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L_p AND BESOV MAXIMAL ESTIMATES FOR SOLUTIONS TO THE SCHRÖDINGER EQUATION

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Abstract. Precise results on L_p and Besov estimates of the maximal function of the solutions to the Schrödinger equation are given. These results contain an improvement of the theorem in Sjölin [10].

1. Introduction. It is well-known that the solution to the Schrödinger equation

$$\frac{\partial u}{\partial t} = -i\triangle u, \quad u(0,x) = f(x), \qquad (x \in \boldsymbol{R}^n,\ t \in \boldsymbol{R}) \tag{1.1}$$

is given by

$$u(t,x) = c_n \iint e^{i(x-y)\xi + it|\xi|^2} f(y)d\xi dy\,.$$

In this note we shall consider estimates of L_2-norm and the Besov type norm of integrals of this kind by means of the Besov norm of f, and give L_p-estimates of their maximal functions.

Our first results are the following two theorems:

THEOREM 1. *Let σ be a positive number, $I = (0,1)$, $\gamma > 1$ and let $1 \le q \le \infty$. Assume that $h(t,\xi)$ is real-valued, measurable, and C^∞ in t and the inequality*

$$\left|\frac{\partial^k h(t,\xi)}{\partial t^k}\right| \le C_k(1 + |\xi|^{k\gamma}) \tag{1.2}$$

holds for any positive integer k, where C_k is a constant independent of t and ξ. Then, the operator T_1 defined by

$$T_1 f(t,x) = c_n \iint_{\boldsymbol{R}^n} e^{i(x-y)\xi + ih(t,\xi)} f(y)d\xi dy\,, \tag{1.3}$$

where $c_n = (2\pi)^{-n}$, is bounded from $B^{\gamma\sigma}_{2,q}(\boldsymbol{R}^n)$ to $B^{\sigma}_{2,q}(I; L_2(\boldsymbol{R}^2))$.

THEOREM 2. *Let h be a real-valued function satisfying the condition* (1.2). *Then, the operator T_1 defined by* (1.3) *is bounded from $B^{\gamma/2}_{2,1}(\boldsymbol{R}^n)$ to $L_2(\boldsymbol{R}^n; L_\infty(I))$, i.e.,*

$$\left(\int_{\boldsymbol{R}^n} \|T_1 f(x,\cdot)\|^2_{L_\infty(I)} dx\right)^{1/2} \le C\|f\|_{B^{\gamma/2}_{2,1}}\,. \tag{1.4}$$

For the operator of the type (1.5) below acting on Sobolev spaces H^s, there are several papers. Carbery [1] and Cowling [2] have prove that T_2 is bounded from $H^s(\boldsymbol{R}^n)$ to $L_2(I; L_2(\boldsymbol{R}^n))$ for $s > a/2$, and Theorem 2 is an improvement of their results. P. Sjölin [10]

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has proved that if $a > 1$ then $s > an/4$ is a sufficient condition for all n, and if $n = 1$ then $s \geq a/4$ is a necessary condition. S. Fukuma [3] has proved that if $f \in H^{1/4}(\boldsymbol{R}^n)$, $n \geq 2$, $q = 4n/(2n-1)$ and if f is radial, then $T_2 f \in L_q(\boldsymbol{R}^n; L_\infty(I))$. C. E. Kenig, G. Ponce and L. Vega ([5], [6], [7]) have indicated the application of the estimate to the dispersive equations.

In this paper we also have the following theorem for the operator of type (1.5):

THEOREM 3. *Let $a > 1$. Then, the operator T_2 defined by*

$$T_2 f(t, x) = c_n \iint e^{i(x-y)\xi + it|\xi|^a} f(y) d\xi dy \tag{1.5}$$

is bounded from $B_{2,1}^{an/4}(\boldsymbol{R}^n)$ to $L_2(\boldsymbol{R}^n; L_\infty(I))$, i.e.,

$$\left(\int_{\boldsymbol{R}^n} \| T_2 f(x, \cdot) \|_{L_\infty(I)}^2 dx \right)^{1/2} \leq C \| f \|_{B_{2,1}^{an/4}(\boldsymbol{R}^n)} .$$

Noting that $H^s \subset B_{2,1}^{an/4}$ if $s > an/4$, this result is an improvement of the theorem in Sjölin [10].

Our L_p-results are as follows:

THEOREM 4. *Let $a > 1$, $I = (0, 1)$, $1 \leq p \leq \infty$ and let*

$$\sigma = \min \left\{ \frac{1}{2} + (n-1) \left| \frac{1}{2} - \frac{1}{p} \right| , \frac{n}{4} + \frac{n}{2} \left| \frac{1}{2} - \frac{1}{p} \right| \right\} .$$

Then, the operator T_2 defined by (1.5) is bounded from $B_{p,1}^{a\sigma}(\boldsymbol{R}^n)$ to $L_p(\boldsymbol{R}^n; L_\infty(I))$, i.e.,

$$\left(\int_{\boldsymbol{R}^n} \| T_2 f(x, \cdot) \|_{L_\infty(I)}^p dx \right)^{1/p} \leq C \| f \|_{B_{p,1}^{a\sigma}(\boldsymbol{R}^n)} .$$

In §2 we shall give a proof of Theorem 1. In §3 we state a lemma needed in the proof of Theorem 2 and prove Theorem 2. In §4 we explain the proof of Theorem 3 and lemmas we used. In §5 we prove the Lemma 2 in the previous section. Finally in §6 we prove Theorem 4.

NOTATIONS. $\hat{f}(\xi) = \int e^{ix\xi} f(x) dx$ (Fourier transform of f); $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, $\nabla = (\partial_1, \dots, \partial_n)$, $\Delta = \sum_{j=1}^n \partial_j^2$, $x = (x_1, \dots, x_n)$; $\partial^\alpha = \prod_{j=1}^\alpha \partial_j^{\alpha_j}$, $x^\alpha = \prod_{j=1}^n x_j^{\alpha_j}$; L_p denotes the usual Lebesgue space on $\boldsymbol{R}^n$ with norm $\| \cdot \|_{L_p(\boldsymbol{R}^n)}$; H^s denotes the Sobolev space defined by $\{ f \in \mathcal{S}'; \| f \|_{H^s} = \| \hat{f}(\xi)(1 + |\xi|^2)^{s/2} \|_{L_2(\boldsymbol{R}^n)} < \infty \}$; $B_{p,q}^\sigma$ denotes Besov spaces with norm $\| \cdot \|_{B_{p,q}^\sigma}$ which is explained, for example, in [11]; $\mathcal{L}(X, Y)$ denotes the space of linear bounded operators from a Banach space X to Y; $L_p(\cdot\,; X)$ denotes the L_p-space of X-valued functions.

2. Proof of Theorem 1. First, consider the case where $q = 2$ and σ is a non-negative integer m. Notice that $B_{2,2}^m = H^m$. It is easy to see that

$$\partial_t^k T_1 f = c_n \iint e^{i(x-y)\xi + ih(t,\xi)} H_k(t, \xi) f(y) d\xi dy$$

with $|H_k(t,\xi)| \le c_k'(1+|\xi|^{k\gamma})$. Hence, by Parseval's formula we have

$$\|T_1 f\|^2_{L_2(I;L_2(\boldsymbol{R}^n))} \le C_0 \int_0^1 \|f\|^2_{L_2} dt = C_0 \|f\|^2_{L_2},$$

and

$$\|\partial_t^k T_1 f\|^2_{L_2(I;L_2(\boldsymbol{R}^n))} \le C_k \int_0^1 dt \|(1+|\xi|^{k\gamma})\hat{f}(\xi)\|^2_{L_2(\boldsymbol{R}^n)} \le C_k \|f\|^2_{H^{k\gamma}(\boldsymbol{R}^n)}.$$

Combining these facts, we obtain that

$$\|T_1 f\|^2_{H^m(I;L_2(\boldsymbol{R}^n))} \le C_m \|f\|_{H^{m\gamma}(\boldsymbol{R}^n)}.$$

Finally, we recall that the Besov spaces are identical with the real interpolation of the Sobolev spaces:

$$(L_2(\Omega;X), H^m(\Omega;X))_{\theta,q} = B^{m\theta}_{2,q}(\Omega;X).$$

Here, X is a Banach space and $(\cdot,\cdot)_{\theta,q}$ denotes the real interpolation spaces. Therefore, the conclusion of the theorem follows from interpolation of linear operators and the fact that T_1 is bounded from $H^{m\gamma}(\boldsymbol{R}^n)$ to $H^m(I;L_2(\boldsymbol{R}^n))$ for any non-negative integer m.

3. Proof of Theorem 2. To get L_2 maximal estimates for the operator of type (1.3) we need the following

LEMMA 1. *Let $I=(0,1)$, $1\le q\le p\le\infty$, and let σ be a positive number. Then, the Besov space $B^\sigma_{p,q}(I;L_p(\boldsymbol{R}^n))$ is continuously imbedded in the space $L_p(\boldsymbol{R}^n;B^\sigma_{p,q}(I))$.*

PROOF. Consider first the case where $0<\sigma<1$. Assume that $u(t,x)\in B^\sigma_{p,q}(I;L_p(\boldsymbol{R}^n))$. Then, by Minkowsky's inequality we see that

$$\begin{aligned}\|\{|u(t,x)|_{B^\sigma_{p,q}(I)}\}\|_{L_p(\boldsymbol{R}^n)} &= \|[\|\{\|u(t+s,x)-u(t,x)\|_{L_p((0,1-s))}\}s^{-\sigma}\}\|_{L_q^*(I)}]\|_{L_p(\boldsymbol{R}^n)}\\ &\le \|[\{\|(\|u(t+s,x)-u(t,x)\|_{L_p((0,1-s))})\|_{L_p(\boldsymbol{R}^n)}\}s^{-\sigma}]\|_{L_q^*(I)}\\ &\le \|[\{\|(\|u(t+s,x)-u(t,x)\|_{L_p(\boldsymbol{R}^n)})\|_{L_p((0,1-s))}\}s^{-\sigma}]\|_{L_q^*(I)}\\ &= |u|_{B^\sigma_{p,q}(I;L_p(\boldsymbol{R}^n))}.\end{aligned}$$

Here $L_q^*(I) := L_q(I, ds/s)$. In the same way we get for the case where $\sigma=1$;

$$\begin{aligned}&\|\{|u(t,x)|_{B^1_{p,q}(I)}\}\|_{L_p(\boldsymbol{R}^n)}\\ &\quad= \|[\|\{\|u(t+2s,x)-2u(t+s,x)+u(t,x)\|_{L_p((0,1-2s))}s^{-1}\}\|_{L_q^*(I)}]\|_{L_p(\boldsymbol{R}^n)}\\ &\quad\le \|\{\|(\|u(t+2s,x)-2u(t+s,x)+u(t,x)\|_{L_p(\boldsymbol{R}^n)})\|_{L_p((0,1-2s))}s^{-1}\}\|_{L_p^*(I)}\\ &\quad= |u|_{B^1_{p,q}(I;L_p(\boldsymbol{R}^n))}.\end{aligned}$$

Consider now the case where $\sigma = k+\theta$, k is a positive integer, and $0<\theta\le 1$. By the facts proved above we have

$$\begin{aligned}\|\{|u(t,x)|_{B^{\sigma}_{p,q}(I)}\}\|_{L_p(\boldsymbol{R}^n)} &= \|\{|\partial_t^k u(t,x)|_{B^{\theta}_{p,q}(I)}\}\|_{L_p(\boldsymbol{R}^n)}\\ &\le |\partial_t^k u|_{B^{\theta}_{p,q}(I;L_p(\boldsymbol{R}^n))}\\ &= |u|_{B^{\sigma}_{p,q}(I;L_p(\boldsymbol{R}^n))}\,.\end{aligned}$$

Noting that the norm of $B^{\sigma}_{p,q}(I;X)$ is given by $\|\cdot\|_{W^k_p(I;X)} + |\cdot|_{B^{\sigma}_{p,q}(I;X)}$ (see T. Muramatu [8]), these estimate gives the proof of Lemma 1.

From Lemma 1 and the imbedding theorem $B^{1/2}_{2,1}(I)\subset L_\infty(I)$ (see Muramatu [9]) it follows that

$$B^{1/2}_{2,1}(I;L_2(\boldsymbol{R}^n))\subset L_2(\boldsymbol{R}^n;B^{1/2}_{2,1}(I))\subset L_2(\boldsymbol{R}^n;L_\infty(I))$$

with continuous inclusions, which, combined with Theorem 1, gives Theorem 2.

4. Proof of Theorem 3. Next, Theorem 3 has been proved if we show that the operator S defined by

$$Sf(x) = \iint e^{i(x-y)\xi + it(x)|\xi|^a} f(y)\,d\xi\,dy\,,$$

where $t(x)$ is a measurable function of $x\in\boldsymbol{R}^n$ with $0\le t(x)\le 1$, is bounded from $B^{an/4}_{2,1}(\boldsymbol{R}^n)$ to $L_2(\boldsymbol{R}^n)$ and its norm is estimated by a constant independent of $t(x)$.

To prove this we need the following partition of unity in ξ-space. Let $\varphi_0\in C^\infty(\boldsymbol{R}^n)$, $\varphi\in C^\infty(\boldsymbol{R}^n)$ and $\psi\in C^\infty(\boldsymbol{R}^n)$ be functions such that

$$\varphi_0(\xi)+\sum_{j=1}^{\infty}\varphi(2^{-j}\xi)=1,\quad 0\le\varphi_0(\xi)\le 1,\quad 0\le\varphi(\xi)\le 1,$$

$$\mathrm{supp}(\varphi_0)\subset\{\xi\in\boldsymbol{R}^n;\ |\xi|<2\},\quad \mathrm{supp}(\varphi)\subset\left\{\xi\in\boldsymbol{R}^n;\ \frac{1}{2}<|\xi|<2\right\}.$$

Put $\varphi_j(\xi):=\varphi(2^{-j}\xi)$, $\psi_0(\xi):=\varphi_0(\xi/2)$, and

$$\psi(\xi):=\varphi\left(\frac{\xi}{2}\right)+\varphi(\xi)+\varphi(2\xi)\,,\quad \psi_j(\xi):=\psi(2^{-j}\xi)\quad \text{for } j\ge 1\,.$$

Then

$$\mathrm{supp}(\psi)\subset\left\{\xi\in\boldsymbol{R}^n;\ \frac{1}{4}<|\xi|<4\right\},\quad \psi(\xi)=1\qquad \text{if}\ \ \frac{1}{2}\le|\xi|\le 2\,,$$

and for $j=0,1,2,\dots$, $\psi_j(\xi)=1$ holds for any $\xi\in\mathrm{supp}(\varphi_j)$. Hence, it follows that

$$\sum_{j=0}^{\infty}\psi_j(\xi)\varphi_j(\xi)=1\,. \tag{4.1}$$

From this identity we see that

$$Sf(x)=\sum_{j=0}^{\infty}S_j f_j(x)\,,$$

where

$$S_j g(x) = c_n \int e^{ix\xi + it(x)|\xi|^a} \psi_j(\xi)\hat{g}(\xi)d\xi\,, \tag{4.2}$$

$$f_j(x) = c_n \int e^{ix\xi} \varphi_j(\xi)\hat{f}(\xi)d\xi\,. \tag{4.3}$$

To estimate $\|S_j\|_{\mathcal{L}(L_2,L_2)}$ we need the following

LEMMA 2. *Let $\psi \in C^\infty$ with support contained in the set $\{\xi; 1/4 < |\xi| < 4\}$, $t(x)$ a measurable function of $x \in \boldsymbol{R}^n$ with $0 \le t(x) \le 1$, j a positive integer, and let $a > 1$. Define the operator S_j by* (4.2). *Then,*

$$\|S_j\|_{\mathcal{L}(L_2(\boldsymbol{R}^n),L_2(\boldsymbol{R}^n))} \le C2^{jan/4}\,, \tag{4.4}$$

where C is a constant independent of j and $t(x)$.

From this lemma we can immediately prove Theorem 3, that is,

$$\|Sf\|_{L_2} \le \sum_{j=0}^{\infty} \|S_j f_j\|_{L_2} \le C\sum_{j=0}^{\infty} 2^{an/4}\|f_j\|_{L_2} \le C'\|f\|_{B^{an/4}_{2,1}(\boldsymbol{R}^n)}\,.$$

5. Proof of Lemma 2. In order to prove Lemma 2 we need several lemmas. We start with recalling the formulas for products and adjoints of Fourier multipliers.

LEMMA 3. *Let X, Y and Z be Hilbert spaces, and let T and S be the operators defined by*

$$Tf(x) = c_n \iint e^{i(x-y)\xi}\hat{K}(\xi)f(y)d\xi dy\,, \quad Sg(x) = c_n \iint e^{i(x-y)\xi}\hat{H}(\xi)g(y)d\xi dy\,,$$

where $\hat{K}(\xi)$ is $\mathcal{L}(X,Y)$-valued functions of $\xi \in \boldsymbol{R}^n$ and $\hat{H}(\xi)$ is $\mathcal{L}(Y,Z)$-valued functions of $\xi \in \boldsymbol{R}^n$ with

$$\sup_\xi \|\hat{K}(\xi)\|_{\mathcal{L}(X,Y)} < \infty\,, \quad \sup_\xi \|\hat{H}(\xi)\|_{\mathcal{L}(Y,Z)} < \infty\,.$$

Then, T^* is the bounded operator from $L_2(\boldsymbol{R}^n; Y)$ to $L_2(\boldsymbol{R}^n; X)$ defined by the formula

$$T^*g(x) = c_n \iint e^{i(x-y)\xi}\hat{K}(\xi)^* g(y)d\xi dy\,,$$

and ST is the bounded operator from $L_2(\boldsymbol{R}^n; X)$ to $L_2(\boldsymbol{R}^n; Z)$ defined by the formula

$$STf(x) = c_n \iint e^{i(x-y)\xi}\hat{H}(\xi)\hat{K}(\xi)f(y)d\xi dy\,.$$

PROOF. Let $f \in \mathcal{S}(\boldsymbol{R}^n; X)$ and $g \in \mathcal{S}(\boldsymbol{R}^n; Y)$. Then, we have

$$
\begin{aligned}
(g, Tf)_{L_2(\boldsymbol{R}^n;Y)} &= c_n \iint (g(x), e^{ix\xi}\hat{K}(\xi)\hat{f}(\xi))_Y dx d\xi \\
&= c_n \int \left(\int e^{-ix\xi} g(x)dx, \hat{K}(\xi)\hat{f}(\xi) \right)_Y d\xi \\
&= c_n \int (\hat{K}(\xi)^* \hat{g}(\xi), \hat{f}(\xi))_X d\xi \\
&= c_n \int \left(\hat{K}(\xi)^* \hat{g}(\xi), \int e^{-ix\xi} f(x)dx \right)_X d\xi \\
&= c_n \int \left(\int e^{ix\xi} \hat{K}(\xi)^* \hat{g}(\xi) d\xi, f(x) \right)_X dx .
\end{aligned}
$$

Therefore, we have

$$T^* g(x) = c_n \int e^{ix\xi} \hat{K}(\xi)^* \hat{g}(\xi) d\xi = c_n \iint e^{i(x-y)\xi} \hat{K}(\xi)^* g(y) d\xi dy .$$

Next, since $\widehat{Tf}(\xi) = \hat{K}(\xi)\hat{f}(\xi)$, it follows that

$$STf(x) = c_n \int e^{ix\xi} \hat{H}(\xi)\widehat{Tf}(\xi)d\xi = c_n \iint e^{i(x-y)\xi} \hat{H}(\xi)\hat{K}(\xi)f(y)d\xi dy .$$

Secondly, we prove the following

LEMMA 4. *Let $\varphi \in C_0^\infty(\boldsymbol{R}^n)$, and ψ a real-valued C^∞-function in a neighborhood of the support of φ. Assume that*

$$|\nabla\psi| \geq C , \quad |\partial^\alpha \psi| \leq C_\alpha |\nabla\psi|$$

hold for any $x \in \operatorname{supp} \varphi$ and any multi-index α. Then

$$\left| \int e^{i\psi(x)} \varphi(x) dx \right| \leq c_{2+2m}(n) C^{-2m} \|\varphi\|_{W_1^{2m}} \tag{5.1}$$

holds for any positive integer m. Here $c_m(n)$ is a constant depend only on n, m and $\{C_\alpha\}_{|\alpha|\leq m}$.

PROOF. If $0 < C < 1$, we have

$$\left| \int e^{i\psi(x)} \varphi(x) dx \right| \leq \|\varphi\|_{L_1} \leq C^{-2m} \|\varphi\|_{L_1} .$$

Therefore, we may assume that $C \geq 1$. Since

$$e^{i\psi(x)} = -|\nabla\psi|^{-2}(\Delta e^{i\psi(x)}) + i(\Delta\psi)|\nabla\psi|^{-2} e^{i\psi(x)} ,$$

we have

$$\int e^{i\psi(x)}\varphi(x)dx = -\int \{\Delta e^{i\psi(x)}\}\varphi(x)|\nabla\psi|^{-2}dx + i\int e^{i\psi(x)}\varphi(x)\Delta\psi(x)|\nabla\psi|^{-2}dx .$$

By integrating by parts we have

$$\int \{\Delta e^{i\psi(x)}\}\varphi(x)|\nabla\psi|^{-2}dx$$
$$= \int e^{i\psi(x)}[\{\Delta\varphi(x)\}|\nabla\psi|^{-2} + \sum_{j=1}^{n} \partial_j\varphi(x)\partial_j(|\nabla\psi|^{-2}) + \varphi(x)\Delta(|\nabla\psi|^{-2})]dx\,,$$

which gives

$$\left|\int \{\Delta e^{i\psi(x)}\}\varphi(x)|\nabla\psi|^{-2}dx\right| \le C_3C^{-2}\|\varphi\|_{W_1^2}\,.$$

By making use of the formula

$$\int e^{i\psi(x)}\varphi(x)\Delta\psi(x)|\nabla\psi|^{-2}dx$$
$$= -\int \{\Delta e^{i\psi(x)}\}\varphi(x)\Delta\psi(x)|\nabla\psi|^{-4}dx + i\int e^{i\psi(x)}\varphi(x)\{\Delta\psi(x)|\nabla\psi|^{-2}\}^2dx\,,$$

we also have the estimate

$$\left|\int e^{i\psi(x)}\varphi(x)\Delta\psi(x)|\nabla\psi|^{-2}dx\right| \le C_4'C^{-3}\|\varphi\|_{W_1^2} + C_2C^{-2}\|\varphi\|_{L_1} \le C_4C^{-2}\|\varphi\|_{W_1^2}\,.$$

Thus we have proved the inequality (5.1) for the case $m = 1$.

From the formula

$$\int e^{i\psi(x)}\varphi(x)dx = \int e^{i\psi(x)}\varphi_1(x)dx\,,$$

where

$$\varphi_1(x) = -\{\Delta\varphi(x)\}|\nabla\psi|^{-2} - \sum_{j=1}^{n} \partial_j\varphi(x)\partial_j(|\nabla\psi|^{-2}) - \varphi(x)\Delta(|\nabla\psi|^{-2})$$
$$- i\Delta\{\varphi(x)\Delta\psi(x)|\nabla\psi|^{-2}\}|\nabla\psi|^{-2} - i\sum_{j=1}^{n} \partial_j\{\varphi(x)\Delta\psi(x)|\nabla\psi|^{-2}\}\partial_j(|\nabla\psi|^{-2})$$
$$- i\varphi(x)\Delta\psi(x)|\nabla\psi|^{-2}(\Delta(|\nabla\psi|^{-2}) - \varphi(x)\{\Delta\psi(x)|\nabla\psi|^{-2}\}^2\,,$$

and the inequality (5.1) for the case $m = 1$ it follows that

$$\left|\int e^{i\psi(x)}\varphi(x)dx\right| \le C_4C^{-2}\|\varphi_1\|_{W_1^2}\,,$$

since $\|\varphi_1\|_{W_1^2} \le C_6'C^{-2}\|\varphi\|_{W_1^4}$. Hence we have the inequality (5.1) for the case $m = 2$. Repeating this argument we get the inequality for arbitrary m.

Next, we prove the following

LEMMA 5. *Let* $\psi \in C^\infty(\boldsymbol{R}^n)$ *with support contained in the set* $\{\xi; 1/4 < |\xi| < 4\}$, $|t| \le 1$, *and let* $N \ge 1$, $a > 1$, $1/N \le |x| \le 2a(4N)^{a-1}$. *Then*

$$\left|\int e^{ix\xi+it|\xi|^a}\psi\left(\frac{\xi}{N}\right)d\xi\right| \le C(n, a, \psi)N^{n/2}|x|^{-n/2}\,.$$

PROOF. Assume that $1 \leq N|x| \leq 2a|t|4^{a-1}N^a$. Apply Lemma 1 in Sjölin [10] to the integral

$$K_N(t,x) = \int e^{ix\xi + it|\xi|^a} \psi\left(\frac{\xi}{N}\right) d\xi = N^n \int e^{iNx\xi + itN^a|\xi|^a} \psi(\xi) d\xi\,.$$

Then, we have

$$\begin{aligned} |K_N(t,x)| &\leq C(n,a,\psi)(|t|N^a)^{-n/2}N^n \\ &\leq C(n,a,\psi)\left(\frac{N|x|}{2a4^{a-1}}\right)^{-n/2} N^n = C'(n,a,\psi)N^{n/2}|x|^{-n/2}\,. \end{aligned}$$

When $|x| > 2a|t|(4N)^{a-1}$,

$$|Nx + atN^a|\xi|^{a-2}\xi| \geq N|x| - a|t|N^a|\xi|^{a-1} \geq N|x|/2 \geq a|t|4^{a-1}N^a\,,$$

$$|\partial_\xi^\alpha(Nx\xi + tN^a|\xi|^a)| \leq C_\alpha|\{Nx + atN^a|\xi|^{a-2}\xi\}| \quad \text{for any } \alpha$$

holds for any $1/4 < |\xi| < 4$, so that by Lemma 4 we get

$$|K_N(t,x)| \leq C'(n,a,\psi)(N|x|)^{-2m}N^n\,.$$

Here, m is the least integer such that $2m > n$. Combining this with the simple inequality $|K_N(t,x)| \leq C(\psi)N^n$, we have

$$\begin{aligned} |K_N(t,x)| &\leq [C(n,a,\psi)N^{n-2m}|x|^{-2m}]^{n/4m}[C(\psi)N^n]^{1-n/4m} \\ &= C(n,a,\psi)N^{n/2}|x|^{-n/2}\,. \end{aligned}$$

Now, let us prove Lemma 2. It is easy to see that

$$S_j^* g(x) = c_n \iint e^{i(x-y)\xi - it(y)|\xi|^a} \psi_j(\xi) g(y) d\xi dy\,,$$

$$\begin{aligned} S_j S_j^* g(x) &= c_n \iint e^{i(x-y)\xi + i\{t(x)-t(y)\}|\xi|^a} \psi_j(\xi)^2 g(y) d\xi dy \\ &= \int K_j(x,y) g(y) dy\,, \end{aligned}$$

where

$$K_j(x,y) = c_n \int e^{i(x-y)\xi + i\{t(x)-t(y)\}|\xi|^a} \psi_j(\xi)^2 d\xi\,.$$

The norm of the integral operator $S_j S_j^*$ is obtained from the inequalities:

$$\int |K_j(x,y)|dx \leq C(n,a,\psi)2^{jan/2}\,, \quad \int |K_j(x,y)|dy \leq C(n,a,\psi)2^{jan/2}\,, \tag{5.2}$$

which can be proved as follows: It is clear that

$$|K_j(x,y)| \leq C_n \int |\psi(2^{-j}\xi)^2| d\xi = C_n 2^{jn} \|\psi\|_{L_2}^2$$

holds for any x and y. Also, we have as in Proof of Lemma 5 that

$$|K_j(x,y)| \leq C(n,a,\psi)2^{j(n-2m)}|x-y|^{-2m}$$

holds for any $|x-y| \geq 2a2^{(2+j)(a-1)}$, where m is an integer such that $2m > n$. Hence, by Lemma 5 we obtain that

$$\begin{aligned}
&\int |K_j(x,y)|dy \\
&\quad \leq C(n,a,\psi)\left\{\int_{|y-x|\leq 2^{-j}} 2^{jn}dy + \int_{2^{-j}\leq |y-x|\leq 2a2^{(2+j)(a-1)}} 2^{jn/2}|y-x|^{-n/2}dy \right. \\
&\qquad \left. + \int_{|y-x|\geq 2a2^{(2+j)(a-1)}} 2^{j(n-2m)}|y-x|^{-2m}dy\right\} \\
&\quad \leq C'(n,a,\psi)2^{jan/2}
\end{aligned}$$

holds for any $x \in \boldsymbol{R}^n$. The second inequality in (5.2) can be proved in the same way. Now, from (5.2) we have

$$\|S_j S_j^*\|_{\mathcal{L}(L_2,L_2)} \leq C^2 2^{jan/2},$$

where C is a constant independent of j and $t(x)$, which gives (4.4), because $\|A\| = \|AA^*\|^{1/2}$ holds for any bounded linear operator A between Hilbert spaces.

6. Proof of Theorem 4. To prove Theorem 4 we start with

LEMMA 6. *Let $\psi \in C^\infty$ with support contained in the set $\{\xi;\ 1/4 < |\xi| < 4\}$, and let j be a positive integer, $I = (0,1)$, $1 \leq p \leq \infty$, and $a > 1$. Define the operator P_j by*

$$P_j : g \to c_n \int e^{ix\xi + it|\xi|^a}\psi(2^{-j}\xi)\hat{g}(\xi)d\xi .$$

Then,

$$\|P_j\|_{\mathcal{L}(L_p(\boldsymbol{R}^n), L_p(\boldsymbol{R}^n; L_\infty(I)))} \leq C2^{jan/2},$$

where C is a constant independent of j.

PROOF. We consider P_j as an integral operator with $\mathcal{L}(\boldsymbol{C}, L_\infty(I))$-valued kernel, i.e.,

$$P_j f(x) = \int \vec{K}_j(x-y)f(y)dy,$$

where

$$\vec{K}_j(x) = K_j(t,x) = c_n \int e^{ix\xi+it|\xi|^a}\psi(2^{-j}\xi)d\xi .$$

Hence, the conclusion follows from the estimate

$$\int \|\vec{K}_j(x)\|_{\mathcal{L}(\boldsymbol{C}, L_\infty(I))}dx = \int \operatorname*{ess.\,sup}_{|t|\leq 1} |K_j(t,x)|dx \leq C(n,a,\psi)2^{jan/2}.$$

It is clear that

$$|K_j(t,x)| \leq \int |\psi(2^{-j}\xi)|d\xi = 2^{jn}\|\psi\|_{L_1}$$

holds for any x. Also, we have as in Proof of Lemma 5 that

$$|K_j(t,x)| \leq C(n,a,\psi)2^{j(n-2m)}|x|^{-2m}$$

holds for any $|x| \geq 2a2^{(2+j)(a-1)}$, where m is an integer such that $2m > n$. Hence, by Lemma 5 we obtain that

$$\begin{aligned}\int \operatorname{ess.}\sup_{|t|\leq 1} |K_j(t,x)|dx &\leq C(n,a,\psi)\left\{\int_{|x|\leq 2^{-j}} 2^{jn}dx + \int_{2^{-j}\leq|x|\leq 2a2^{(2+j)(a-1)}} 2^{jn/2}|x|^{-n/2}dx\right.\\ &\qquad \left. + \int_{|x|\geq 2a2^{(2+j)(a-1)}} 2^{j(n-2m)}|x|^{-2m}dx\right\}\\ &\leq C'(n,a,\psi)2^{jan/2}.\end{aligned}$$

PROOF OF THEOREM 4. The results for the case $p = 2$ are given in Theorem 2 and Theorem 3. Next, consider the case where $p = 1$. It follows from the identity (4.1) that

$$T_2 f = \sum_{j=0}^{\infty} P_j f_j ,$$

where

$$P_j : g \to v(t,x) = c_n \int e^{ix\xi + it|\xi|^a}\psi_j(\xi)\hat{g}(\xi)d\xi ,$$

$$f_j(x) = c_n \int e^{ix\xi}\varphi_j(\xi)\hat{f}(\xi)d\xi .$$

By this formula we see that

$$\|T_2 f\|_{L_1(\boldsymbol{R}^n;L_\infty(I))} \leq \sum_{j=0}^{\infty} \|P_j f_j\|_{L_1(\boldsymbol{R}^n;L_\infty(I))} ,$$

which gives with the aid of Lemma 6 that

$$\|T_2 f\|_{L_1(\boldsymbol{R}^n;L_\infty(I))} \leq C\sum_{j=0}^{\infty} 2^{jan/2}\|f_j\|_{L_1(\boldsymbol{R}^n)} \leq C'\|f\|_{B^{an/2}_{1,1}(\boldsymbol{R}^n)} .$$

In the same way we have

$$\begin{aligned}\|T_2 f\|_{L_\infty(\boldsymbol{R}^n;L_\infty(I))} &\leq \sum_{j=0}^{\infty} \|P_j f_j\|_{L_\infty(\boldsymbol{R}^n;L_\infty(I))}\\ &\leq C\sum_{j=0}^{\infty} 2^{jan/2}\|f_j\|_{L_\infty(\boldsymbol{R}^n)}\\ &\leq C'\|f\|_{B^{an/2}_{\infty,1}(\boldsymbol{R}^n)} .\end{aligned}$$

For the case $1 < p < 2$ (the case $2 < p < \infty$) the result follows from that for the cases $p = 1, 2$ (the cases $p = 2, \infty$) and the complex interpolation:

$$[B^{\sigma_0}_{p_0,q_0}, B^{\sigma_1}_{p_1,q_1}]_\theta = B^{\sigma}_{p,q}$$

with

$$\sigma = (1-\theta)\sigma_0 + \theta\sigma_1\,, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\,, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}\,.$$

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