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PERIODIC SOLUTIONS FOR DISSIPATIVE-REPULSIVE SYSTEMS

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Abstract. It is proved that every dissipative-repulsive periodic system admits a periodic solution, which is comparable with some well-known results due to Yoshizawa, Hale and Lopes, and Burton and Zhang for dissipative systems.

1. Introduction. Consider an ordinary differential equation

$$x' = f(t,x)\,, \tag{1}$$

where $f : R\times R^n \to R^n$ is continuous and locally Lipschitz in the space variable x; moreover, $f(t+\omega, x) = f(t,x)$ for all t and some $\omega > 0$. Denote by $x(t, t_0, x_0)$ the solution of (1) with the initial value $x(t_0) = x_0$. Then (1) is said to be dissipative if there is a $B > 0$ such that

$$\limsup_{t\to\infty} |x(t, t_0, x_0)| < B$$

for all t_0 and x_0.

It is well-known that such systems (including more general systems, e.g., functional differential equations and evolution equations) admit periodic solutions ([15], [6], [2], [13] and the references therein). For the existence of periodic solutions of nondissipative systems, the situation becomes more complicated ([14], [3], [4], [5], [9] and [11]).

There are some systems whose solutions exhibit the following regular behavior; some components are dissipative, and other components are repulsive relative to some states. Naturally, it is concerned whether the similar results hold for these systems or not. In the present paper, we will provide a simple and clear conclusion; such systems also admit periodic solutions, via the Brouwer degree theory.

The plan of the paper is as follows. In Section 2, we first consider ordinary differential equations. Then in Section 3, we deal with functional differential equations. Finally in Section 4, we discuss the equilibrium problems, similar to Hutson's one ([8]).

2. Ordinary differential equations. In the following, let $x = (y, z)$, $y \in R^m$, $z \in R^l$ with $m + l = n$, and let $x(t, x_0) = x(t, 0, x_0)$. We give the exact definition on the dissipative-repulsive system as follows.

DIFINITION 1. The equation (1) is said to be dissipative-repulsive if there exist B, d, $r_0 > 0$ and a continuous ω-periodic function $g : R \to R^l$ with $|g(t)| < r_0$ $(t \in R)$ such that

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for any $a \geq d$, $b' \geq r_0$, there are $b \geq r_0$ and $T = T(a, b') > 0$ such that the following hold for all $|y_0| \leq a$:

i) $|y(t, x_0)| \leq B$, whenever $t \geq T$ and $|z_0| \leq b$;

ii) $|z(t, x_0) - g(t)| > 0$, whenever $0 \leq t \leq T$ and $b \leq |z_0| \leq b + b'$;
$|z(t, x_0)| > b$, whenever $t \geq T$ and $b \leq |z_0| \leq b + b'$.

We have the following.

THEOREM 1. *If equation* (1) *is dissipative-repulsive, then it admits an* ω*-periodic solution.*

PROOF. Denote by $S_1(\sigma)$ and $S_2(\sigma)$ the open balls centered at the origin with radius σ, respectively in R^m and R^l. Now we put $a = B + r_0 + 1$, $b' = h$ in Definition 1, and set $D = S_1(a) \times S_2(b)$, where h satisfies

$$|x(t, x_0)| < h \quad \text{for any} \quad t \in [0, \omega] \quad \text{and} \quad x_0 \in \bar{D}\,.$$

Consider the Poincaré map $P(x_0) = x(\omega, x_0)$. Fix a prime number N such that $N\omega \geq T(a, b') + \omega$. Since (1) is dissipative-repulsive, from i) and ii) we have

$$P^N(x) \neq x \quad \text{for any} \quad x \in \partial D\,.$$

We claim that for each fixed point p of P^N in $\bar{D}$,

$$P(p) \in D\,.$$

If this fails, then there would be a fixed point $p \in D$ such that $P(p) \notin D$. Write

$$P(p) = P^{N+1}(p) = x((N+1)\omega, p) = q = (q_1, q_2)\,.$$

Clearly, $(N+1)\omega > T(a, b')$ and $a > B$. Thus, by the definition of a, i) and the construction of D, we derive that $q_1 \in S_1(a)$, and hence $b \leq |q_2| \leq b + b'$. From the choice of N it follows that

$$z((N-1)\omega, q) \notin \bar{S}_2(b)\,,$$

which implies that

$$p = P^{N-1}(q) \notin D\,,$$

a contradiction. By a modular degree theorem ([16]),

$$\deg(\mathrm{id} - P, D, 0) = \deg(\mathrm{id} - P^N, D, 0) \pmod{N}\,. \tag{2}$$

We have to prove

$$|\deg(\mathrm{id} - P^N, D, 0)| = 1\,.$$

Once this is true, then from (2) it follows that $\deg(\mathrm{id} - P, D, 0) \neq 0$. Hence P has a fixed point p in D, and $x(t, p)$ is an ω-periodic of (1), as desired. To see this true, consider the homotopy:

$$H_1(y_0, z_0, \mu) = (y_0 - \mu y(N\omega, (1-\mu)y_* + \mu y_0, z_0), \mu z_0 - z(N\omega, (1-\mu)y_* + \mu y_0, z_0))\,,$$

where any $y_* \in S_1(a)$ is chosen and $\mu \in [0, 1]$. By i) and ii),

$$0 \notin H_1(\partial D \times [0, 1])\,,$$

which implies

$$\begin{aligned}\deg(\mathrm{id} - P^N, D, 0) &= \deg(H_1(\cdot, 1), D, 0) = \deg(H_1(\cdot, 0), D, 0)\\ &= \deg(-z(N\omega, y_*, \cdot), S_2(b), 0)\,.\end{aligned} \tag{3}$$

To calculate that degree, we consider another homotopy

$$H_2(z_0, \mu) = z(\mu N\omega, y_*, z_0) - g(\mu N\omega)\,,$$

where $\mu \in [0, 1]$. From ii) we have

$$0 \notin H_2(\partial S_2(b) \times [0, 1])\,,$$

which implies

$$\deg(z(N\omega, y_*, \cdot) - g(0), S_2(b), 0) = \deg(\mathrm{id} - g(0), S_2(b), 0) = \deg(\mathrm{id}, S_2(b), 0) = 1\,.$$

This together with (3) implies the desired conclusion. The proof is complete.

3. Functional differential equations.

In this section, we will prove an analogous result for functional differential equations. Since the phase space is infinite-dimensional for such systems, the Poincaré type map looses compactness in case of larger delay. Hence we have to treat the system under consideration as a finite-dimensional one.

Consider the functional differential equation

$$x' = F(t, x_t)\,, \tag{4}$$

where $F : R \times C \to R^n$ is continuous and locally Lipschitz in the second variable; moreover $F(t + \omega, \varphi) = F(t, \varphi)$ for all (t, φ) and takes any bounded sets in C into bounded sets in R^n. Here $\omega > 0$ is given, $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$ with $r > 0$; $C = C([-r, 0], R^n)$ with the usual supremum norm $|\cdot|$. In the following, denote by $x(t, s, \varphi)$ the solution of (4) with initial value $x_s = \varphi$. For simplicity, we let $x(t, \varphi) = x(t, 0, \varphi)$.

DEFINITION 2. The equation (4) is said to be dissipative-repulsive if there exist B, d, $r_0 > 0$ and a continuous ω-function $g : R \to S_2(r_0)$ such that for any $a \geq d$, $b' \geq r_0$, there are $b \geq r_0$, $M = M(a)$, $T = T(a, b') > 0$ such that the following holds:

i) $|x(t, \varphi)| \leq M$, whenever $t \geq 0$ and $|\varphi| \leq a + b$;

ii) $|y(t, \varphi)| \leq B$, whenever $t \geq T$, $|\psi_1| \leq a$ and $|\psi_2| \leq b$, where $\varphi = (\psi_1, \psi_2)$;

iii) $|z(s, \varphi) - g(s)| > 0$, whenever $0 \leq s \leq T$, $|\psi_1| \leq a$ and $b \leq |\psi_2(0)| \leq b'$; and $|z(t, \varphi)| > b$, whenever $t \geq T$, $|\psi_1| \leq a$ and $b \leq |\psi_2(0)| \leq b'$.

THEOREM 2. *If equation* (4) *is dissipative-repulsive, then it admits an ω-periodic solution.*

PROOF. Put $a = B + r_0 + 1$, $b = b(B + r_0 + 1)$, $M_0 = M(B + r_0 + 1)$ and $b' = M_1 = M(M_0 + 2)$. Since F maps any bounded sets in C into bounded sets in R^n, there is a constant $L > 0$ such that for any $t \in R$ and $|\varphi| \leq 2M_1$,

$$|F(t, \varphi)| \leq L - 1\,.$$

Set

$$S = \{\varphi \in C : |\varphi| \le M_0, |\varphi(s_1) - \varphi(s_2)| \le L|s_1 - s_2|, s_i \in [-r, 0], i = 1, 2\},$$

and

$$S' = \{\varphi \in C : |\varphi| \le 2M_1, |\varphi(s_1) - \varphi(s_2)| \le L|s_1 - s_2|, s_i \in [-r, 0], i = 1, 2\}.$$

Then S and S' are compact. Take any partition

$$t_0 = 0 > t_1 > \cdots > t_k = -r, \quad t_i - t_{i+1} = \frac{r}{k} = \Delta, \quad i = 0, \dots, k-1.$$

Given $\varphi \in S$, define

$$\bar\varphi^k(s) = \varphi(t_i) - \frac{s - t_i}{\Delta}[\varphi(t_{i+1}) - \varphi(t_i)], \quad s \in [t_{i+1}, t_i], \quad i = 0, \dots, k-1;$$

$$\tilde\varphi^k = (\varphi(t_k), \dots, \varphi(t_0)) \in R^{(k+1)n}. \tag{5}$$

Note that for each $\varphi \in S$, there is an i such that

$$|\varphi - \bar\varphi^k| = \max_{[t_{i+1}, t_i]} |\varphi - \bar\varphi^k| \le 2L\Delta \to 0, \tag{6}$$

as $k \to \infty$ uniformly on S.

Define $f(t, \tilde\varphi^k) = F(t, \bar\varphi^k)$ and consider the delayed equation

$$x'(t) = f(t, x(t + t_k), \dots, x(t + t_0)). \tag{7}$$

Denote by $x_k(t, s, \bar\varphi^k)$ the solution of (7) with initial value $(x_k)_{t_0} = \bar\varphi^k$ and let $x_k(t, \bar\varphi^k) = x_k(t, 0, \bar\varphi^k)$. Clearly, they are unique and continuous in initial values. In particular, by (6) and (7), we have

$$x_k(t, s, \bar\varphi^k) \to x(t, s, \varphi), \tag{8}$$

as $k \to \infty$ uniformly for $\varphi \in S$ and t, s on any finite interval.

Indeed,

$$|x_k(t, \bar\varphi^k) - x(t, \varphi)| \le |x_k(t, \bar\varphi^k) - x_k(t, \varphi)| + |x_k(t, \varphi) - x(t, \varphi)| = I_1 + I_2.$$

Since F is locally Lipschitz in φ and S' is compact, there is a $K > 0$ such that

$$|F(t, \varphi) - F(t, \psi)| \le K|\varphi - \psi| \quad \text{for any} \quad \varphi, \psi \in S'.$$

If $t \in [-r, 0]$, then

$$I_1 \le |\bar\varphi^k - \varphi| \le 2L\Delta, \quad I_2 = 0.$$

If $t > 0$, then

$$\begin{aligned} \max_{[0,t]} I_1 &\le \int_0^t |F(s, \overline{(x_k)_s^k}(\cdot, \bar\varphi^k)) - F(s, \overline{(x_k)_s^k}(\cdot, \varphi))| \mathrm{d}s \\ &\le K \int_0^t |\overline{(x_k)_s^k}(\cdot, \bar\varphi^k) - \overline{(x_k)_s^k}(\cdot, \varphi)| \mathrm{d}s \\ &\le 2L\Delta K t + K \int_0^t \max_{[0,s]} |\bar x_k(\tau, \bar\varphi^k) - \bar x_k(\tau, \varphi)| \mathrm{d}s, \end{aligned} \tag{9}$$

which together with Gronwall's inequality implies

$$I_1 \le 2L\varDelta K t e^{Kt}\,;$$

similarly, we estimate I_2. These estimates also hold for $x(t, s, \varphi)$ and $x_k(t, s, \bar{\varphi}^k)$. This implies (8).

Let $T = T(M_1, M_1)$ and choose a prime N such that $N\omega > r + \omega + T$. Then from (8) we may assume that the following hold on S.

iv) $|x_k(t, \bar{\varphi}^k)| \le M_0 + 1/2$, whenever $t \in [0, (N+2)\omega] = J$ and $|\varphi| \le a + b$;

v) $|y_k(t, \bar{\varphi}^k)| \le B + 1/2$, whenever $t \in [N\omega - r, (N+1)\omega]$, $|\psi_1| \le a$ and $|\psi_2| \le b$, where $\varphi = (\psi_1, \psi_2)$;

vi) $|z(s, \bar{\varphi}^k) - g(s)| > 0$ and $|z(t, \bar{\varphi}^k)| > b$, whenever $s \in J$, $t \in [N\omega, (N+1)\omega)$, $|\psi_1| \le a$ and $b \le |\psi_2(0)| \le M_1 + M_0$.

Define $\varPi(\tilde{\varphi}^k)(t) = (\tilde{x}_k)_t(\cdot, \bar{\varphi}^k)$. Clearly, it is continuous in t and φ, since

$$(\tilde{x}_k)_t(\cdot, \bar{\varphi}^k) = (x_k(t + t_k, \bar{\varphi}^k), \dots, x_k(t + t_0, \bar{\varphi}^k))\,.$$

By iv)–vi) we have that on S,

1) $\varPi(\tilde{\varphi}^k)(t) \in S$, whenever $t \in J$ and $|\varphi| \le a + b$;

2) $|(\tilde{y}_k)_t(\cdot, \bar{\varphi}^k) \le B + 1$, whenever $t \in [N\omega, (N+1)\omega]$, $|\psi_1| \le a$ and $|\psi_2| \le b$, where $\varphi = (\psi_1, \psi_2)$;

3) $|(\tilde{z}_k)_s(\cdot, \bar{\varphi}^k) - \tilde{g}_s| > 0$ and $|(\tilde{z}_k)_t(\cdot, \bar{\varphi}^k)| > b$, whenever $s \in J$, $t \in [N\omega, (N+1)\omega)$, $|\psi_1| \le a$ and $b \le |\psi_2(0)| \le M_1 + M_0$.

1) and 2) are obvious. Note that for $\varphi \in S$, $x(t, \varphi) \in S'$; hence

$$|\overline{(x_k)_t}(\cdot, \bar{\varphi}^k) - (x_k)_t(\cdot, \bar{\varphi}^k)| \le 2L\varDelta \quad \text{for any} \quad t \in [0, (N+2)\omega]\,,$$

which together with (8) implies 3).

Let $D = (S_1(a) \times S_2(b))^{k+1}$. For any $p \in \bar{D}$, we have

$$\bar{p}(t) = p_i - \frac{s - t_{i-1}}{\varDelta}(p_{i+1} - p_i) \quad \text{for } s \in [t_{i+1}, t_i]\,, \quad i = 0, \dots, k-1\,.$$

Set

$$p_*(t) = p_i - (s - t_{i-1})\alpha\left(\frac{p_{i+1} - p_i}{\varDelta}\right) \quad \text{for } s \in [t_{i+1}, t_i]\,, \quad i = 0, \dots, k-1\,,$$

where $\alpha : R^n \to \bar{S}(0, L)$ is the usual continuous retract.

By 2), 3) and a similar argument as in the proof of Theorem 1, we obtain

$$\deg(\varPi(\cdot_*)(N\omega), D, 0) = 1\,. \tag{10}$$

Define

$$P(p) = (\tilde{x}_k)_\omega(\cdot, p_*) \quad \text{for any} \quad \bar{p} \in S'\,.$$

Then $P : \bar{D} \to R^{(k+1)n}$ is continuous, because of $|p_* - q_*| \le |\bar{p} - \bar{q}|$ for all p, q. Moreover, P is well defined for $|\bar{p}| \le a + b$.

Note that

$$(x_k)_\omega(\cdot, (x_k)_{i\omega}(\cdot, \bar{\varphi}^k)) = (x_k)_{(i+1)\omega}(\cdot, \bar{\varphi}^k)\,,$$

and on S,

$$P^{i+1}(p) = P^i \circ (\tilde{x}_k)_\omega(\cdot, p_*) = P^{i-1} \circ (\tilde{x}_k)_\omega(\cdot, \overline{(x_k)_\omega}(\cdot, p_*)\omega) .$$

Hence

$$\begin{aligned} |P^2(p) - \Pi(p_*)(2\omega)| \le\ & |(\bar{x}_k)_\omega(\cdot, (\bar{x}_k)_\omega(\cdot, p_*)) - (x_k)_\omega(\cdot, (\bar{x}_k)_\omega(\cdot, p_*))| \\ & +|(x_k)_\omega(\cdot, (\bar{x}_k)_\omega(\cdot, p_*)) - (x_k)_\omega(\cdot, (x_k)_\omega(\cdot, p_*))| \\ & +|(x_k)_{2\omega}(\cdot, p_*) - (\bar{x}_k)_{2\omega}(\cdot, p_*)| \\ \le\ & 4L\Delta + 2L\Delta K e^{K\omega} . \end{aligned} \tag{11}$$

Generally,

$$\begin{aligned} |P^i(p) - \Pi(p_*)(i\omega)| \le\ & |(\bar{x}_k)_\omega(\cdot, (\bar{x}_k)_\omega(\cdot, \cdots, (\bar{x}_k)_\omega(\cdot, p_*) \cdots)) \\ & -(x_k)_\omega(\cdot, (\bar{x}_k)_\omega(\cdot, \cdots, (\bar{x}_k)_\omega)(\cdot, p_*) \cdots))| + \cdots \\ & +|(x_k)_\omega(\cdot, (x_k)_\omega(\cdot, \cdots, (\bar{x}_k)_\omega(\cdot, p_*) \cdots)) \\ & -(x_k)_\omega(\cdot, (x_k)_\omega(\cdot, \cdots, (x_k)_\omega(\cdot, p_*) \cdots))| \\ & +|(x_k)_{i\omega}(\cdot, p_*) - (\bar{x}_k)_{i\omega}(\cdot, p_*)| \\ \le\ & 4L\Delta + 2L\Delta K^2 e^{2K\omega} + \cdots + 2L\Delta K^{i-1} e^{(i-1)K\omega} = \varepsilon_{ik} . \end{aligned} \tag{12}$$

Hence

$$P^i(p) \in S(\Pi(p_*)(i\omega), \varepsilon_{ik}) , \quad 0 \le i \le N+2 , \tag{13}$$

where $S(p, s)$ denotes the open ball of $R^{(k+1)n}$ centered at p with radius s.

By (10) and (13), for k large enough,

$$\deg(P^N, D, 0) = \deg(\Pi(\cdot_*)(N\omega), D, 0) = 1 . \tag{14}$$

From 2), 3) and (13) we have that P^N has no fixed point in ∂D. By (13) and the choice of N, T and M_1, for k large enough, each fixed point p of P^N in $\bar{D}$ satisfies that $\bar{p} = p_*$. We claim that $P(p) \in D$. If this fails, then there would hold: $q = P(p) = (q_1, q_2) \notin D$. By 2), 3) and (13), $q_1 \in (S_1(a))^{k+1}$, $q_2 \notin (S_2(b))^{k+1}$. Note that $|x(t, p_*)| \le M_0$, $t \in [0, \omega]$. By 3), (13) and the choice of N,

$$z((N-1)\omega, q) \notin \bar{S}_2(b) ,$$

and hence

$$p = P^{N-1}(q) \notin D ,$$

a contradiction. By the modular degree theorem and (14),

$$0 \ne \deg(P, D, 0) = \deg(P^N, D, 0) \pmod N .$$

Hence P has a fixed point $p_k \in D$, i.e.,

$$p_k = P(p_k) = (\tilde{x}_k)_\omega(\cdot, (p_k)_*) ,$$

which shows that $\bar{p}_k = (p_k)_* \in S$. Applying the Arzela-Ascoli theorem, we may assume

$$p_k \to \varphi \quad \text{as} \quad k \to \infty ,$$

in C. Note

$$|\overline{(x_k)^k_\omega}(\cdot, \bar{\varphi}) - \overline{(x_k)^k_\omega}(\cdot, \bar{p}_k)| \le |\bar{\varphi} - \bar{p}_k| e^{K\omega} .$$

Then

$$
\begin{aligned}
|x_\omega(\cdot,\varphi) - \varphi| \leq\ & |x_\omega(\cdot,\varphi) - (x_k)_\omega(\cdot,\varphi)| \\
& + |(x_k)_\omega(\cdot,\varphi) - (x_k)_\omega(\cdot,\bar{\varphi})| + |(x_k)_\omega(\cdot,\bar{\varphi}) - \overline{(x_k)^k}_\omega(\cdot,\bar{\varphi})| \\
& + |\overline{(x_k)^k}_\omega(\cdot,\bar{\varphi}) - \overline{(x_k)^k}_\omega(\cdot,\bar{p}_k)| + |\bar{\varphi} - \bar{p}_k| \\
\leq\ & 6L\Delta K e^{K\omega} + (1 + e^{K\omega})|\bar{\varphi} - \bar{p}_k| \to 0\,,
\end{aligned} \tag{15}
$$

as $k \to \infty$. From uniqueness it follows that

$$x(t+\omega,\varphi) = x(t,\varphi) \quad \text{for any} \quad t \in R\,,$$

that is, $x(t,\varphi)$ in an ω-periodic solution of (4). This completes the proof.

4. Equilibria for nonpermanent systems. Deterministic modelling in the biological sciences often reduces to ordinary differential equations, e.g., an ecological differential equation with state space R^n_+:

$$x_i' = x_i f_i(x) = F(x)\,. \tag{16}$$

We assume F satisfies a local Lipschitz condition. For such a system, it is important to discover if permanence implies the existence of equilibria. A positive answer has been proved by several authors [7] and [8].

Roughly speaking, the system (16) is said to be permanent if there are $A_i < B_i$, $i = 1,\dots,n$, such that for any solution $x(t,x_0)$, $x_0 \in R^n_+$, there is a $T = T(x_0) > 0$ for which

$$A_i \leq x_i(t,x_0) \leq B_i\,, \quad i = 1,\dots,n\,,$$

whenever $t \geq T$. Hence such a system is dissipative.

In this section, we will prove that dissipative-repulsive systems also admit equilibria. Let us state our result as follows.

THEOREM 3. *Let* $0 < A_i < B_i < \infty$, $i = 1,\dots,n$, *and* $0 < m, l, m + l = n$. *Let* $A_i < c_i < B_i$, $i = m+1,\dots,n$. *Assume that for any* $a_i' < a_i < A_i$ *and* $B_i < b_i < b_i'$, $i = 1,\dots,n$, *there is a* $T > 0$ *such that*

i) $x(s,x_0) \in R^n_+$, *whenever* $t \geq 0$ *and* $x_0 \in R^n_+$;

ii) $x_i(t,x_0) \in [A_i,B_i]$, $i = i,\dots,m$, *whenever* $t \geq T$ *and* $(x_0)_i \in [a_i,b_i]$, $i = 1,\dots,n$;

iii) $x_i(s,x_0) \neq c_i$ *and* $x_i(t,x_0) \notin [a_i,b_i]$, $i = m+1,\dots,n$, *whenever* $s \geq 0$, $t \geq T$, $(x_0)_i \in [a_i,b_i]$, $i = 1,\dots,m$, *and* $a_j' \leq (x_0)_j \leq a_j$ *or* $b_j \leq (x_0)_j \leq b_j'$ *for some* $m+1 \leq j \leq n$.

Then (16) *admits an equilibrium* p *with* $A_i \leq p_i \leq B_i$, $i = 1,\dots,n$.

PROOF. Put

$$\omega = \frac{1}{k}\,, \quad k > 1\,, \quad a_i = \frac{1}{2}A_i\,, \quad a_i' = h_-\,, \quad b_i = B_i + 1\,, \quad b_i' = b_i + h_+\,,$$

where h_- and h_+ satisfy

$$h_- \leq x_i(t,x_0) \leq h_+\,,$$

whenever $t \in [0,1]$ and $a_i \le (x_0)_i \le b_i$, $i = 1, \dots, n$. Set $P(x_0) = x(\omega, x_0)$, $D = (a_1, b_1) \times \cdots \times (a_n, b_n)$. By a similar argument to that in the proof of Theorem 1, P has a fixed point $p_k \in D$. By compactness, we may assume $p_k \to p$ as $k \to \infty$. Since $x(1/k, p_k) = p_k$, it follows that $F(p) = 0$, as desired.

Finally, let us make some comments.

REMARK 1. The existence of the continuous curve $g(t)$ in above theorems is necessary, otherwise the following is a counterexample:

Consider the equation

$$y' = -y, \quad z' = 1.$$

Clearly, $y(t, x_0)$ is dissipative, $z(t, x_0) = z_0 + t$ is repulsive, there is no such continuous periodic curve $g(t)$, and this equation also has no periodic solution.

REMARK 2. The case $m = n$ corresponds to some well-known theorems for dissipative systems [15], [6] and [2].

REMARK 3. For functional differential equations with infinite delay, there should be some similar results. This will require a correspondent phase space theory [1].

REMARK 4. If we combine our approach with some theories about differential inclusions (e.g., [10] and [12]), then we can obtain similar results for differential inclusions.

5. Example.

Consider the system

$$\begin{aligned} y' &= -y^3 + e^{-z^2} + \sin t = F_1(t, y, z), \\ z' &= z + y^2 + \cos t = F_2(t, y, z). \end{aligned} \tag{17}$$

Set

$$B = 2, \quad d = 0, \quad r_0 = 6, \quad g(t) \equiv 0.$$

Note

$$\operatorname{sgn} a \cdot F_1(t, a, z) < -5 \quad \text{for any} \quad t, z \in R \quad \text{and} \quad |a| \ge 2.$$

Then

$$|y(t, y_0, z_0)| < y_0 \quad \text{for any} \quad t \ge 0, \quad |y_0| \ge 3 \quad \text{and} \quad z_0 \in R, \tag{18}$$

and

$$|y(t, y_0, z_0)| \le 2 \quad \text{for any} \quad t \ge T_1 = \frac{a}{10\pi}, \quad |y_0| \le a \quad \text{and} \quad z_0 \in R. \tag{19}$$

Put

$$b = a^2 + 2.$$

Then

$$|z(t, y_0, z_0)| > b \quad \text{for any} \quad t \ge 0, \quad |y_0| \le a \quad \text{and} \quad z_0 \ge b. \tag{20}$$

By (19) and (20), we can apply Theorem 1 to conclude that (17) admits a 2π-periodic solution.

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