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GLOBALLY DEFINED LINEAR CONNECTIONS ON THE REAL LINE AND THE CIRCLE

Dedicated to Professor Thomas James Willmore on his seventy-seventh birthday

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Abstract. We classify globally defined linear connections on the real line as well as on the circle up to diffeomorphisms. We also prove that such connections can be realized by affine immersions into the affine plane.

1. Linear connections on $\boldsymbol{R}^1$ and S^1. On the real line $\boldsymbol{R}^1$ with its usual coordinate system $\{x\}$, we denote by d/dx the vector field (or its value at a point). A linear connection ∇ can be determined by a function Γ such that

$$\nabla_x \frac{d}{dx} = \Gamma(x) \frac{d}{dx}\,, \tag{1}$$

where ∇_x denotes covariant differentiation relative to d/dx.

Since $\boldsymbol{R}^1$ is 1-dimensional, every connection ∇ on $\boldsymbol{R}^1$ must be locally flat. This means that for each point x_0, there is a neighborhood with a flat local coordinate system, $\{\bar{x}\}$, relative to which we have

$$\nabla_{\bar{x}} \frac{d}{d\bar{x}} = 0\,. \tag{2}$$

We deal with the question: How do we find such a local coordinate system $\{\bar{x}\}$?

This is related to the question of finding a geodesic relative to ∇, that is, of solving the equation

$$\frac{d^2x}{dt^2} + \Gamma(x(t))\left(\frac{dx}{dt}\right)^2 = 0\,, \tag{3}$$

with the initial conditions, for convenience, say,

$$x(0) = 0\,, \quad \left(\frac{dx}{dt}\right)(0) = 1\,. \tag{4}$$

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The standard method of solving (3) is the following. Let $Q(x)$ be a function such that $Q'=\Gamma Q$, that is,

$$Q(x)=\exp\left(\int_0^x \Gamma(u)du\right). \tag{5}$$

By virtue of (3) we have

$$\frac{d}{dt}\left(Q(x(t))\frac{dx}{dt}\right)=0$$

and hence

$$Q(x(t))\frac{dx}{dt}=a \ \ (a\colon \text{const}),$$

assuming $dx/dt\neq 0$. Since $x(0)=0$ and $(dx/dt)(0)=Q(0)=1$, we get $a=1$. Therefore

$$\int_0^t Q(x(t))\frac{dx}{dt}dt=t.$$

The integral may be written simply as

$$t=\int_0^x Q(u)du.$$

We recover the affine parameter t from this integral.

Now we prove

THEOREM 1. *For a linear connection on* $\boldsymbol{R}^1$ *given by* (1), *a flat local coordinate system* $\{\bar{x}\}$ *around the point* 0 *can be obtained by*

$$\bar{x}(x)=\int_0^x Q(u)du, \tag{6}$$

where Q *is given by* (5). *Furthermore, the inverse function* $x(\bar{x})$ *satisfies the equation of a geodesic*:

$$d^2x/dt^2+\Gamma(x(t))(dx/dt)^2=0,$$

if we write t *for* $\bar{x}$.

PROOF. We recall how the function Γ transforms under a change of coordinate system. The function $\bar{\Gamma}$ for the coordinate system $\{\bar{x}\}$ is given by

$$\bar{\Gamma}=\frac{d^2x}{d\bar{x}^2}\frac{d\bar{x}}{dx}+\Gamma\frac{dx}{d\bar{x}}. \tag{7}$$

Now from (6) we get

$$\frac{d\bar{x}}{dx}=Q(x)\,,\quad \frac{dx}{d\bar{x}}=Q^{-1}(x)$$

and

$$\frac{d^2x}{d\bar{x}^2}\frac{d\bar{x}}{dx}=-Q'Q^{-2}Q^{-1}Q=-Q'Q^{-2}\,.$$

Substituting these into (7), we find $\bar{\Gamma}=0$ by virtue of $Q'=\Gamma Q$. The additional assertion follows from (7) by setting $\bar{\Gamma}=0$.

We present three examples to illustrate the discussions.

EXAMPLE 1. In the case where $\Gamma=0$, we have $Q(x)=1$. Hence $\bar{x}=x$. The geodesic is written simply as $x=t$, $-\infty<t<\infty$.

EXAMPLE 2. Let us consider the linear connection on $\boldsymbol{R}^1$ defined by $\Gamma=1$, and determine the geodesic $x(t)$ with initial conditions (4). We have

$$Q(x)=\exp\left(\int_0^x du\right)=e^x$$

and

$$\bar{x}=\int_0^x e^x dx=e^x-1=t\,.$$

The geodesic is given by

$$x=\ln(t+1)\,,\qquad -1<t<\infty\,.$$

This shows that the geodesic cannot be extended beyond -1 so the connection is not complete. It also means that the flat coordinate system $\{\bar{x}\}$ is confined to $(-1,\infty)$.

EXAMPLE 3. We next consider the linear connection given by $\Gamma(x)=-2x/(1+x^2)$. We have

$$Q(x)=\exp\left(-\int_0^x\frac{2u}{1+u^2}\,du\right)$$

and a flat coordinate system is given by

$$\bar{x}=\int_0^x\frac{1}{1+u^2}\,du=\arctan x\,.$$

The geodesic with the initial condition (4) is given by

$$x=\tan t\,,\qquad t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right).$$

The connection is not complete and the flat coordinate system is confined to $(-\pi/2, \pi/2)$.

Now going back to Theorem 1, we write

$$t=\int_0^x Q(u)du \tag{8}$$

using t instead of $\bar{x}$ on the left-hand side of (6). Since $dt/dx=Q(x)>0$, the range of the function $\mu: x\mapsto t$ is an interval I in $\boldsymbol{R}^1$. It falls into three kinds: the total space $\boldsymbol{R}^1$, a half-line (c, ∞) or (∞, c) (with 0 in it), or a bounded interval (with 0 in it). The inverse function $\tau=\mu^{-1}$ takes the interval I onto $\boldsymbol{R}^1$. We now show that τ maps the ordinary flat connection ∇^0 upon ∇. Since ∇^0 is flat, the condition on the connections is equivalent to

$$\nabla_{\tau_*(d/dt)}\tau_*(d/dt)=0 .$$

This equation is satisfied as in Theorem 1. By an *affine parametrization* of $(\boldsymbol{R}^1, \nabla)$ we mean a triple (I, ∇^0, τ), where I is an open interval with the ordinary flat connection ∇^0 (for which t is a flat coordinate) and τ is a connection-preserving diffeomorphism of (I, ∇^0) onto $(\boldsymbol{R}^1, \nabla)$.

Let us now assume that the given $(\boldsymbol{R}^1, \nabla)$ has an affine parametrization (I, ∇^0, τ), where $I=(c, \infty)$ and τ is a connection-preserving diffeomorphism from (I, ∇^0) onto $(\boldsymbol{R}^1, \nabla)$. It is easy to obtain a connection-preserving diffeomorphism from (I, ∇^0) onto $((-1, \infty), \nabla^0)$, an affine parametrization for the connection in Example 2. There is also a connection-preserving diffeomorphism between $((-1, \infty), \nabla^0)$ and $(\boldsymbol{R}^1, \nabla)$ in Example 2. Combining these diffeomorphisms we obtain a connection-preserving diffeomorphism between $(\boldsymbol{R}^1, \nabla)$ and the connection of Example 2.

This argument also holds in the case where $(\boldsymbol{R}^1, \nabla)$ has a bounded interval as the interval of affine parametrization. The case where the interval of affine parametrization is $(-\infty, \infty)$ is self-evident. We have thus proved

THEOREM 2. *Every linear connection globally defined on $\boldsymbol{R}^1$ can be obtained by a diffeomorphism from one of the models in Examples* 1, 2, *and* 3.

We may rephrase this result as follows.

COROLLARY 1. *Given a linear connection ∇ on $\boldsymbol{R}^1$, there exists a global coordinate y such that the given connection is equal to the one defined by*

$$\nabla_Y Y=0\,, \quad \nabla_Y Y=Y \quad or \quad \nabla_Y Y=-\frac{2y}{1+y^2}\,Y\,,$$

where $Y=d/dy$.

We have also

COROLLARY 2. *Each linear connection globally defined on $\boldsymbol{R}^1$ is the Levi-Civita connection of a certain Riemannian metric on $\boldsymbol{R}^1$.*

In order to prove this result, suppose g is the Riemannian metric defined by

$$g(X, X) = e^{2\lambda},$$

where $X = d/dx$ and λ is a smooth function. We may assume $\lambda(0) = 0$. The Levi-Civita connection ∇ for g satisfies

$$2g(\nabla_X X, X) = X(e^{2\lambda}),$$

which implies that Γ for ∇ is equal to $X\lambda$, that is, $d\lambda/dx$. In the earlier computation we now have

$$\int_0^x \Gamma(u)du = \lambda(x)$$

and

$$Q(x) = e^{\lambda(x)}.$$

If we take the Riemannian metric g with $\lambda(x) = 0$, then we get $\Gamma = 0$, leading to the connection in Example 1. If we take the metric g with $\lambda(x) = x$, then we get $\Gamma = 1$, namely, the connection in Example 2. The connection in Example 3 is obtained if we take $\lambda(x) = -\ln(x^2+1)$ so that $\Gamma(x) = -2x/(x^2+1)$. This shows that the models in Examples 1, 2, and 3 are metric connections. We already know that an arbitrary $(\boldsymbol{R}^1, \nabla)$ admits a connection-preserving diffeomorphism ϕ to one of the models in Examples 1, 2, and 3. If g denotes the pull-back of the metric in each model by ϕ, we see that ∇ is the Levi-Civita connection for g.

We shall now consider linear connections on the circle $\boldsymbol{S}^1$. The universal covering X of $\boldsymbol{S}^1$ is diffeomorphic to $\boldsymbol{R}^1$ and the given connection ∇ on $\boldsymbol{S}^1$ induces a linear connection $\tilde{\nabla}$ on X. We know that $(X, \tilde{\nabla})$ admits an affine parametrization (I, ∇^0, τ). Thus the fundamental group $\pi_1(\boldsymbol{S}^1)$ acting on X as a group of affine transformations of $(X, \tilde{\nabla})$ also acts on I as a group, say, G of affine transformations of (I, ∇^0). Hence the possibility of I being a bounded interval is excluded and we are left with the two cases: $I = \boldsymbol{R}^1$ and $I = (-1, \infty)$.

In the first case, G is generated by a transformation of the form $g(x) = ax + b$. If $a \neq 1$, then it has a fixed point; hence, $g(x) = x + b$ is a translation. This means that $(\boldsymbol{S}^1, \nabla)$ is diffeomorphic to $(I/G = \boldsymbol{R}^1/\boldsymbol{Z}, \nabla^0)$, where ∇^0 is the ordinary flat connection on $\boldsymbol{R}^1$; that is, there exists a diffeomorphism $\phi\colon \boldsymbol{S}^1 \to \boldsymbol{R}^1/\boldsymbol{Z}$ such that $\nabla = \phi^*\nabla^0$.

In the second case where $I = (-1, \infty)$, we may instead consider the half-line $R^+ = (0, \infty)$ with the ordinary flat connection. The group G is generated by multiplication $\varphi_a\colon x \mapsto ax$ $(a \neq 1)$, and we get $I/G = \boldsymbol{R}^+/\langle\varphi_a\rangle$. The quotient can be identified with $\boldsymbol{R}^1/\boldsymbol{Z}_a$, where $\boldsymbol{Z}_a$ is the additive subgroup generated by translation $x \mapsto x + \log a$; the induced connection $\boldsymbol{R}^1$ is that in Example 2. We have proved the following theorem.

THEOREM 3. *For any linear connection* ∇ *on the circle* $\boldsymbol{S}^1$, *there exists a diffeomorphism* ϕ *of* $\boldsymbol{S}^1$ *to* $\boldsymbol{R}^1/\boldsymbol{Z}$ *or to* $\boldsymbol{R}^+/\langle\varphi_a\rangle$ *such that* $\nabla=\phi^*\nabla^0$, *where* ∇^0 *is the respective flat connection. The connecion* ∇ *is complete in the former case and incomplete in the latter case.*

REMARK. The first connection is metric, because we can transfer the usual metric on $\boldsymbol{R}^1$ to X. The second connection is not metric; if it were, it would be complete because $\boldsymbol{S}^1$ is compact. We note that a non-complete connection was constructed also in [AM]. See also [KN, p. 292].

2. Affine immersion of linear connections on $\boldsymbol{R}^1$ and $\boldsymbol{S}^1$. Let I be an open interval of $\boldsymbol{R}$ and ∇ a linear connection on I. An immersion f of I into the affine plane $\boldsymbol{R}^2$ with the ordinary flat connection D and the determinant function is called an *affine immersion* if there exists a certain vector field ξ along the immersion and transversal to the immersed curve, we have the equation

$$D_X f_* X = f_*(\nabla_X X) + h(X, X)\xi\,, \tag{9}$$

where X is a vector field on I and h is a quadratic function on X. If h is nowhere 0, the immersion is called *nondegenerate*; geometrically, it is equivalent to the condition that the immersed curve has no inflection points.

We now recall the classical treatment of nondegenerate curves in $\boldsymbol{R}^2$. Refer to [NS, p. 2]. An immersion $f\colon t\mapsto x(t)$ is nondegenerate if and only if $\det[dx/dt, d^2x/dt^2]$ never vanishes. In this case, we may define a new parameter $s=s(t)$, called an *affine arclength parameter* by

$$s(t)=\int_0^t \det[dx/dt, d^2x/dt^2]^{1/3}dt\,.$$

We set

$$e_1(s)=f_*(d/ds)\,,\quad e_2(s)=de_1/ds\,.$$

Then we have

$$\det[e_1, e_2]=1\,,\quad de_2/ds=-ke_1\,,$$

where $k=k(s)$ is a function called *affine curvature*.

In this notation, the terms in (9) are given by

$$X=d/ds\,,\quad f_*X=e_1(s)\,,\quad \nabla_X X=0\,,\quad h(X, X)=1\,,\quad \xi=e_2\,. \tag{10}$$

Thus (J, s), where $J=s(I)$, is an affine parametrization. The vector field $\xi=e_2$ is called an affine normal, uniquely determined up to orientation.

Let us consider the following problem. Given a connection ∇ on an open interval

I, does I admit a nondegenerate immersion into $\boldsymbol{R}^2$ in such a way that ∇ is induced as in (9) by using the affine normal ξ? If it does, we say that (I, ∇) is *realizable* in $\boldsymbol{R}^2$. Let us begin with examples.

EXAMPLE 4. Let $I=\boldsymbol{R}$ and let ∇ be the ordinary connection. The immersion $f(t)=(t, (1/2)t^2)$ is a paraboloid and, for $\xi=(0, 1)$, we have

$$D_X f_* X = h(X, X)\xi$$

for $X=d/dt$. Hence the model in Example 1 is realizable.

EXAMPLE 5. Let $f(t)=(-1/s, (1/20)s^4)$ be an immersion of the half-line $(0, \infty)$. Then it is easy to see that s is an affine arclength parameter and

$$e_1=(1/s^2, (1/5)s^3)\,, \quad e_2=(-2/s^3, (3/5)s^2)\,.$$

We have $k=-6/s^2$. This example shows that the half-line with the ordinary flat connection, that is, the model in Example 2 is realizable.

The last example is the following.

EXAMPLE 6. Consider the immersion $x(t)=(t^{-p}, (1-t)^{-p})$ of the bounded interval $(0, 1)$. Since

$$\frac{dx}{dt}=(-pt^{-p-1}, p(1-t)^{-p-1})$$

and

$$\frac{d^2x}{dt^2}=(p(p+1)t^{-p-2}, p(p+1)(1-t)^{-p-2})\,,$$

we see that

$$\det[dx/dt, d^2x/dt^2]=-p^2(p+1)(t(1-t))^{-p-2}\,.$$

Hence if $p\neq 0, -1$, the immersion is nondegenerate and the affine arclength parameter is defined by

$$\frac{ds}{dt}=-(p^2(p+1))^{1/3}(t(1-t))^{-(p+2)/3}\,.$$

This means that the range J of s is bounded if $0<p<1$. Assuming, say, $p=1/2$, we have a bounded open interval J with the ordinary flat connection that can be immersed into $\boldsymbol{R}^2$. Hence the model in Example 3 is realizable. (We remark that the affine curvature is negative and tends to $-\infty$ at the both ends.)

Because of Theorem 2, we have the following.

THEOREM 4. *The real line with any linear connection is realizable as a closed nondegenerate curve in the affine plane* $\boldsymbol{R}^2$.

We finally treat affine immersions of the circle with a linear connection. If the linear connection is ordinary, then any ellipse lying in the affine plane realizes the connection. The affine curvature is constant and positive.

On the other hand, for any nondegenerate immersion of $\boldsymbol{S}^1$, the affine arclength parameter should extend infinitely on both sides. Hence the induced connection must be complete. In view of Theorem 3, we have the following.

THEOREM 5. *The circle with any linear connection is realizable in* $\boldsymbol{R}^2$ *if and only if the connection is complete.*

REMARK. We thank the referee for pointing out a gap in the statement of the original Theorem 5.

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