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# HARDY SPACES AND MAXIMAL OPERATORS ON REAL RANK ONE SEMISIMPLE LIE GROUPS I

Dedicated to Professor Satoru Igari on his sixtieth birthday

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**Abstract.** Let $G$ be a real rank one connected semisimple Lie group with finite center. As well-known the radial, heat, and Poisson maximal operators satisfy the $L^p$-norm inequalities for any $p > 1$ and a weak type $L^1$ estimate. The aim of this paper is to find a subspace of $L^1(G)$ from which they are bounded into $L^1(G)$. As an analogue of the atomic Hardy space on the real line, we introduce an atomic Hardy space on $G$ and prove that these maximal operators with suitable modifications are bounded from the atomic Hardy space on $G$ to $L^1(G)$.

**1. Introduction.** The study of Hardy spaces $H^p$ originated in the 1910's in the setting of Fourier series and was developed by the so-called complex variable methods. In the 1970's these spaces were completely characterized by various maximal operators without using complex variables and the study was advanced by the so-called real variable methods. Atomic characterization of $H^p$ was also given at the same time. Since the real variable methods have no need for the complex structure, the Hardy space theory cound be generalized to one on locally compact groups $G$ such as compact Lie groups and the Heisenberg groups. Nowadays, this fruitful $H^p$ theory has been extended to the spaces $X$ of homogeneous type in the sense that they satisfy the so-called doubling condition: There exists $c > 0$ such that for each $x \in X$ and $t > 0$

$$|B(2t, x)| \le c|B(t, x)|,$$

where $B(t, x)$ is the ball with redius $t$ centered at $x$ and $|B(x, r)|$ is the volume of the ball. Roughly speaking, on $X$ Hardy spaces $H^p(X)$ are characterized by the radial maximal operator, and the heat and Poisson maximal operators are bounded from $H^p(X)$ to $L^p(X)$ for any $1 \le p < \infty$ (cf. [3]). However, when the space is not of homogeneous type, little work has been done. In this paper, looking at the example of semisimple Lie groups $G$, we shall consider the Hardy space theory on $G$ of nonhomogeneous type. Actually, on $G$, $|B(t, x)|$ has exponential growth order (cf. Lemma 2.1 below), hence $G$ is not of homogeneous type. Our goal is to introduce an atomic Hardy space $H^1_{p,0}(G)$ on $G$ and show that the modified radial, heat, and Poisson maximal operators are strongly bounded from $H^1_{p,0}(G)$ to $L^1(G)$ under suitable conditions.

This paper is organized as follows. We suppose that $G$ is a real rank one connected semisimple Lie group with finite center and $G = K \exp \mathfrak{p}$ the Cartan decomposition of $G$. For

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each $g = k \exp X$ ($k \in K$, $X \in \mathfrak{p}$) let $\sigma(g)$ denote the norm of $X$ with respect to the Euclidean structure of $\mathfrak{p}$ induced from the Killing form. Let $dg$ be a Haar measure on $G$, $dk$ the one on $K$ with total mass 1, and $ds$ the Lebesgue measure on the Lie algebra $\mathfrak{a}$ of $A$. Then $dg$ is decomposed as $dg = \Delta(s)dkdsdk'$ relative to the Cartan decomposition $G = KCL(A^+)K$ of $G$. We identify $A = \exp \mathfrak{a}$ with $\boldsymbol{R}$. Let $B(t)$ denote the ball with redius $t > 0$ centered at the origin: $B(t) = \{x \in G; \sigma(x) \leq t\}$ and $|B(t)| = \int_0^r \Delta(s)ds$ the volume of the ball.

The Hardy-Littlewood maximal operator $M_{\mathrm{HL}}$ on $G$ is defined as follows.

$$
\begin{aligned}
M_{\mathrm{HL}} f(x) &= \sup_{t>0} |B(t)|^{-1} |f| * \chi_t(x) \\
&= \sup_{t>0} |B(t)|^{-1} \int_G f(xg^{-1}) \chi_t(g) dg \qquad (1) \\
&= \sup_{t>0} |B(t)|^{-1} \int_{B(t)} |f(xg^{-1})| dg,
\end{aligned}
$$

where $\chi_t$ is the characteristic function of $B(t)$. As well-known, $M_{\mathrm{HL}}$ satisfies the maximal theorem: For any $1 < p \leq \infty$, $M_{\mathrm{HL}}$ is of type $(p, p)$, and is of weak type $(1, 1)$, that is, it maps $L^p(G)$ into itself and $L^1(G)$ into weak $L^1$ functions on $G$ (see [7]). In §3 we shall obtain a pointwise estimate: If $a$ is a function on $G$ supported on $B(r)$ with $\|a\|_\infty \leq |B(r)|^{-1}$, then

$$
M_{\mathrm{HL}} a(x) \leq |B(\sigma(x))|^{-1} . \qquad (2)
$$

We fix a smooth and compactly supported $K$-bi-invariant function $\phi$ on $G$ and, identifying it with an even function on $\boldsymbol{R}$, we define the dilation $\phi_t$ ($t > 0$) of $\phi$ by

$$
\phi_t(s) = \frac{1}{t} \Delta(s)^{-1} \Delta\left(\frac{s}{t}\right) \phi\left(\frac{s}{t}\right) \quad (s \in \boldsymbol{R}),
$$

where $\Delta$ is the density of the Haar measure $dg$ related to the Cartan decomposition of $G$. Then, a radial maximal operator $M_\phi$ is defined as

$$
M_\phi f(x) = \sup_{t>0} |f * \phi_t(x)| .
$$

As shown by [4, Theorem 3.4], $M_\phi f(x)$ is dominated by $c(M_{\mathrm{HL}} f(x) + |f| * E(x))$, where $E(x) = e^{-2\rho\sigma(x)}$ and hence, the radial maximal operator $M_\phi$ is also of type $(p, p)$ for any $1 < p \leq \infty$, and is of weak type $(1, 1)$.

We now introduce an atom on $G$. Let $1 < p \leq \infty$. We say that a function $a$ on $G$ is a $(1, p, 0)$-atom provided that

(i) $a$ is supported on $B(r)$ for some $r > 0$,

(ii) if $r \leq 1$, then $\|a\|_p \leq |B(r)|^{1/p-1}$ and $\int_G a(g)dg = 0$,

(iii) if $r > 1$, then $\|a\|_p \leq |B(r)|^{-1}$.

In the Euclidean case the moment condition $\int_{\boldsymbol{R}} a(g)dg = 0$ of an atom $a$ on $\boldsymbol{R}$ essentially yields the integrability of a radial maximal function of the atom (cf. [3, Theorem 2.9]). However, in our case $M_\phi a$ is not integrable on $G$, because the density $\Delta(x)$ cancels the order of decay obtained in (2) (see Remark 4.7). In §4, by using (2) we shall obtain some pointwise estimates of $a * \phi_t(x)$ and thereby deduce the following weak equi-integrability of $M_\phi a$: For each $\varepsilon > 0$

$$\int_G M_\phi a(g)(1+\sigma(g))^{-\varepsilon} dg \le c\,, \tag{3}$$

where $c$ is independent of the $(1, p, 0)$-atom $a$ on $G$. As an easy consequence, a refinement of [4, Proposition 4.1] follows: If we define a modified radial maximal operator $M_\phi^\varepsilon$ on $G$ by

$$M_\phi^\varepsilon f(x) = \sup_{t>0}(1+t)^{-\varepsilon}|f * \phi_t(x)|\,,$$

then for each $\varepsilon > 0$ we have

$$\int_G M_\phi^\varepsilon a(g) dg \le c\,,$$

where $c$ is independent of the $(1, p, 0)$-atom $a$ on $G$. In §5 we shall consider a left translation of each $(1, p, 0)$-atom $a$ on $G$: $a_x(g) = a(xg)$, $(x, g \in G)$. Then we shall introduce an atomic Hardy space $H^1_{p,0}(G)$ as a collection of these translations. The above estimate implies that $M_\phi^\varepsilon$ is bounded from $H^1_{p,0}(G)$ to $L^1(G)$ (see Theorem 5.3).

We shall treat the same problem for the (modified) heat and Poisson maximal operators $M_{\mathrm{H}}^\varepsilon$ and $M_{\mathrm{P}}^\varepsilon$ on $G$, which are defined respectively by

$$M_{\mathrm{H}}^\varepsilon f(x) = \sup_{t>0}(1+t)^{-\varepsilon}|f * h_t(x)| \quad \text{and} \quad M_{\mathrm{P}}^\varepsilon f(x) = \sup_{t>0}(1+t)^{-\varepsilon}|f * p_t(x)|\,,$$

for each $\varepsilon \ge 0$, where $h_t$ and $p_t$ are the heat and Poisson kernels on $G/K$, respectively. We denote $M_{\mathrm{H}}^0$ (resp. $M_{\mathrm{P}}^0$) by $M_{\mathrm{H}}$ (resp. $M_{\mathrm{P}}$) for simplicity. As shown by [6, Chap. III] and [1, Corollary 3.2], $M_{\mathrm{H}}$ and $M_{\mathrm{P}}$ also satisfy the maximal theorem. In §6, applying the sophisticated estimates for $h_t$ and $p_t$ obtained in [1], we shall prove that the inequality (3) for $M_{\mathrm{H}}$ (resp. $M_{\mathrm{P}}$) holds for $\varepsilon > 1/2$ (resp. $\varepsilon > 0$). This implies that $M_{\mathrm{H}}^\varepsilon$ and $M_{\mathrm{P}}^\varepsilon$ are bounded from $H^1_{p,0}(G)$ to $L^1(G)$ provided $\varepsilon > 1/2$ and $\varepsilon > 0$, respectively (see Theorem 6.1 and Theorem 6.4).

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**2. Notation and preliminaries.** Let $G = KAN$ be a connected semisimple Lie group with finite center and suppose that $\dim A = 1$. Let $\mathfrak{a}$ be the Lie algebra of $A$ and $\mathfrak{a}^*$ the dual space of $\mathfrak{a}$. Let $\gamma$ be the positive simple root of $(G, A)$, and $m_1, m_2$ the multiplicities of $\gamma$ and $2\gamma$, respectively. We put $2\rho = m_1 + 2m_2$ and $2\alpha = m_1 + m_2 - 1$. Let $H$ be the element in $\mathfrak{a}$ such that $\gamma(H) = 1$. In the following we identify $A$, $\mathfrak{a}$, and $\mathfrak{a}^*$ with $\boldsymbol{R}$ as $s \mapsto a_s = \exp(sH)$, $sH$ and $s\gamma$, respectively. According to the Cartan decomposition $G = KCL(A^+)K$, $A^+ = \{a_s; s > 0\}$, we define $\sigma : G \to \boldsymbol{R}^+$ by $g \in Ka_{\sigma(g)}K$. Then $\sigma$ is $K$-bi-invariant and

$$|\sigma(x) - \sigma(y)| \le \sigma(xy) \le \sigma(x) + \sigma(y) \tag{4}$$

for $x, y \in G$ (cf. [8, 8.1.2]). Let $dg$ be a Haar measure on $G$ normalized as

$$\int_G f(g)dg = \int_K \int_0^\infty \int_K f(ka_sk')\Delta(s)dkdsdk'\,, \tag{5}$$

where $dk$ is the Haar measure on $K$ such that $\int_K dk = 1$, $ds$ is the Lebesgue measure on $\boldsymbol{R}$, and $\Delta(s) = (\sinh s)^{m_1}(\sinh 2s)^{m_2}$ (cf. [2, (2.4)]). We use the notation $L^p(G)$ to stand for the space $L^p(G, dg)$ and we denote the norm by $\|\cdot\|_p$. Let $C_c^\infty(G)$ be the space of all $C^\infty$ functions with compact support on $G$. We denote by $L^p(G//K)$ and $C_c^\infty(G//K)$ respectively the subspaces of $L^p(G)$ and $C_c^\infty(G)$ consisting of all $K$-bi-invariant functions on $G$. Then we identify each $K$-bi-invariant function $f$ on $G$ with an even function on $\boldsymbol{R}$, which we denote also by the same letter:

$$f(g) = f(a_{\sigma(g)}) = f(\sigma(g)) \quad (g \in G).$$

We recall some basic facts on the spherical Fourier analysis on $G$. For a survey of this subject we refer to [2] and [8, Chap. 9]. Let $\Omega$ be the Laplace-Beltrami operator on $G/K$ and let $\phi_\lambda(x)$ $(x \in G, \lambda \in \mathfrak{a}^*)$ be the spherical function on $G$ such that $\phi_\lambda(e) = 1$ and $\Omega\phi_\lambda(x) = -(\lambda^2 + \rho^2)\phi_\lambda(x)$. For each $f \in L^1(G//K)$, the spherical Fourier transform $\hat{f}(\lambda)$ is defined by

$$\hat{f}(\lambda) = \int_G f(g)\bar{\phi}_\lambda(g)dg \quad (\lambda \in \mathfrak{a}^*).$$

Then $\hat{f}(w\lambda) = \hat{f}(\lambda)$ for $w \in W$, the Weyl group of $(G, A)$. Since $\dim A = 1$, this means that $\hat{f}(\lambda)$ is an even function on $\boldsymbol{R}$. When $f$ belongs to $C_c^\infty(G//K)$, the spherical Fourier transform $f \mapsto \hat{f}$ has an inversion formula of the form

$$f(g) = c\int_{\mathfrak{a}^*} \hat{f}(\lambda)\phi_\lambda(g)|C(\lambda)|^{-2}d\lambda \quad (g \in G), \tag{6}$$

where $c$ is a constant and $C(\lambda)$ is Harish-Chandra's $C$-function. This transform has an $L^2$-extension, that is, it gives an isometric isomorphism between $L^2(G//K)$ and $L^2_W(\boldsymbol{R}, |C(\lambda)|^{-2}d\lambda)$, the space of even functions in $L^2(\boldsymbol{R}, |C(\lambda)|^{-2}d\lambda)$.

In the following, we follow the custom of using the letter "$c$" to denote a constant which might be different at each occurrence.

Let $B(s)$ denote the open ball with radius $s > 0$ centered at the origin and $|B(s)|$ the volume $\int_0^s \Delta(u)du$ of the ball.

LEMMA 2.1. *The density $\Delta(s)$ and the volume $|B(s)|$ have the following properties.*

(i) $\Delta(s) \sim e^{2\rho s}$ $(s \geq 1)$,

(ii) $\Delta(s) \sim s^{2\alpha+1}$ $(s \leq 1)$,

(iii) $|B(s)| \sim e^{2\rho s}$ $(s \geq 1)$,

(iv) $|B(s)| \sim s^{2(\alpha+1)}$ $(s \leq 1)$,

(v) $|B(s)|' = \Delta(s)$,

*where the symbol "$\sim$" means that the ratio of the left hand side to the right hand side is bounded below and above by a positive constant, and the prime in* (v) *means the derivative with respect to $s$.*

LEMMA 2.2. *Suppose that $x, y \in \boldsymbol{R}^+$ and $x - y \geq 1$. Then for each $q \geq 1$*

$$\int_K \Delta(\sigma(a_x k a_y^{-1}))^{-q}dk \leq ce^{-2\rho q x}e^{2\rho(q-1)y}.$$

PROOF. If $x-y \geq 1$ and $y \leq 1$, then (4) and Lemma 2.1(i) imply that $\Delta(\sigma(a_x k a_y^{-1}))^{-q} \leq e^{-2\rho q(x-y)} \leq c e^{-2\rho q x}$ for all $k \in K$, so we may assume that $x, y, x-y \geq 1$. We recall the kernel form of the product of spherical functions $\phi_\lambda$ (see [2, (4.2)]):

$$\phi_\lambda(g_1)\phi_\lambda(g_2) = \int_G K(g_1, g_2, g_3)\phi_\lambda(g_3)dg_3 \quad (g_1, g_2, g_3 \in G).$$

Applying (6), we see that for $f \in C_c^\infty(G//K)$,

$$\begin{aligned}
\int_K f(g_1 k g_2)dk &= c\int_{\mathfrak{a}^*} \hat{f}(\lambda)\left(\int_K \phi_\lambda(g_1 k g_2)dk\right)|C(\lambda)|^{-2}d\lambda \\
&= c\int_{\mathfrak{a}^*} \hat{f}(\lambda)\phi_\lambda(g_1)\phi_\lambda(g_2)|C(\lambda)|^{-2}d\lambda \\
&= \int_G K(g_1, g_2, g_3)\left(c\int_{\mathfrak{a}^*} \hat{f}(\lambda)\phi_\lambda(g_3)|C(\lambda)|^{-2}d\lambda\right)dg_3 \\
&= \int_G K(g_1, g_2, g_3) f(g_3)dg_3 .
\end{aligned}$$

Here $K(g_1, g_2, g_3) = 0$ if $\sigma(g_3)$ satisfies $\sigma(g_3) \leq |\sigma(g_1)-\sigma(g_2)|$ or $\sigma(g_3) \geq \sigma(g_1)+\sigma(g_2)$ (see [2, (4.17)]). Therefore, approximating $\Delta(\sigma(g))^{-q}$, $q \geq 1$, by functions in $C_c^\infty(G//K)$, we may replace $f$ in the above equations with $\Delta(\sigma(g))^{-q}$. Then it follows from (5) and Lemma 2.1(i) that

$$\begin{aligned}
\int_K \Delta(\sigma(a_x k a_y^{-1}))^{-q}dk &= \int_G K(a_x, a_y, g)\Delta(\sigma(g))^{-q}dg \\
&\leq \int_{x-y}^{x+y} K(a_x, a_y, a_z)e^{2\rho(1-q)z}dz .
\end{aligned}$$

Since $K(a_x, a_y, a_z) = O(e^{-\rho(x+y+z)})$ provided $x, y, z \geq 1$ (see [2, (4.14)]), the desired result follows. $\square$

**3. The Hardy-Littlewood maximal operator.** We keep the notation in the previous sections. We shall treat the Hardy-Littlewood maximal operator $M_{\mathrm{HL}}$ on $G$ defined by (1) and prove the estimate (2).

PROPOSITION 3.1. *Suppose that a function $a$ on $G$ is supported on $B(r)$ and $\|a\|_\infty \leq |B(r)|^{-1}$. Then*

$$M_{\mathrm{HL}}a(x) \leq \min(|B(r)|^{-1}, |B(\sigma(x))|^{-1}) \quad (x \in G).$$

PROOF. Without loss of generality, we may assume that $a(x) = |B(r)|^{-1}\chi_r(x)$, where $\chi_r$ is the characteristic function of the ball $B(r)$. We shall show that the supremum over $t > 0$ of

$$F(t) = F_{x,r}(t) = |B(r)|^{-1}|B(t)|^{-1}\chi_r * \chi_t(x)$$

is dominated by $\min(|B(r)|^{-1}, |B(\sigma(x))|^{-1})$. Clearly, $F(t) \leq |B(t)|^{-1}$ and $|B(r)|^{-1}$, and hence we may assume that $\sigma(x) \geq r$. Since $F(t) = 0$ for $t < \sigma(x) - r$, to obtain the desired estimate it suffices to prove that $F(t)$ is increasing on the interval $\sigma(x) - r \le t \le \sigma(x)$. If we put

$$I(x, r, y) = \int_K \chi_r(xky^{-1})dk \quad (x, y \in G), \tag{7}$$

then we see that

$$F(t) = |B(r)|^{-1}|B(t)|^{-1} \int_{\sigma(x)-r}^{t} I(x, r, a_s)\Delta(s)ds .$$

Here we note that, as a function of $s$, $I(x, r, a_s)$ is increasing on $\sigma(x) - r \le s \le \sigma(x)$. Therefore, since $|B(t)|' = \Delta(t)$ and $\int_{\sigma(x)-r}^{t} \Delta(s)ds \le |B(t)|$, it follows that

$$\begin{aligned}
|B(r)|\,|B(t)|F'(t) &= I(x, r, a_t)\Delta(t) - |B(t)|'\,|B(t)|^{-1} \int_{\sigma(x)-r}^{t} I(x, r, a_s)\Delta(s)ds \\
&\ge I(x, r, a_t)\Delta(t) \left(1 - |B(t)|^{-1} \int_{\sigma(x)-r}^{t} \Delta(s)ds\right) \\
&\ge 0
\end{aligned}$$

□

COROLLARY 3.2. *Suppose that a function $a$ on $G$ is supported on $B(z, r)$, the ball with redius $r$ centered at $z$, and $\|a\|_\infty \le |B(r)|^{-1}$.*

(i) *For every $\lambda > 0$,*

$$|\{x \in G; M_{\mathrm{HL}}a(x) > \lambda\}| \le \lambda^{-1},$$

(ii) *For every $1 < p \le \infty$,*

$$\|M_{\mathrm{HL}}a\|_p \le \left(\frac{p}{p-1}\right)^{1/p} |B(r)|^{1/p-1} .$$

PROOF. Since $b(x) = a(zx)$ is supported on $B(r)$ and $\|b\|_\infty \le |B(r)|^{-1}$, it follows from Proposition 3.1 that

$$M_{\mathrm{HL}}a(x) = M_{\mathrm{HL}}b(z^{-1}x) \le \min(|B(r)|^{-1}, |B(\sigma(z^{-1}x))|^{-1}) .$$

Let $S(\lambda) = \{x \in G; M_{\mathrm{HL}}a(x) > \lambda\}$. Obviously, if $\lambda > |B(r)|^{-1}$, then $S(\lambda)$ is empty, and if $\lambda \le |B(r)|^{-1}$, then $S(\lambda) \subset B(z, r_\lambda) = \{x \in G; |B(\sigma(z^{-1}x))|^{-1} > \lambda\}$. Therefore, $|S(\lambda)| \le |B(z, r_\lambda)| = |B(r_\lambda)| = \lambda^{-1}$. Moreover, it follows that

$$\int_G |M_{\mathrm{HL}}a(x)|^p dx = p \int_0^{|B(r)|^{-1}} \lambda^{p-1} S(\lambda)d\lambda \le \frac{p}{p-1}|B(r)|^{1-p} .$$

□

**4. The radial maximal operator and atoms.** Let $\phi$ be a $K$-bi-invariant, differentiable function on $G$. We say that $\phi$ belongs to the class $\mathcal{A}_\delta$ ($\delta \ge 0$) if it satisfies, as an even function on $\boldsymbol{R}$,

$$\text{(8)} \qquad \begin{aligned} &\text{(i)} \quad C_{\phi,0} = \|\phi\Delta\|_1 \le 1\,, \\ &\text{(ii)} \quad C_{\phi,1} = \|(\phi\Delta)(s)|s|(1+|s|)^{\delta}\|_{\infty} \le 1\,, \\ &\text{(iii)} \quad C_{\phi,2} = \|(\phi\Delta)'(s)|s|^2(1+|s|)^{\delta}\|_{\infty} \le 1\,. \end{aligned}$$

For each $\phi \in \mathcal{A}_\delta$ we define the dilation $\phi_t$ ($t > 0$) of $\phi$ and the corresponding modified radial maximal operator $M_\phi^\varepsilon$ ($\varepsilon \ge 0$) on $G$ by

$$\phi_t(s) = \frac{1}{t}\Delta(s)^{-1}\Delta\left(\frac{s}{t}\right)\phi\left(\frac{s}{t}\right) \quad (s \in \boldsymbol{R}),$$

$$M_\phi^\varepsilon f(x) = \sup_{t>0}(1+t)^{-\varepsilon}|f * \phi_t(x)| \quad (x \in G).$$

Then, as explained in §1, the maximal operator satisfies

$$M_\phi^\varepsilon f(x) \le M_\phi f(x) \le c(M_{\mathrm{HL}} f(x) + |f| * E(x)),$$

and hence it satisfies the maximal theorem on $G$.

We first obtain some estimates for $\phi_t * a$ when $a$ is supported on a ball $B(r)$.

PROPOSITION 4.1. *Let $\phi \in \mathcal{A}_\delta$. Suppose that a function a on G is supported on $B(r)$, $\|a\|_1 \le 1$, and if $r > 1$, then $\|a\|_p \le |B(r)|^{-1}$ for some $p > 1$. Then*

$$|a * \phi_t(x)| \le \frac{c}{\sigma(x) - r}\left(1 + \frac{\sigma(x)-r}{t}\right)^{-\delta}\Delta(\sigma(x))^{-1} \quad (\sigma(x) \ge r_0),$$

*where $r_0 = 2r$ if $r \le 1$, and $r_0 = r+1$ if $r > 1$.*

PROOF. Let $r \le 1$ and $\sigma(x) \ge 2r$. Then it follows from (ii) of (8) that

$$\begin{aligned} |a * \phi_t(x)| &\le \frac{1}{t}C_{\phi,1}\int_{\sigma(x)-r}^{\sigma(x)+r}\frac{t}{s}\left(1+\frac{s}{t}\right)^{-\delta}\Delta(s)^{-1}\int_K\int_K |a(xka_s^{-1}k')|dkdk'\Delta(s)ds \\ &\le \frac{c}{\sigma(x)-r}\left(1+\frac{\sigma(x)-r}{t}\right)^{-\delta}\Delta(\sigma(x)-r)^{-1}\|a\|_1\,. \end{aligned}$$

If $\sigma(x) \ge 2$, then $\sigma(x) - r \ge 1$, and hence, $\Delta(\sigma(x)-r)^{-1} \sim e^{-2\rho(\sigma(x)-r)} \sim \Delta(\sigma(x))^{-1}$ by (i) of Lemma 2.1. Moreover, if $2r \le \sigma(x) < 2$, then $\sigma(x) - r \ge \sigma(x) - \sigma(x)/2 = \sigma(x)/2$, which implies that $\Delta(\sigma(x)-r)^{-1} \le \Delta(\sigma(x)/2)^{-1} \sim \Delta(\sigma(x))^{-1}$ by (ii) of Lemma 2.1. Therefore, the desired estimate follows.

Let $r > 1$ and $\sigma(x) \ge r+1$. We note that, if $s < r$, then by (4), $\sigma(a_s^{-1}kx) \ge \sigma(x) - s \ge \sigma(x) - r > 1$ for all $k \in K$. Therefore, using Hölder's inequality three times, we see from

(ii) of (8), Lemma 2.2, and (i) of Lemma 2.1 that $|a * \phi_t(x)|$ is dominated by

$$
\begin{aligned}
&\frac{1}{t}C_{\phi,1}\int_0^r\int_K\int_K |a(ka_sk')|dk'\frac{t}{\sigma(x)-r}\left(1+\frac{\sigma(x)-r}{t}\right)^{-\delta}\Delta(\sigma(a_s^{-1}k^{-1}x))^{-1}dk\Delta(s)ds\\
&\leq \frac{1}{t}C_{\phi,1}\frac{t}{\sigma(x)-r}\left(1+\frac{\sigma(x)-r}{t}\right)^{-\delta}\int_0^r\left(\int_K\left(\int_K|a(ka_sk')|dk'\right)^p dk\right)^{1/p}\\
&\quad\times\left(\int_K\Delta(\sigma(a_s^{-1}k^{-1}x))^{-q}dk\right)^{1/q}\Delta(s)ds \qquad \left(\frac{1}{p}+\frac{1}{q}=1\right)\\
&\leq \frac{c}{\sigma(x)-r}\left(1+\frac{\sigma(x)-r}{t}\right)^{-\delta}\left(\int_0^r\int_K\int_K|a(ka_sk')|^p dk'dk\Delta(s)ds\right)^{1/p}\\
&\quad\times e^{-2\rho\sigma(x)}\left(\int_0^r e^{2\rho(q-1)s}\Delta(s)ds\right)^{1/q}\\
&\leq \frac{c}{\sigma(x)-r}\left(1+\frac{\sigma(x)-r}{t}\right)^{-\delta}\|a\|_p e^{2\rho r}e^{-2\rho\sigma(x)}.
\end{aligned}
$$

Since $\|a\|_p \leq |B(r)|^{-1} \sim e^{-2\rho r}$ $(r > 1)$ and $e^{-2\rho\sigma(x)} \sim \Delta(\sigma(x))^{-1}$ $(\sigma(x) > 1)$, we are done. □

REMARK 4.2. In the proof of Proposition 4.1, when $r > 1$, we used Hölder's inequality to divide the integral over $K$ into the ones of $\int_K |a(ka_sk')|dk'$ and $\Delta(\sigma(a_s^{-1}k^{-1}x))^{-1}$. If $a$ is left $K$-invariant on $G$, then this process is not necessary and we can directly apply Lemma 2.2 with $q = 1$ to $\int_K \Delta(\sigma(a_s^{-1}k^{-1}x))^{-1}dk$. In this case, $\|a\|_p e^{2\rho r}$ in the last inequality can be replaced by $\|a\|_1 \leq 1$, and therefore the assumption $\|a\|_p \leq |B(r)|^{-1}$ is not necessary.

PROPOSITION 4.3. *Let $\phi \in \mathcal{A}_\delta$. Suppose that a function a on G is supported on $B(r)$, $\|a\|_1 \leq 1$, and $\int_G a(g)dg = 0$. Then*

$$|a * \phi_t(x)| \leq \frac{cr}{\sigma(x)-r}\left(1+\frac{\sigma(x)-r}{t}\right)^{-\delta} M_{\mathrm{HL}}a(x) \quad (x \in G).$$

PROOF. For simplicity we put $\Phi = \phi\Delta$ and

$$A(x,y) = \int_K\int_K a(xky^{-1}k')dkdk' \quad (x, y \in G).$$

Clearly, as a function of $s$, the support of $A(x, a_s)$ is contained in the interval $[\sigma(x)-r, \sigma(x)+r]$, and $\int_0^\infty |A(x,a_s)|\Delta(s)ds \leq \|a\|_1 \leq 1$. Moreover, it follows from the moment condition that

$$\int_0^\infty A(x,a_s)\Delta(s)ds = \int_G a(g)dg = 0.$$

Therefore, by integration by parts, we see that

$$\begin{aligned}
&a * \phi_t(x)\\
&= \frac{1}{t}\int_0^\infty \Phi\left(\frac{s}{t}\right)\Delta(s)^{-1}A(x,a_s)\Delta(s)ds\\
&= \frac{1}{t}\int_0^\infty \left(\Phi\left(\frac{s}{t}\right)\Delta(s)^{-1}\right)'\int_0^s A(x,a_u)\Delta(u)duds\\
&= \int_{\sigma(x)-r}^{\sigma(x)+r}\left(-\frac{1}{t^2}\Phi'\left(\frac{s}{t}\right)\Delta(s)^{-1}+\frac{1}{t}\Phi\left(\frac{s}{t}\right)\Delta(s)'\Delta(s)^{-2}\right)\int_0^s A(x,a_u)\Delta(u)duds\,.
\end{aligned}$$

Here we note that

$$|B(s)|^{-1}\left|\int_0^s A(x,a_u)\Delta(u)du\right| = |B(s)|^{-1}|a*\chi_s(x)| \le M_{\mathrm{HL}}a(x)\,. \tag{9}$$

Since $|B(s)|\Delta(s)^{-1} \sim s/(1+s)$ and $\Delta(s)'\Delta(s)^{-1} \sim (1+s)/s$, it follows from (ii) and (iii) of (8) that

$$\begin{aligned}
|a*\phi_t(x)| &\le c\int_{\sigma(x)-r}^{\sigma(x)+r}\left(\frac{1}{t^2}\left|\Phi'\left(\frac{s}{t}\right)\frac{s}{1+s}\right|+\frac{1}{t}\left|\Phi\left(\frac{s}{t}\right)\right|\right)dsM_{\mathrm{HL}}a(x)\\
&\le \frac{cr}{\sigma(x)-r}\left(1+\frac{\sigma(x)-r}{t}\right)^{-\delta}(C_{\phi,2}+C_{\phi,1})M_{\mathrm{HL}}a(x)\\
&\le \frac{cr}{\sigma(x)-r}\left(1+\frac{\sigma(x)-r}{t}\right)^{-\delta}M_{\mathrm{HL}}a(x)\,.
\end{aligned}$$

□

PROPOSITION 4.4. *Let $\phi$ and $a$ be as above, and suppose that $r \le 1$. Then*

$$|a*\phi_t(x)| \le \frac{cr}{\sigma(x)}\left(1+\frac{\sigma(x)-r}{t}\right)^{-\delta}|B(\sigma(x))|^{-1} \quad (\sigma(x)\ge 2r)\,.$$

PROOF. Since $r \le 1$ and $\sigma(x) \ge 2r$, it follows that $|B(\sigma(x)-r)|^{-1} \le c|B(\sigma(x))|^{-1}$ (see the proof of Proposition 4.1). Therefore, we can replace the estimate (9) by

$$|B(s)|^{-1}\left|\int_0^s A(x,a_u)\Delta(u)du\right| \le |B(\sigma(x)-r)|^{-1}\|a\|_1\|\chi_s\|_\infty \le c|B(\sigma(x))|^{-1}\,.$$

The rest of the proof is the same as in the proof of Proposition 4.3. □

Let $1 < p \le \infty$. We say that a function $a$ on $G$ is a $(1,p,0)$-atom provided that

(i) $a$ is supported on $B(r)$ for some $r > 0$,
(ii) if $r \le 1$, then $\|a\|_p \le |B(r)|^{1/p-1}$ and $\int_G a(g)dg = 0$,
(iii) if $r > 1$, then $\|a\|_p \le |B(r)|^{-1}$.

Then, combining the estimates obtained in Propositions 4.1, 4.3, and 4.4, we can obtain the following.

THEOREM 4.5. *Let $\phi \in \mathcal{A}_0$, $\varepsilon > 0$, and $1 < p \leq \infty$. Then for every $(1, p, 0)$-atom $a$ on $G$,*

$$\int_G M_\phi a(g)(1+\sigma(g))^{-\varepsilon} dg \leq c\,,$$

*where $c$ is independent of $a$.*

PROOF. Let $r_0$ be as in Proposition 4.1. Since $M_\phi$ is of type $(p, p)$, it follows that

$$\int_{B(r_0)} M_\phi a(g) dg \leq \|M_\phi a\|_p |B(r_0)|^{1-1/p} \leq c\|a\|_p |B(r_0)|^{1-1/p} \leq c\,. \tag{10}$$

Hence, $M_\phi a$ is equi-integrable on $B(r_0)$. Let us consider the integrability in the exterior $B(r_0)^c$ of $B(r_0)$. We note that $\|a\|_1 \leq 1$, and without loss of generality we may assume that $0 < \varepsilon \leq 1$. If $r \leq 1$, then Proposition 4.4 with $\delta = 0$ yields that

$$\int_{B(2r)^c} M_\phi a(g)(1+\sigma(g))^{-\varepsilon} dg \leq c \int_{2r}^{\infty} \frac{r}{s} \frac{1}{(1+s)^\varepsilon} \frac{1+s}{s} ds \leq c \int_2^\infty \frac{1}{s^2}(1+s)^{1-\varepsilon} ds \leq c\,,$$

and if $r > 1$, then Proposition 4.1 with $\delta = 0$ gives

$$\int_{B(r+1)^c} M_\phi a(g)(1+\sigma(g))^{-\varepsilon} dg \leq c \int_{r+1}^{\infty} \frac{1}{s-r} \frac{1}{(1+s)^\varepsilon} ds \leq c \int_1^\infty \frac{1}{s(1+s)^\varepsilon} ds \leq c\,.$$

□

COROLLARY 4.6. *Let $\phi \in \mathcal{A}_\varepsilon$ $(\varepsilon > 0)$ and $1 < p \leq \infty$. Then for every $(1, p, 0)$-atom $a$ on $G$,*

$$\int_G M_\phi^\varepsilon a(g) dg \leq c\,,$$

*where $c$ is independent of $a$.*

PROOF. We modify the proof of Theorem 4.5. Since the estimate (10) similarly holds in this case, the equi-integrability of $M_\phi^\varepsilon a$ on $B(r_0)$ follows. Let $\sigma(g) \geq r_0$. Here we note that if $t > 1$, then $(1+t)^{-\varepsilon}(1+(\sigma(x)-r)/t)^{-\varepsilon} = (t/1+t)^\varepsilon (t+\sigma(g)-r)^{-\varepsilon} \leq (1+\sigma(g)-r)^{-\varepsilon}$, and if $t \leq 1$, then $(1+t)^{-\varepsilon}(1+(\sigma(g)-r)/t)^{-\varepsilon} \leq (1+\sigma(g)-r)^{-\varepsilon}$, and hence

$$(1+t)^{-\varepsilon}|a * \phi_t(g)| \leq \left(1 + \frac{\sigma(g)-r}{t}\right)^\varepsilon (1+(\sigma(g)-r))^{-\varepsilon}|a * \phi_t(g)|\,.$$

Then, applying Propositions 4.1 and 4.4 with $\delta = \varepsilon$ to the right hand side, we see that for $\sigma(g) \geq r_0$, $M_\phi^\varepsilon a(g) \leq cr\sigma(g)^{-1}B(\sigma(g))^{-1}(1+(\sigma(g)-r))^{-\varepsilon}$ if $r \leq 1$ and $c(\sigma(g)-r)^{-1}\Delta(\sigma(g))^{-1}(1+(\sigma(g)-r))^{-\varepsilon}$ if $r > 1$. Therefore, as in the proof of Theorem 4.5, we have the equi-integrability of $M_\phi^\varepsilon a$ outside $B(r_0)$. □

REMARK 4.7. (1) In the Euclidean case, each function $a$ on $\boldsymbol{R}$ supported on $[-r, r]$ with $\|a\|_\infty \leq (2r)^{-1}$ satisfies $M_\phi a(x) \leq cM_{\mathrm{HL}}a(x) \leq c|x|^{-1}$, and furthermore if $a$ satisfies the moment condition $\int_{\boldsymbol{R}} a(x)dx = 0$, then $M_\phi a(x) \leq cr|x|^{-2}$. This estimate yields the integrability of $M_\phi a$ on $|x| > 2r$ (cf. [3, Theorem 2.9]). On the other hand, let $a$ be a function on $G$ supported on $B(r)$ with $\|a\|_\infty \leq |B(r)|^{-1}$. Then $M_{\mathrm{HL}}a$ satisfies (2) and, if $a$ satisfies the moment condition $\int_G a(g)dg = 0$, then $M_\phi a(x) \leq cr\sigma(x)^{-1}|B(\sigma(x))|^{-1}$ (see Proposition

4.4). Since this estimate is not enough to obtain the integrability of $M_\phi a$ on $\sigma(x) > 2r$, some modification seems to be necessary to obtain the integrability of $M_\phi a$ on $G$.

(2) As pointed out in Remark 4.2, if we restricted to left $K$-invariant $(1, p, 0)$-atoms on $G$, then we can replace $|B(r)|^{-1}$ in (iii) of the definition of the $(1, p, 0)$-atoms on $G$ by $|B(r)|^{1/p-1}$ as in (ii).

**5. Atomic Hardy spaces.** We retain the notation in the previous sections. Since each atom $a$ on $G$ is supported on a ball centered at the origin, in order to obtain a wide class of functions which satisfy the estimates in Theorem 4.5 and Corollary 4.6, we need to translate each atom. For a function $f$ on $G$ we define the translation and the average over $K$ as follows.

$$f_x(g) = f(xg) \quad (x \in G),$$
$$f^\sharp(g) = \int_K f(gk)dk,$$
$$f^\flat(g) = \int_K \int_K f(kgk')dkdk'.$$

Then we introduce an atomic Hardy space $H^1_{p,0}(G)$ on $G$ as follows.

DEFINITION 5.1. Let $1 < p \le \infty$. We define

$$H^1_{p,0}(G) = \left\{ f = \sum_i \lambda_i a_{i,x_i};\ a_i \text{ is a } (1, p, 0)\text{-atom on } G,\ x_i \in G, \text{ and } \sum_i |\lambda_i| < \infty \right\},$$

and $\|f\|_{1,p,0} = \inf \sum_i |\lambda_i|$, where the infimum is taken over all such representations $f = \sum_i \lambda_i a_{i,x_i}$. Furthermore, we define $H^{1,\sharp}_{p,0}(G)$ and $H^{1,\flat}_{p,0}(G)$ as the spaces consisting of $f^\sharp$ and $f^\flat$ of $f$ in $H^1_{p,0}(G)$, respectively, and we define the norms in the same way as in $H^1_{p,0}(G)$.

Let $a$ be a $(1, p, 0)$-atom on $G$ and $x \in G$. Since $\|(a_x)^\sharp\|_1$ and $\|(a_x)^\flat\|_1$ are bounded by $\|a_x\|_1 = \|a\|_1 \le 1$, it follows that

$$H^{1,\sharp}_{p,0}(G) \subset H^1_{p,0}(G) \subset L^1(G), \quad H^{1,\flat}_{p,0}(G) \subset L^1(G),$$

and $\|f\|_1 \le \|f\|_{1,p,0}$ for all $f \in H^1_{p,0}(G)$ (resp. $H^{1,\flat}_{p,0}(G)$). Here we note that

$$a_x * \phi_t(g) = a^\sharp * \phi_t(xg),$$

and

$$(a_x)^\flat * \phi_t(g) = \int_K a^\sharp * \phi_t(xkg)dk.$$

In particular, for $\varepsilon \ge 0$, $\|M^\varepsilon_\phi a_x\|_1$ and $\|M^\varepsilon_\phi (a_x)^\flat\|_1$ are bounded by $\|M^\varepsilon_\phi a^\sharp\|_1$. Since $a^\sharp$ is a $(1, p, 0)$-atom on $G$, Theorem 4.5 and Corollary 4.6 yield the following.

THEOREM 5.2. *Let $\varepsilon > 0$ and $\phi \in \mathcal{A}_0$. Then the radial maximal operator $M_\phi$ satisfies*

$$\int_G M_\phi f(g)(1 + \sigma(g))^{-\varepsilon} dg \le c\|f\|_{1,p,0}$$

*for all $f \in H^1_{p,0}(G)$ (resp. $H^{1,\flat}_{p,0}(G)$).*

THEOREM 5.3. *Let $\varepsilon > 0$ and $\phi \in \mathcal{A}_\varepsilon$. Then the modified radial maximal operator $M_\phi^\varepsilon$ satisfies*

$$\int_G M_\phi^\varepsilon f(g)dg \le c\|f\|_{1,p,0}$$

*for all $f \in H_{p,0}^1(G)$ (resp. $H_{p,0}^{1,\flat}(G)$).*

We shall give a characterization for $H_{p,0}^{1,\flat}(G)$ without using the translation and the average over $K$ of $(1, p, 0)$-atoms on $G$. Let $x \in G$ and $r > 0$. We set the domain $R(x,r)$ as

$$R(x,r) = \{g \in G; \sigma(x) - r \le \sigma(g) \le \sigma(x) + r\}$$

and for a function $f$ on $G$ supported on $R(x,r)$ we put

$$\|f\|_{x,r,p} = \left(\int_G |f(g)|^p I(x, r_0, g^{-1})^{1-p} dg\right)^{1/p},$$

where $I$ is given by (7) and $r_0$ is the same as that in Proposition 4.1. When $p = \infty$, $\|f\|_{x,r,\infty}$ means $\|f(g)I(x, r_0, g^{-1})^{-1}\|_\infty$. Then we say that a function $a$ on $G$ is a $(1, p, 0, \natural)$-atom provided that

(i) $a$ is $K$-bi-invariant and supported on $R(x,r)$ for some $x \in G$ and $r > 0$,
(ii) if $r \le 1$, then $\|f\|_{x,r,p} \le |B(r)|^{1/p-1}$ and $\int_G a(g)dg = 0$,
(iii) if $r > 1$, then $\|f\|_{x,r,p} \le |B(r)|^{-1}$.

By using these $\natural$-atoms on $G$ we define an atomic Hardy space $H_{p,0}^{1,\natural}(G)$ on $G$ as follows.

DEFINITION 5.4. Let $1 < p \le \infty$. We define

$$H_{p,0}^{1,\natural}(G) = \left\{ f = \sum_i \lambda_i a_i;\ a_i \text{ is a } (1, p, 0, \natural)\text{-atom on } G \text{ and } \sum_i |\lambda_i| < \infty \right\},$$

and $\|f\|_{1,p,0,\natural} = \inf \sum_i |\lambda_i|$, where the infimum is taken over all such representations $f = \sum_i \lambda_i a_i$.

THEOREM 5.5. *$H_{p,0}^{1,\flat}(G) = H_{p,0}^{1,\natural}(G)$ and $\|f\|_{1,p,0} \sim \|f\|_{1,p,0,\natural}$.*

PROOF. To prove $H_{p,0}^{1,\natural} \subset H_{p,0}^{1,\flat}$ it suffices to show that each $(1, p, 0, \natural)$-atom $a$ on $G$ is contained in $H_{p,0}^{1,\flat}(G)$ and $\|a\|_{1,p,0} \le 1$. Suppose that $a$ is supported on $R(x,r)$ and put

$$b(g) = \frac{a(x^{-1}g)}{I(x, r_0, (x^{-1}g)^{-1})} \chi_{r_0}(g) \quad (g \in G).$$

Since $a$ and $I(x, r, \cdot)$ are $K$-bi-invariant on $G$, it follows that

$$\int_G b(g)dg = \int_G \frac{a(g)}{I(x, r_0, g^{-1})} \left(\int_K \chi_{r_0}(xkg)dk\right) dg = \int_G a(g)dg = 0 \quad \text{if } r \le 1.$$

Moreover, since

$$\|b\|_p^p = \int_G \left|\frac{a(g)}{I(x, r_0, g^{-1})}\right|^p \int_K \chi_{r_0}(xkg)dkdg \le \int_G |a(g)|^p I(x, r_0, g^{-1})^{1-p} dg = \|a\|_{x,r,p}$$

and $\|b\|_\infty \leq \|a(g)I(x, r_0, g^{-1})^{-1}\|_\infty = \|a\|_{x,r,\infty}$, it is easy to see that $b$ is a $(1, p, 0)$-atom on $G$. Here we note that

$$(b_x)^\flat(g) = \int_K b(xkg)dk = \frac{a(g)}{I(x, r_0, g^{-1})} \int_K \chi_{r_0}(xkg)dk = a(g)\,,$$

and thus, $a \in H^{1,\flat}_{p,0}(G)$ with $\|a\|_{1,p,0} \leq 1$.

Next, to prove $H^{1,\flat}_{p,0} \subset H^{1,\natural}_{p,0}$ it is enough to show that $(a_x)^\flat \in H^{1,\natural}_{p,0}(G)$ with $\|(a_x)^\flat\|_{1,p,0,\natural} \leq 1$ for each $(1, p, 0)$-atom $a$ on $G$. Suppose that $a$ is supported on $B(r)$. Clearly, $(a_x)^\flat$ is supported on $R(x, r)$, and if $r \leq 1$, then

$$\int_G (a_x)^\flat(g)dg = \int_G \int_K a^\sharp(xkg)dkdg = \int_G a(g)dg = 0\,.$$

Moreover, since

$$|(a_x)^\flat(g)| = \left|\int_K a^\sharp(xkg)\chi_{r_0}(xkg)dk\right| \leq \left(\int_K |a^\sharp(xkg)|^p dk\right)^{1/p} \left(\int_K \chi_{r_0}(xkg)dk\right)^{1-1/p}$$

and $\|(a_x)^\flat\|_\infty \leq \|a^\sharp\|_\infty I(x, r_0, g^{-1})$, it follows that

$$\left(\int_G |(a_x)^\flat(g)|^p I(x, r_0, g^{-1})^{1-p} dg\right)^{1/p} \leq \left(\int_G \int_K |a^\sharp(xkg)|^p dkdg\right)^{1/p} \leq \|a^\sharp\|_p \leq \|a\|_p\,.$$

Therefore, $(a_x)^\flat \in H^{1,\natural}_{p,0}(G)$ with $\|(a_x)^\flat\|_{1,p,0,\natural} \leq 1$. □

**6. The heat and Poisson maximal operators.** We define the modified heat maximal operator $M^\varepsilon_{\mathrm{H}}$ $(\varepsilon \geq 0)$ on $G$ by

$$M^\varepsilon_{\mathrm{H}} f(x) = \sup_{t>0}(1+t)^{-\varepsilon}|e^{t\Omega} f(x)| = \sup_{t>0}(1+t)^{-\varepsilon}|f * h_t(x)|\,,$$

where $e^{t\Omega}$ is the heat diffusion semigroup over $G/K$ realized by the convolution with the heat kernel $h_t$, and we denote $M^0_{\mathrm{H}}$ by $M_{\mathrm{H}}$ for simplicity. As mentioned in §1, $M^\varepsilon_{\mathrm{H}}$ $(\varepsilon \geq 0)$ satisfies the maximal theorem.

First we shall prove the following.

THEOREM 6.1. *Let $\varepsilon > 1/2$. Then*

$$\int_G M_{\mathrm{H}} f(g)(1 + \sigma(g))^{-\varepsilon} dg \leq c\|f\|_{1,p,0}$$

*and*

$$\int_G M^\varepsilon_{\mathrm{H}} f(g) dg \leq c\|f\|_{1,p,0}$$

*for all $f \in H^1_{p,0}(G)$ (resp. $H^{1,\flat}_{p,0}(G)$).*

PROOF. We note that the argument preceding to Theorem 5.2 is also applicable to $M^\varepsilon_{\mathrm{H}}$ with $\phi_t$ replaced by $h_t$. Therefore, to deduce the desired estimates it is enough to show that for each $(1, p, 0)$-atom $a$ on $G$, $M_{\mathrm{H}} a$ and $M^\varepsilon_{\mathrm{H}} a$ satisfy, respectively,

$$\int_G M_{\mathrm{H}} a(g)(1 + \sigma(g))^{-\varepsilon} dg \leq c\,, \tag{11}$$

$$\int_G M_{\mathrm{H}}^{\varepsilon} a(g)dg \le c\,, \tag{12}$$

for some constant $c$ independent of $a$ (cf. Theorem 4.5 and Corollary 4.6). Then, we need the estimates for $a * h_t(x)$ corresponding to those in Propositions 4.1 and 4.4. In order to obtain the estimates we shall use the ones for $h_t$ and $h_t'$ obtained in [1, Theorem 3.1]:

$$|h_t(s)| \le ce^{-2\rho|s|}e^{-(2\rho t-|s|)^2/4t}\begin{cases} t^{-n/2}(1+|s|)^{n-1} & (t \le 1)\,, \\ |s|^{-1/2}\left(\dfrac{|s|}{t}\right)^{\alpha_i} & (1 \le t)\,, \end{cases}$$

where $n = \dim G/K = m_1 + m_2 + 1 = 2\alpha + 2$ and $\alpha_i$ $(1 \le i \le 3)$ depend on the three regions in $[0,\infty) \times [1,\infty)$ to which $(|s|, t)$ belongs: explicitly, they are given as $\alpha_1 = 1/2$ if $|s| \le \sqrt{t}$, $\alpha_2 = 1$ if $\sqrt{t} \le |s| \le t$, and $\alpha_3$ is the smallest integer $> n - 1/2$ if $1 \le t \le |s|$ (see [1, Fig. 5]) and when $1 \le t$ and $|s| \le \sqrt{t}$, we used the fact that $(1+|s|)/t \le 2$. Similarly,

$$|h_t'(s)| \le ce^{-2\rho|s|}e^{-(2\rho t-|s|)^2/4t}\begin{cases} t^{-n/2}(1+|s|)^{n-1}\left(1+\dfrac{|s|}{t}\right) & (t \le 1)\,, \\ |s|^{-1/2}\left(\dfrac{|s|}{t}\right)^{\beta_i} & (1 \le t)\,, \end{cases}$$

where $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$, and $\beta_3 = \alpha_3 + 1$ according to the regions on which the $\alpha_i$ depend. Here we note that $F(t) = t^{-l}e^{-(2\rho t-|s|)^2/4t}$ $(l > 0)$ has a maximum at $t_0 \sim |s|/2\rho$ if $|s| \ge 1$ and at $t_0 \sim |s|^2$ if $|s| \le 1$. Moreover, $F(t)$ is increasing on $(0, t_0]$ and decreasing on $[t_0, \infty)$. Therefore, we can take constants $C_\delta$ $(\delta \ge 0)$, $C_{\delta,k}$ $(k = 0, 1)$, and $C$ so that

$$\left(\frac{|s|}{t}\right)^{\gamma+\delta} e^{-(2\rho t-|s|)^2/4t} \le C_\delta \quad (1 \le t,\, \gamma = \alpha_i, \beta_i,\, 1 \le i \le 3)\,,$$

$$\left(\frac{1+|s|}{t}\right)^{n+\delta}\left(1+\frac{|s|}{t}\right)^{k} e^{-(2\rho t-|s|)^2/4t} \le C_{\delta,k} \quad (t \le 1,\, |s| \ge 1)\,,$$

$$\left(\frac{1+|s|}{t}\right)^{n/2}\left(1+\frac{|s|}{t}\right) e^{-(2\rho t-|s|)^2/4t} \le C|s|^{-n/2}\left(1+\frac{1}{|s|}\right)^{n/2+1} \sim |s|^{-(n+1)}$$

$(t \le 1, |s| \le 3)$, where $C_\delta$ and $C_{\delta,k}$ are independent of $(|s|, t)$ and $C$ of $t$. Hence, we have

$$|h_t(s)| \le ce^{-2\rho|s|}\begin{cases} C_{\delta,0}t^{n/2+\delta}(1+|s|)^{-(1+\delta)} & (t \le 1,\, |s| \ge 1)\,, \\ C_\delta|s|^{-1/2}\left(\dfrac{|s|}{t}\right)^{-\delta} & (1 \le t)\,, \end{cases} \tag{13}$$

$$|h_t'(s)| \le ce^{-2\rho|s|}\begin{cases} C|s|^{-(n+1)} & (t \le 1,\, |s| \le 3)\,, \\ C_{\delta,1}t^{n/2+\delta}(1+|s|)^{-(1+\delta)} & (t \le 1,\, |s| \ge 1)\,, \\ C_\delta|s|^{-1/2}\left(\dfrac{|s|}{t}\right)^{-\delta} & (1 \le t)\,. \end{cases} \tag{14}$$

LEMMA 6.2. *Let $r \geq 1$ and $\delta \geq 0$. Then for $\sigma(x) \geq r_0 = r+1$*

$$|a * h_t(x)| \leq ce^{2\rho\sigma(x)} \begin{cases} (1+\sigma(x)-r)^{-(1+\delta)} & (t \leq 1), \\ (\sigma(x)-r)^{-1/2}\left(\dfrac{\sigma(x)-r}{t}\right)^{-\delta} & (1 \leq t). \end{cases}$$

PROOF. We shall recall the proof of Proposition 4.1 and note that

$$|a * h_t(x)| \leq \int_0^r \int_K \int_K |a(ka_sk')|dk'|h_t(\sigma(a_s^{-1}k^{-1}x))|dk\Delta(s)ds\,.$$

Since $\sigma(a_s^{-1}kx) \geq \sigma(x) - s \geq 1$ for $k \in K$, we can substitute (13) into $|h_t(\sigma(a_s^{-1}k^{-1}x))|$ and hence

$$|h_t(\sigma(a_s^{-1}kx))| \leq c\Delta(\sigma(x)-r)^{-1} \begin{cases} (1+\sigma(x)-r)^{-(1+\delta)} & (t \leq 1), \\ (\sigma(x)-r)^{-1/2}\left(\dfrac{\sigma(x)-r}{t}\right)^{-\delta} & (1 \leq t). \end{cases}$$

The rest of the proof is the same as that in Proposition 4.1. □

LEMMA 6.3. *Let $r \leq 1$ and $\delta \geq 0$. Then for $\sigma(x) \geq r_0 = 2r$*

$$|a * h_t(x)| \leq cre^{-2\rho\sigma(x)} \begin{cases} (\sigma(x)-r)^{-(n+1)} & (t \leq 1, \sigma(x) \leq 2), \\ (1+\sigma(x)-r)^{-(1+\delta)} & (t \leq 1, \sigma(x) \geq 2), \\ (\sigma(x)-r)^{-1/2}\left(\dfrac{\sigma(x)-r}{t}\right)^{-\delta} & (1 \leq t). \end{cases}$$

PROOF. We recall the proof of Propositions 4.3 and 4.4. Integration by parts yields that

$$|a * h_t(x)| \leq \int_{\sigma(x)-r}^{\sigma(x)+r} |h_t'(s)| \int_0^s |A(x, a_u)|\Delta(u)duds\,.$$

Since $\sigma(x) - r \leq s \leq \sigma(x) + r \leq 3$ if $\sigma(x) \leq 2$ and $s \geq \sigma(x) - r \geq 1$ if $\sigma(x) \geq 2$, we can substitute (14) into $|h_t'(s)|$. Then, replacing $|s|^{-l}$ ($l \geq 0$) with $(\sigma(x)-r)^{-l}$, we can deduce the desired estimate from the same arguments as those in Propositions 4.1 and 4.3. □

Now, we return to the proof of (11) and (12). Since $M_{\mathrm{H}}$ is of type $(p, p)$, $1 < p \leq \infty$, $M_{\mathrm{H}}$ satisfies (10) instead of $M_\phi$ and thereby $M_{\mathrm{H}}^\varepsilon$ ($\varepsilon \geq 0$) is equi-integrable on $B(r_0)$. Let us consider the integrals of $M_{\mathrm{H}}a(g)(1+\sigma(g))^{-\varepsilon}$ and $M_{\mathrm{H}}^\varepsilon a(g)$ in the exterior of $B(r_0)$. Clearly, without loss of generality, we may assume that $1/2 < \varepsilon \leq 1$. Then we shall show the equi-integrability for the local and global parts of the maximal operator $M_{\mathrm{H}}^\varepsilon$:

$$M_{\mathrm{H},0}a(x) = \sup_{0<t\leq 1} |a * h_t(x)| \quad \text{and} \quad M_{\mathrm{H},1}^\varepsilon a(x) = \sup_{1\leq t<\infty} (1+t)^{-\varepsilon}|a * h_t(x)|\,.$$

Let $r \leq 1$. Then Lemma 6.3 with $\delta = 0, \varepsilon$, together with the fact that $\Delta(s) \sim |s|^{2\alpha+1} = |s|^{n-1}$ if $|s| \leq 2$ and $\Delta(s) \sim e^{-2\rho|s|}$ if $|s| \geq 2$ (see Lemma 2.2), yields that

$$\begin{aligned} (15) \quad \int_{B(2r)^c} M_{\mathrm{H},0}a(g)dg &\leq cr\left(\int_{2r}^2 \frac{1}{(s-r)^{n+1}} s^{2\alpha+1}ds + \int_2^\infty \frac{1}{(1+s-r)^{1+\varepsilon}}ds\right) \\ &\leq c(1+r^{1-\varepsilon}) \leq c \end{aligned}$$

and

$$\int_{B(2r)^c} M_{\mathrm{H},1}a(g)(1+\sigma(g))^{-\varepsilon}dg \le cr\int_{2r}^{\infty}\frac{1}{(s-r)^{1/2}(1+s)^{\varepsilon}} \le cr^{3/2-\varepsilon} \le c\,. \tag{16}$$

Let $r > 1$. It follows from Lemma 6.2 with $\delta = 0$ that

$$\int_{B(r+1)^c} M_{\mathrm{H},0}a(g)dg \le \int_{r+1}^{\infty}\frac{1}{(1+s-r)^{1+\varepsilon}}ds \le c \tag{17}$$

and

$$\int_{B(r+1)^c} M_{\mathrm{H},1}a(g)(1+\sigma(g))^{-\varepsilon}dg \le c\int_{r+1}^{\infty}\frac{1}{(s-r)^{1/2}(1+s)^{\varepsilon}}ds \le c\,. \tag{18}$$

Therefore, $M_{\mathrm{H},0}a(g)$ and $M_{\mathrm{H}}a(g)(1+\sigma(g))^{-\varepsilon}$ are equi-integrable outside $B(r_0)$. This completes the proof of (11).

As for (12) it remains to show the equi-integrability of $M^{\varepsilon}_{\mathrm{H},1}a(g)$ on $B(r_0)^c$. Let $\sigma(g) \ge r_0$ and $1 \le t < \infty$. Since $(1+t)^{\varepsilon}((\sigma(g)-r)/t)^{\varepsilon} \ge (\sigma(g)-r)^{\varepsilon}$, it follows that

$$(1+t)^{-\varepsilon}|a * h_t(g)| \le \left(\frac{\sigma(g)-r}{t}\right)^{\varepsilon}(\sigma(g)-r)^{-\varepsilon}|a * h_t(g)|\,.$$

Then, applying Lemma 6.3 and Lemma 6.2 with $\delta = \varepsilon$ to the right hand side, we see that if $r \le 1$, then $M^{\varepsilon}_{\mathrm{H},1}a(g) \le cre^{-2\rho\sigma(g)}(\sigma(g)-r)^{-1/2-\varepsilon}$ and if $r \ge 1$, then $M^{\varepsilon}_{\mathrm{H},1}a(g) \le ce^{-2\rho\sigma(g)}(\sigma(g)-r)^{-1/2-\varepsilon}$. Therefore, as in (16) and (18), the equi-integrability of $M^{\varepsilon}_{\mathrm{H},1}a$ on $B(r_0)^c$ follows. This completes the proof of (12) and finally, Theorem 6.1. □

Next, we shall consider the same problem for the modified Poisson maximal operator $M^{\varepsilon}_{\mathrm{P}}$ ($\varepsilon \ge 0$) on $G$ defined as

$$M^{\varepsilon}_{\mathrm{P}}f(x) = \sup_{t>0}(1+t)^{-\varepsilon}|e^{t\sqrt{\Omega}} * f(x)| = \sup_{t>0}(1+t)^{-\varepsilon}|f * p_t(x)|\,,$$

where $p_t$ is the Poisson kernel of $e^{t\sqrt{\Omega}}$, and we denote $M^0_{\mathrm{P}}$ by $M_{\mathrm{P}}$ for simplicity. Then, $M^{\varepsilon}_{\mathrm{P}}$ ($\varepsilon \ge 0$) satisfies the maximum theorem.

We recall the estimates for $p_t$ and $p'_t$ obtained in [1, Theorem 6.1, (6.3) and (6.4)]:

$$|p_t(s)| \le c\begin{cases} t(t^2+s^2)^{-n/2-1/2} + t(t^2+s^2)^{-n/2+1/2} & (t \le 1, |s| \le 1)\,, \\ \dfrac{1}{\sqrt{t}}\left(\dfrac{1+|s|}{t}\right)e^{-\rho t}e^{-\rho|s|} & (1 \le t, |s| \le \sqrt{t})\,, \\ \dfrac{1}{\sqrt{s}}\left(\dfrac{t}{t+|s|}\right)e^{-\rho|s|}e^{-\rho(t^2+s^2)^{1/2}} & (|s| \ge 1, \sqrt{t}) \end{cases}$$

and for $|p'_t(s)|$ we replace the first line on the right hand side by $(t+|s|)^{-(n+1)}$ and $t/(t+|s|)$ in the third line by $(1+t)/(t+|s|)$. Let $\delta \ge 0$ and we note that

$$(t+|s|)^{-(n+1)} \le |s|^{-(n+1)}\,,$$

$$\frac{1}{\sqrt{|s|}}\left(\frac{t}{t+|s|}\right) \le |s|^{-3/2} \quad (t \le 1)\,,$$

$$\frac{1}{\sqrt{t}}\left(\frac{1+|s|}{t}\right)\left(\frac{\sqrt{|s|}}{t}\right)^{\delta} e^{-\rho t} \leq |s|^{-2-3\delta/2} e^{-\rho|s|^2} \quad (1 \leq |s| \leq \sqrt{t}),$$

$$\frac{1}{\sqrt{|s|}}\left(\frac{1+t}{t+|s|}\right)\left(\frac{\sqrt{|s|}}{t}\right)^{\delta} e^{-\rho(t^2+s^2)^{1/2}} \leq C|s|^{-1} e^{-\rho|s|} \quad (1 \leq t, |s| \geq \sqrt{t}),$$

where $C$ is independent of $t$ and we used the fact that the left hand side in the last inequality takes a maximum at $t \sim \sqrt{|s|}$. Then, it easily follows that

$$|p_t(s)| \leq ce^{-2\rho|s|} \begin{cases} s^{-3/2} & (t \leq 1, |s| \geq 1), \\ |s|^{-1}\left(\dfrac{\sqrt{|s|}}{t}\right)^{-\delta} & (1 \leq t, |s| \geq 1), \end{cases}$$

and

$$|p_t'(s)| \leq ce^{-2\rho|s|} \begin{cases} |s|^{-(n+1)} & (t \leq 1, |s| \leq 1), \\ |s|^{-3/2} & (t \leq 1, |s| \geq 1), \\ |s|^{-1}\left(\dfrac{\sqrt{|s|}}{t}\right)^{-\delta} & (1 \leq t, |s| \geq 1). \end{cases}$$

Then, letting $\delta = 0$ and $\delta = \varepsilon > 0$ and repeating the same arguments that yielded Theorem 6.1, we obtain the following.

THEOREM 6.4. *If $\varepsilon > 0$, then*

$$\int_G M_{\mathrm{P}} f(g)(1+\sigma(g))^{-\varepsilon} dg \leq c\|f\|_{1,p,0},$$

*and*

$$\int_G M_{\mathrm{P}}^{\varepsilon} f(g) dg \leq c\|f\|_{1,p,0}$$

*for all $f \in H^1_{p,0}(G)$ (resp. $H^{1,\flat}_{p,0}(G)$).*

REMARK 6.5. It follows from (15) and (17), together with the corresponding estimates for $M_{\mathrm{P},0}$ that the local maximal operators $M_{\mathrm{H},0}$ and $M_{\mathrm{P},0}$ are bounded from $H^1_{p,0}(G)$ (resp. $H^{1,\flat}_{p,0}(G)$) to $L^1(G)$, that is,

$$\|M_{\mathrm{H},0} f\|_1 \leq c\|f\|_{1,p,0} \quad \text{and} \quad \|M_{\mathrm{P},0} f\|_1 \leq c\|f\|_{1,p,0}$$

for all $f \in H^1_{p,0}(G)$ (resp. $H^{1,\flat}_{p,0}(G)$).

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