text{for} \quad v\in V(G)\,.$$

The main result of this section is the following

THEOREM 4. *For a smooth function* $f$ *on* $M$*, the following two conditions are equivalent.*

(1) $f$ *is a admissible on* $(M, F)$.

(2) *There is a volume element* $dV$ *on* $M$ *so that* $G_{dV}(f)$ *is admissible on* $G(M, F)$.

PROOF. First assume that $f$ is admissible. By definition, there is a Riemannian metric $g$ on $M$ so that $f=-H$, where $H$ is the mean curvature function of $F$ with respect to $g$. Set $dV=dV(M, g)$, that is, the volume element of $(M, g)$. We show that $G_{dV}(f)$ is admissible. Let $K$ be a minimal $(+)$-subgraph of $G$. Set $D_K=\bigcup_{v\in V(K)} D_v$. By the above lemma, $D_K$ is a minimal $(+)$-fcd. Using Rummler's formula (Proposition R) we have

$$\sum_{v\in V(K)} G_{dV}(f)(v)=\int_{D_K} f\,dV=\int_{\partial D_K}\chi_F>0\,,$$

since $D_K$ is a $(+)$-fcd. Thus $G_{dV}(f)(v)>0$ for some $v\in V(K)$. Similary, we also have $G_{dV}(f)(v)<0$ for some $v\in V(K)$ when $K$ is a $(-)$-subgraph.

We prove the converse. By Theorem O, it is sufficient to show that $f(x)>0$ somewhere in any minimal $(+)$-fcd and $f(y)<0$ somewhere in any minimal $(-)$-fcd. Let $D$ be a minimal $(+)$-fcd. By the above lemma, $G(D)$ is a minimal $(+)$-subgraph. Thus, there is a vertex $v\in V(G(D))$ so that $G_{dV}(f)(v)>0$, as $G_{dV}(f)$ is admissible on $G(M, F)$. By definition, this means that $\int_{D_v} f\,dV>0$. Therefore there must be a point

$x \in D_v \subset D$ with $f(x) > 0$. Similary, there must be a point $y \in D_v \subset D$ with $f(y) < 0$ for any minimal $(-)$-fcd $D$. This completes the proof.

COROLLARY 2. *For any admissible function $h$ on $G(M, F)$, there are an admissible function $f$ and a volume element $dV$ on $M$ so that $h = G_{dV}(f)$.*

PROOF. Let $h$ be an admissible function on $G(M, F)$. If there are a smooth function $f$ and a volume element $dV$ on $M$ so that $h = G_{dV}(f)$, then, by the above theorem, $f$ is automatically admissible. Thus, we have only to show the existence of a smooth function $f$ and a volume element $dV$ on $M$ so that $h = G_{dV}(f)$. Choose an arbitrary volume element $dV$ on $M$ and fix it. Set $f_1 \equiv 0$. For each $v \in V(G)$, deform $f_1$ smoothly on $\mathrm{Int}\, D_v \subset M$ so that $h(v) = \int_{D_v} f_1 dV$, and set the resulting smooth function $f$. It is easy to see $h = G_{dV}(f)$.

**6. Concluding remarks.** The viewpoint given above enables us to translate many notions on foliated manifolds into the ones on graphs. We shall discuss on this point briefly.

Let $G = (V, E)$ be an oriented graph. Set

$$C^0(G) = \{f: V \to \boldsymbol{R}\} \quad \text{and} \quad C^1(G) = \{\phi: E \to \boldsymbol{R}\}.$$

We call $g_G = (g_V, g_E)$ a Riemannian label, where $g_V: V \to \boldsymbol{R}_+$ and $g_E: E \to \boldsymbol{R}_+$ are functions with positive real values. Define inner products on $C^0(G)$ and $C^1(G)$ by

$$\langle f_1, f_2 \rangle = \sum_{v \in V} g_V(v) f_1(v) f_2(v) \qquad \text{for} \quad f_1, f_2 \in C^0(G)$$

and

$$\langle \phi_1, \phi_2 \rangle = \sum_{e \in E} g_E(e) \phi_1(e) \phi_2(e) \qquad \text{for} \quad \phi_1, \phi_2 \in C^1(G).$$

Recall the boundary operator $d: C^0(G) \to C^1(G)$ defined by

$$df([x, y]) = f(y) - f(x) \qquad \text{for} \quad f \in C^0(G) \text{ and an oriented edge } [x, y] \in E.$$

Define the coboundary operator $\delta: C^1(G) \to C^0(G)$ by

$$\delta\phi(v) = \frac{1}{g_V(v)} \sum_{e_v} \operatorname{sgn}(e_v) g_E(e_v) \phi(e_v),$$

where the summation is taken over all edges $e_v \in E$ adjacent to $v$, $\operatorname{sgn}(e_v) = +1$ if $v$ is the terminal point of $e_v$, and $\operatorname{sgn}(e_v) = -1$ if $v$ is the initial point of $e_v$. It is easy to see that

$$\langle df, \phi \rangle = \langle f, \delta\phi \rangle \qquad \text{for} \quad f \in C^0(G) \text{ and } \phi \in C^1(G).$$

This enables us to define the so-called Laplacians $\Delta_V^g: C^0(G) \to G^0(G)$ and $\Delta_E^g: C^1(G) \to C^1(G)$ by

$$\Delta_V^g(f) = \delta df \quad \text{and} \quad \Delta_E^g(\phi) = d\delta\phi.$$

If we choose $g_V = g_E = 1$, then $\Delta_V^g$ is the standard Laplacian on graphs (cf. Biggs [1], Urakawa [14]). Note that the definition of $\delta$ involves the orientation of edges, however, the definitions of $\Delta^g$'s work without orientation of edges.

Finally, we mention the so-called Stokes' Theorem. For an oriented graph $G$ with a Riemannian label $g = (g_V, g_E)$, define the integrations of $f$ and $\phi$ by

$$\int_K f = \sum_{v \in V(K)} g_V(v) f(v) \qquad \text{for} \quad f \in C^0(G)$$

and

$$\int_K \phi = \sum_{e \in E(K)} g_E(e) \phi(e) \qquad \text{for} \quad \phi \in C^1(G)\,,$$

where $K = V(K) \cup E(K)$ is a set of vertices and edges with orientations in $G$. Here the following convention is used: If the orientation of $e \in E(K)$ is opposite to the one of the corresponding edge $e' \in E(G)$, then define $\phi(e) = -\phi(e')$. For a full subgraph $H$, we have the well-known Stokes' Theorem:

$$\int_H \delta\phi = \int_{\partial H} \phi \qquad \text{for} \quad \phi \in C^1(G)\,,$$

where $\partial H$ is the set of oriented edges $e$ such that $\partial e \cap V(H) \neq \varnothing$ and $\partial e \cap (V(G) - V(H)) \neq \varnothing$. $e \in \partial H$ is oriented from $H$ to the complement of $H$, that is, outward from $H$.

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