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KÄHLER-EINSTEIN METRICS FOR MANIFOLDS WITH NONVANISHING FUTAKI CHARACTER

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Abstract. In this paper, we generalize the concept of Kähler-Einstein metrics for Fano manifolds with nonvanishing Futaki character. Similar to Kähler-Einstein metrics, these new metrics have various nice properties. In addition, the equations for the metrics are in general neither those of extremal Kähler metrics nor those of Kähler-Ricci solitons.

1. Introduction. For an n-dimensional compact complex connected manifold M with $c_1(M)_{\boldsymbol{R}} > 0$, we consider the set $\mathcal{K}$ of all Kähler forms in the class $2\pi c_1(M)_{\boldsymbol{R}}$ such that the associated groups of the isometries are maximal compact subgroups* in the identity component $\mathrm{Aut}^0(M)$ of the holomorphic automorphisms of M. Let $\omega \in \mathcal{K}$. To such ω, we can associate a real-valued smooth function f_ω on M such that

$$\mathrm{Ric}(\omega) = \omega + \sqrt{-1}\partial\bar{\partial} f_\omega \quad \text{and} \quad \int_M e^{f_\omega}\omega^n = \int_M \omega^n , \tag{1.1}$$

where $\mathrm{Ric}(\omega) = \sqrt{-1}\bar{\partial}\partial \log(\omega^n)$. Let $\mathfrak{g} := H^0(M, \mathcal{O}(TM))$ be the complex Lie algebra of all holomorphic vector fields on M. Then the Futaki character $F : \mathfrak{g} \to \boldsymbol{C}$ defined by

$$F(X) := (\sqrt{-1}\,)^{-1} \int_M (X f_\omega)\omega^n/n! , \qquad X \in \mathfrak{g} , \tag{1.2}$$

is independent of the choice of ω in $\mathcal{K}$. The group $G := \mathrm{Aut}(M)$ of all holomorphic automorphisms of M has the Lie algebra $\mathfrak{g}$ above. We now write

$$\omega = \sqrt{-1} \sum_{\alpha,\beta} g_{\alpha\bar{\beta}}\, dz^\alpha \wedge dz^{\bar{\beta}}$$

in terms of a system $(z^1, z^2, \dots, z^n)$ of holomorphic local coordinates on M. To each complex-valued smooth function φ on M, we can associate a complex vector field $\mathrm{grad}^{\boldsymbol{C}}_\omega \varphi$ of type $(1, 0)$ on M by

$$\mathrm{grad}^{\boldsymbol{C}}_\omega \varphi := \frac{1}{\sqrt{-1}} \sum_{\alpha,\beta} g^{\bar{\beta}\alpha} \frac{\partial \varphi}{\partial z^{\bar{\beta}}} \frac{\partial}{\partial z^\alpha} .$$

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* Just to define the concept of Kähler-Einstein forms for manifolds with nonvanishing Futaki character, the maximal compactness of the group of the isometries is not necessary, as long as the condition (1.4) is guaranteed. Because, by an argument similar to [C1; p.109], the condition (1.4) automatically implies the maximal compactness, in $\mathrm{Aut}^0(M)$, of the group of the isometries.

Then the vector space $\tilde{\mathfrak{g}}_\omega := \{\varphi \in C^\infty(M)_{\boldsymbol{C}} ; \operatorname{grad}^{\boldsymbol{C}}_\omega \varphi \in \mathfrak{g}, \int_M \varphi\omega^n = 0\}$ has a natural structure of a complex Lie algebra in terms of the Poisson bracket by ω. Moreover,

$$\tilde{\mathfrak{g}}_\omega \cong \mathfrak{g}\,, \quad \varphi \leftrightarrow \operatorname{grad}^{\boldsymbol{C}}_\omega \varphi, \tag{1.3}$$

is an isomorphism of complex Lie algebras. By abuse of terminology, we say that ω is a *Kähler-Einstein* form on M if

$$1 - e^{f_\omega} \in \tilde{\mathfrak{g}}_\omega\,, \tag{1.4}$$

where Kähler-Einstein forms are often called Kähler-Einstein metrics, since a Kähler form and the associated Kähler metric are used interchangeably throughout this paper. Take a *Kähler-Einstein* form ω on M in the sense of (1.4). If $F = 0$, by setting $\varphi = 1 - e^{f_\omega}$, we have $\operatorname{grad}^{\boldsymbol{C}}_\omega \varphi \in \mathfrak{g}$ by (1.3), and in view of (1.2),

$$0 = F(\operatorname{grad}^{\boldsymbol{C}}_\omega \varphi) = \int_M e^{f_\omega}(\bar\partial f_\omega, \bar\partial f_\omega)\,\omega^n/n!\,,$$

which implies $f_\omega = 1$ by (1.1), i.e., ω is a Kähler-Einstein form in an ordinary sense. It is thus expected that, even for $F \neq 0$, the above definition of Kähler-Einstein forms has some good meaning.

In this paper, we shall show that even for $F \neq 0$, Kähler-Einstein forms in the above sense have several nice properties as those in an ordinary sense.

This work, except Section 6, was done during my stay at International Centre for Mathematical Sciences (ICMS), Edinburgh in 1997. I thank especially Professor Michael Singer who invited me to give lectures at ICMS on various subjects of Kähler-Einstein metrics.

2. Extremal Kähler vector fields. For $\mathcal{K}$ as in Section 1, let $\omega \in \mathcal{K}$. Let σ_ω be the scalar curvature $\sum g^{\bar\alpha\beta} R_{\beta\bar\alpha}$ of ω, where $R_{\beta\bar\alpha} := -(\partial^2/\partial z^\beta \partial z^{\bar\alpha})(\log \omega^n)$. Moreover, we put

$$\tilde{\mathfrak{k}}_\omega := \{\varphi \in \tilde{\mathfrak{g}}_\omega\,;\ \varphi \text{ is a real-valued function on } M\,\}\,.$$

Then its natural image, denoted by $\mathfrak{k}$, in $\mathfrak{g}$ by the isomorphism in (1.3) coincides with the space of all Killing vector fields on the Kähler manifold (M, ω). Let $\tilde{\mathfrak{k}}^\perp_\omega$ be the orthogonal complement of $\tilde{\mathfrak{k}}_\omega$ in the Hilbert space $L^2(M, \omega)_{\boldsymbol{R}}$ of all real-valued L^2 functions on (M, ω), and let $\operatorname{pr} : L^2(M, \omega)_{\boldsymbol{R}}\ (= \tilde{\mathfrak{k}}_\omega \oplus \tilde{\mathfrak{k}}^\perp_\omega) \to \tilde{\mathfrak{k}}_\omega$ be the corresponding orthogonal projection. Then the image $v^\omega \in \mathfrak{k}$ of the element $\operatorname{pr}(\sigma_\omega - n)$ in $\tilde{\mathfrak{k}}_\omega$ by the isomorphism $\tilde{\mathfrak{g}}_\omega \cong \mathfrak{g}$ in (1.3) is called the *extremal Kähler vector field* on M, which is unique up to conjugacy in $\mathfrak{g}$ (see [FM]). We here show that v^ω is simultaneously the image of $\operatorname{pr}(1 - e^{f_\omega})$ by the isomorphism $\tilde{\mathfrak{g}}_\omega \cong \mathfrak{g}$. The proof of this is reduced to showing the following:

THEOREM 2.1. $\operatorname{pr}(\sigma_\omega - n) = \operatorname{pr}(1 - e^{f_\omega})$.

Before proving this, we fix notation. For the Kähler manifold (M, ω), let $\square_\omega$ denote its complex Laplacian $\sum g^{\bar\alpha\beta}\partial^2/\partial z^\beta \partial z^{\bar\alpha}$ on functions. By setting

$$\tilde{\Box}_\omega := \Box_\omega + \sum_{\alpha,\beta} g^{\bar\beta\alpha} \frac{\partial f_\omega}{\partial z^\alpha} \frac{\partial}{\partial z^{\bar\beta}} ,$$

$$\mathrm{Ker}_F(\tilde{\Box}_\omega + 1) := \{\varphi \in C^\infty(M)_F\, ;\ (\tilde{\Box}_\omega + 1)\varphi = 0\}, \quad (\text{where } F = \boldsymbol{R}, \boldsymbol{C}),$$

we see that $\mathrm{Ker}_F(\tilde{\Box}_\omega + 1)$ forms a Lie algebra in terms of the Poisson bracket by the Kähler form ω. Note that the equalities (cf. [F1])

$$\int_M (\tilde{\Box}_\omega \varphi_1)\bar\varphi_2 e^{f_\omega}\omega^n = \int_M \varphi_1 \overline{(\tilde{\Box}_\omega \varphi_2)} e^{f_\omega}\omega^n = -\int_M e^{f_\omega}(\bar\partial\varphi_1, \bar\partial\varphi_2)\omega^n \tag{2.2}$$

holds for all $\varphi_1, \varphi_2 \in C^\infty(M)_{\boldsymbol{C}}$. Hence, all eigenvalues of the operator $-\tilde{\Box}_\omega$ on the space $C^\infty(M)_{\boldsymbol{C}}$ of complex functions on M are nonnegative and real. We naturally have isomorphisms of Lie algebras

$$\mathrm{Ker}_{\boldsymbol{R}}(\tilde{\Box}_\omega + 1) \cong \tilde{\mathfrak{k}}_\omega \cong \mathfrak{k}, \qquad \varphi \leftrightarrow \varphi - \hat\varphi \leftrightarrow \mathrm{grad}^{\boldsymbol{C}}_\omega \varphi , \tag{2.3}$$

$$\mathrm{Ker}_{\boldsymbol{C}}(\tilde{\Box}_\omega + 1) \cong \tilde{\mathfrak{g}}_\omega \cong \mathfrak{g}, \qquad \varphi \leftrightarrow \varphi - \hat\varphi \leftrightarrow \mathrm{grad}^{\boldsymbol{C}}_\omega \varphi , \tag{2.4}$$

where $\hat\varphi := (\int_M \omega^n)^{-1} \int_M \varphi\, \omega^n$. If $\mathfrak{g} \neq \{0\}$, then the first positive eigenvalue of the operator $-\tilde{\Box}_\omega$ on the space $C^\infty(M)_{\boldsymbol{C}}$ of complex functions on M is 1.

PROOF OF THEOREM 2.1. Let ψ be an arbitrary element of $\tilde{\mathfrak{k}}_\omega$. Then ψ is written as $\varphi - \hat\varphi$ for some $\varphi \in \mathrm{Ker}_{\boldsymbol{R}}(\tilde{\Box}_\omega + 1)$. We now have

$$\begin{aligned}\int_M \psi(1 - e^{f_\omega})\omega^n &= \int_M \varphi(1 - e^{f_\omega})\omega^n = \int_M (\tilde{\Box}_\omega\varphi)e^{f_\omega}\omega^n - \int_M (\tilde{\Box}_\omega\varphi)\omega^n \\ &= -\int_M (\tilde{\Box}_\omega\varphi)\omega^n = -(\bar\partial\varphi\, , \bar\partial f_\omega)_{L^2} = (\psi\, , \Box_\omega f_\omega)_{L^2} = \int_M \psi(\sigma_\omega - n)\omega^n .\end{aligned}$$

It is now easy to see that $\mathrm{pr}(\sigma_\omega - n) = \mathrm{pr}(1 - e^{f_\omega})$, as required. □

This theorem shows that $v^\omega = \mathrm{grad}^{\boldsymbol{C}}_\omega \mathrm{pr}(\sigma_\omega - n) = \mathrm{grad}^{\boldsymbol{C}}_\omega \mathrm{pr}(1 - e^{f_\omega}) \in \tilde{\mathfrak{k}}_\omega$. Recall that the corresponding real vector field $v^\omega_{\boldsymbol{R}} := v^\omega + \bar v^\omega$ on M satisfies

$$\exp(2\pi m v^\omega_{\boldsymbol{R}}) = \mathrm{id}_M \tag{2.5}$$

for some positive integer m (see [FM]), and that the equality $v^\omega_{\boldsymbol{R}} = 0$ holds if and only if Futaki's obstruction vanishes. In particular, we have:

(1) If $v^\omega_{\boldsymbol{R}} \neq 0$, then $v^\omega_{\boldsymbol{R}}$ generates an S^1-action on M.

(2) $\alpha_M := \max_M \mathrm{pr}(\sigma_\omega - n) = \max_M \mathrm{pr}(1 - e^{f_\omega})$ is independent of the choice of ω in $\mathcal{K}$, so that α_M is a holomorphic invariant of the Fano manifold M.

Since we have $\int_M \mathrm{pr}(1 - e^{f_\omega})\omega^n = 0$ by $\mathrm{pr}(1 - e^{f_\omega}) \in \tilde{\mathfrak{g}}_\omega$, the inequality $\alpha_M \geq 0$ always holds. The identity (2.5) together with (1) and (2) above is called the *strict periodicity* of the extremal Kähler vector fied on M.

3. Obstruction of Futaki's type. Let $\mathcal{E}$ be the set of all Kähler-Einstein forms in $\mathcal{K}$ in the sense of Section 1, i.e., $\mathcal{E} := \{\omega \in \mathcal{K}\, ;\ 1 - e^{f_\omega} \in \tilde{\mathfrak{g}}_\omega\}$. Let $\omega \in \mathcal{E}$. Then by

$1 - e^{f_\omega} = \mathrm{pr}(1 - e^{f_\omega})$ and $e^{f_\omega} > 0$,

$$1 > \max_M \mathrm{pr}(1 - e^{f_\omega}) = \alpha_M,$$

where α_M is the holomorphic invariant of M defined in the previous section. Therefore, we have the following analogue of Futaki's obstruction:

THEOREM 3.1. *If $\mathcal{E} \neq \emptyset$, then $\alpha_M < 1$.*

REMARK 3.2. The Futaki character $F : \mathfrak{g} \to \boldsymbol{C}$ of M vanishes if and only if $\alpha_M = 0$. This is true even when $\mathcal{E}$ is empty. To see this, let $\omega \in \mathcal{K}$. If $F = 0$, then by [FM], the extremal Kähler vector field v^ω is zero, i.e., $\mathrm{pr}(1 - e^{f_\omega}) = 0$, and hence $\alpha_M = 0$. On the other hand, if $\alpha_M = 0$, then by

$$0 = \alpha_M \geq \mathrm{pr}(1 - e^{f_\omega})$$

and $\int_M \mathrm{pr}(1 - e^{f_\omega})\omega^n = 0$, we have $\mathrm{pr}(1 - e^{f_\omega}) = 0$, i.e., $F = 0$ as required.

REMARK 3.3. There is actually a Fano manifold for which this obstruction of Futaki's type does not vanish. This will be discussed in detail in Section 6. The existence of such an example suggests that the concept of Kähler-Einstein forms in the sense of Section 1 will be more closely related to the stability of complex manifolds than the concept of Kähler-Ricci solitons (see for instance [K1], [G1] for Kähler-Ricci solitons).

4. Obstruction of Matsushima's type. If $\mathcal{E} \neq \emptyset$, we have a decomposition theorem of the Lie algebra $\mathfrak{g}$ as shown for extremal Kähler metrics by [C1]. In this section, we use the same notation as in the previous sections. Let $\mathfrak{k}^{\boldsymbol{C}}$ be the complexification of $\mathfrak{k}$ in $\mathfrak{g}$. For nonnegative rational numbers μ, put

$$\mathfrak{g}(\mu) := \mathrm{Ker}\{\mathrm{ad}(\sqrt{-1}\, v^\omega) - \mu \mathrm{id}_{\mathfrak{g}}\} = \{X \in \mathfrak{g}\, ; [\sqrt{-1} v^\omega , X] = \mu X\}.$$

We set further $\lambda_0 := 0$ for simplicity. Then the Lie subalgebra $\mathfrak{g}(\lambda_0)$ of $\mathfrak{g}$ is just the centralizer $Z_{\mathfrak{g}}(v^\omega)$ of v^ω in $\mathfrak{g}$.

THEOREM 4.1. *Assume $\mathcal{E} \neq \emptyset$ and let $\omega \in \mathcal{E}$. For some nonnegative integer r, there exists a sequence of rational numbers $0 = \lambda_0 < \lambda_1 < \lambda_2 \cdots < \lambda_r$ such that*

(1) $\mathfrak{k}^{\boldsymbol{C}} = \mathfrak{g}(\lambda_0) = Z_{\mathfrak{g}}(v^\omega)$;

(2) $\mathfrak{g}$ is, as a vector space, nothing but the direct sum $\bigoplus_{i=0}^r \mathfrak{g}(\lambda_i)$.

PROOF. Note that $\mathfrak{g} = \mathfrak{k}^{\boldsymbol{C}} \ltimes \mathfrak{u}$ for the unipotent radical $\mathfrak{u}$ of $\mathfrak{g}$. If $\mathfrak{u} = \{0\}$, then we are done. Therefore, we may assume that $\mathfrak{u} \neq \{0\}$. By the strict periodicity of the extremal Kähler vector field on M, there exists an increasing sequence of rational numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_r$ for some $r > 0$ such that $\mathfrak{u}$ is, as a vector space, written as a direct sum $\bigoplus_{i=1}^r \mathfrak{g}(\lambda_i)$. Then by $\mathfrak{k}^{\boldsymbol{C}} \subset Z_{\mathfrak{g}}(v^\omega)$ (cf. [FM]), the proof of Theorem 4.1 is now reduced to showing $\lambda_1 > 0$.

Choosing an element $u \neq 0$ in $\mathfrak{g}(\lambda_1)$, we have

$$[\sqrt{-1}\, v^\omega, u] = \lambda_1 u\,. \tag{4.2}$$

On the other hand, there uniquely exist elements u_0, v_0 of $\mathrm{Ker}_{\boldsymbol{C}}(\tilde{\square}_\omega + 1)$ such that u, v^ω are respectively the images in $\mathfrak{g}$ of u_0, v_0 by the Lie algebra isomorphism $\mathrm{Ker}_{\boldsymbol{C}}(\tilde{\square}_\omega + 1) \cong \mathfrak{g}$ in

(2.4). In view of $\omega \in \mathcal{E}$, we have

$$1 - e^{f_\omega} = \mathrm{pr}(1 - e^{f_\omega}) = v_0 - \hat{v}_0 ,$$

where $\hat{v}_0$ is a constant as in Section 2. We first observe that the complex conjugate $\bar{u}_0$ of $u_0 \neq 0$ does not belong to $\mathrm{Ker}_{\boldsymbol{C}}(\tilde{\square}_\omega + 1)$. Because otherwise, both the real part $\mathrm{Re}\, u_0$ and the imaginary part $\mathrm{Im}\, u_0$ of u_0 would belong to $\mathrm{Ker}_{\boldsymbol{R}}(\tilde{\square}_\omega + 1)$, and hence $u \in \mathfrak{k}^{\boldsymbol{C}}$ in contradiction to $0 \neq u \in \mathfrak{u}$. On the other hand, by $u_0 \in \mathrm{Ker}_{\boldsymbol{C}}(\tilde{\square}_\omega + 1)$,

$$\int_M \bar{u}_0 e^{f_\omega} \omega^n = \overline{\int_M u_0 e^{f_\omega} \omega^n} = -\overline{\int_M (\tilde{\square}_\omega u_0) e^{f_\omega} \omega^n} = 0 .$$

Note that all eigenvalues of $-\tilde{\square}_\omega$ are nonnegative real numbers and its first positive eigenvalue is 1 (cf. [F1]). Therefore,

$$\int_M (-\tilde{\square}_\omega \bar{u}_0) u_0 \, e^{f_\omega} \omega^n > \int_M |u_0|^2 e^{f_\omega} \omega^n . \tag{4.4}$$

Again by $u \in \mathrm{Ker}_{\boldsymbol{C}}(\tilde{\square}_\omega + 1)$,

$$\int_M (-\tilde{\square}_\omega u_0) \bar{u}_0 \, e^{f_\omega} \omega^n = \int_M |u_0|^2 e^{f_\omega} \omega^n . \tag{4.5}$$

By (4.4) and (4.5), $\int_M \{(-\tilde{\square}_\omega \bar{u}_0) u_0 + (\tilde{\square}_\omega u_0) \bar{u}_0\} e^{f_\omega} \omega^n > 0$. This inequality together with (2.2) implies

$$\int_M 2\sqrt{-1}\{(\mathrm{Im}\tilde{\square}_\omega) u_0\} \bar{u}_0 e^{f_\omega} \omega^n > 0 , \tag{4.6}$$

where $\mathrm{Re}\tilde{\square}_\omega$, $\mathrm{Im}\tilde{\square}_\omega$ are respectively the real part and the imaginary part of $\tilde{\square}_\omega$, so that $\tilde{\square}_\omega = \mathrm{Re}\tilde{\square}_\omega + \sqrt{-1}\, \mathrm{Im}\tilde{\square}_\omega$. A simple computation shows that

$$2\sqrt{-1}\{(\mathrm{Im}\tilde{\square}_\omega) u_0\} \, e^{f_\omega} = (\sqrt{-1})^{-1} [e^{f_\omega} , u_0] = [\sqrt{-1}\, v_0, u_0] , \tag{4.7}$$

where the Poisson bracket is defined as in [FM]. By (4.2), $[\sqrt{-1}\, v_0, u_0] = \lambda_1 u_0$. In view of (4.6) and (4.7), we now obtain $\lambda_1 > 0$ as required. □

This decomposition theorem is regarded as a variant of Matsushima's obstruction. I have, however, a suspicion that Theorem 4.1 is true for any Fano manifold M even if $\mathcal{E}$ is empty. We therefore pose the following:

PROBLEM 4.8. Either prove the above decomposition theorem for every Fano manifold M or disprove it by giving a counter-example.

REMARK 4.9. If M is an n-dimensional toric Fano variety, there are affirmative answers. Namely, the above decomposition theorem is true in the follwing cases:

(1) $n = 2$ and M is an orbifold (cf. [N1]);
(2) $n = 3$ and M is nonsingular (cf. [S1], [N1]);
(3) $n = 4$ and M is nonsingular (cf. [N1]).

5. Existence. In this section, by solving an ODE, we give an example of a Kähler-Einstein form on a manifold with nonvanishing Futaki character (cf. [C1], [H1], [KS], [G1], [M1]).

Let W be a k-dimensional compact complex connected manifold with a Kähler-Einstein form ϕ in an ordinary sense such that $\mathrm{Ric}(\phi) = \phi$. Let L be a holomorphic line bundle over W with Hermitian fibre metric h such that all eigenvalues, say $\mu_1, \mu_2, \dots, \mu_k$, of the curvature form $\mathrm{Ric}(h) = \sqrt{-1}\bar{\partial}\partial \log h$ are constant with respect to ϕ on W. Using vector bundles and locally free sheaves interchangeably, we shall now define a vector bundle $E := \mathcal{O}_W \oplus L$ of rank 2 over W. Consider the associated projective bundle $\tilde{M} := \boldsymbol{P}.(E)$ consisting of all those lines in the fibres of E which hit the locus of the zero section of E. Then L and L^{-1} are naturally regarded as a Zariski-open subset of $\tilde{M}$ in such a way that

$$\tilde{D}_0 = \boldsymbol{P}.(\mathcal{O}_W \oplus \{0\}) = \tilde{M} \setminus L^{-1}\,,$$
$$\tilde{D}_\infty = \boldsymbol{P}.(\{0\} \oplus L\,) = \tilde{M} \setminus L\,,$$

where $\tilde{D}_0$ and $\tilde{D}_\infty$ are respectively called the zero section and the infinity section of the total space $\boldsymbol{P}.(E)$. Consider the Hermitian norm

$$\rho : L \to \boldsymbol{R}_{\geq 0}\,, \quad \ell \mapsto \rho(\ell) := \|\ell\|_h\,,$$

on L induced by h. Then ρ is regarded as a function on the Zariski-open subset $\tilde{M} \setminus \tilde{D}_\infty$ of $\tilde{M}$. Note that the $\boldsymbol{C}^*$-action on L by scalar multiplication extends naturally to a $\boldsymbol{C}^*$-action on $\tilde{M}$ such that both $\tilde{D}_0$ and $\tilde{D}_\infty$ are the fixed point set of the $\boldsymbol{C}^*$-action on $\tilde{M}$. As in [M1], we consider a smooth $\boldsymbol{C}^*$-equivariant blowing-down M of $\tilde{M}$ possibly with the case $M = \tilde{M}$. More precisely, let $\sigma : \tilde{M} \to M$ be the $\boldsymbol{C}^*$-equivariant blowing-up of the Fano manifold M along the nonsingular center $D_0 := \sigma(\tilde{D}_0)$ and $D_\infty := \sigma(\tilde{D}_\infty)$. Put $n_0 := \mathrm{codim}_M D_0$ and $n_\infty := \mathrm{codim}_M D_\infty$. Moreover, put $\tilde{M}^0 := \tilde{M} \setminus (\tilde{D}_0 \cup \tilde{D}_\infty)$ and $M^0 := M \setminus (D_0 \cup D_\infty)$. Since σ maps $\tilde{M}^0$ isomorphically onto M^0, we hereafter identify $\tilde{M}^0$ with M^0. Let $r : M^0 (= \tilde{M}^0) \to \boldsymbol{R}$ be the smooth function defined by

$$r = -\log(\rho^2)\,. \tag{5.1}$$

Let $\mathrm{p} : \tilde{M} \to W$ be the natural projection. Then the pullback $\mathrm{p}^*\phi$ is, when restricted to $\tilde{M}^0$, regarded as a 2-form on M^0.

We are looking for a Kähler-Einstein form ω on M in the sense defined in Section 1. Put $n := k + 1 = \dim_{\boldsymbol{C}} M$. The form ω is written as $\mathrm{Ric}(\eta)$ for some Kähler form η on M in the class $2\pi c_1(M)_{\boldsymbol{R}}$. By restricting η to M^0, we now write η^n as

$$\eta^n = \sqrt{-1} n e^{-y} (\mathrm{p}^*\phi)^k \partial r \wedge \bar{\partial} r\,, \tag{5.2}$$

where $y = y(r)$ is a smooth function in r. By the same computation as in [M1], we have $\omega = \mathrm{p}^*\phi + y'(r)\mathrm{Ric}(h) + \sqrt{-1} y''(r) \partial r \wedge \bar{\partial} r$, and hence

$$\omega^n = \mathrm{Ric}(\eta)^n = \sqrt{-1} \left\{ n y''(r) \prod_{i=1}^{k} (1 + \mu_i y'(r)) \right\} (\mathrm{p}^*\phi)^k \partial r \wedge \bar{\partial} r\,. \tag{5.3}$$

Note that $y'(r)$ extends to a real-valued smooth function, denoted also by $y' = y'(r)$ by abuse of terminology, on M. This $y'(r)$, which maps M onto the closed interval $[-n_\infty, n_0]$, defines the moment map for the Kähler manifold (M, ω) with respect to the $\boldsymbol{C}^*$-action on M. Then with the notation in Section 2, we have $y'(r) \in \mathrm{Ker}_{\boldsymbol{R}}(\tilde{\Box}_\omega + 1)$ and in particular

$$y' - C_0 \in \tilde{\mathfrak{g}}_\omega\,, \tag{5.4}$$

for some real constant C_0, so that $\mathrm{grad}^{\boldsymbol{C}}_\omega y'(r) \in \mathfrak{g}$. Look at the identities (5.2), (5.3) together with the equality $e^{f_\omega} = \eta^n/\omega^n$. Then as the equation for ω to be a Kähler-Einstein form in the sense of Section 1, we have the following:

$$e^{-y}\left\{y''\prod_{i=1}^{k}(1+\mu_i y')\right\}^{-1} = C_1 + C_2 y'\,, \tag{5.5}$$

where C_1, C_2 are real constants which will be specified later. In fact, if (5.5) holds, then by combining $C_1 + C_2 y' = e^{-y}\{y''\prod_{i=1}^k(1+\mu_i y')\}^{-1} = e^{f_\omega}$ with (5.4), we obtain $e^{f_\omega} - 1 \in \tilde{\mathfrak{g}}_\omega$, i.e., ω is a Kähler-Einstein form in the sense of Section 1, as required. For brevity, we put

$$b_\alpha := \int_{-n_\infty}^{n_0} p^\alpha \prod_{i=1}^{k}(1+\mu_i p)dp\,, \quad \alpha = 0, 1, 2\,.$$

Note that the equality $0 < 1 + \mu_i p$ holds for all i whenever $-n_\infty < p < n_0$ (see for instance [M1]). Then by the Schwarz inequality, $b_1^2 < b_0 b_2$ holds. Moreover, by $\int_M e^{f_\omega}\omega^n = \int_M \omega^n$ and $e^{f_\omega} = C_1 + C_2 y'$, we have

$$C_1 b_0 + C_2 b_1 = b_0\,. \tag{5.6}$$

On the other hand, by $y'(r) \in \mathrm{Ker}_{\boldsymbol{R}}(\tilde{\Box}_\omega + 1)$ together with (2.2), we obtain $\int_M y'(r)e^{f_\omega}\omega^n = -\int_M(\tilde{\Box}_\omega y'(r))e^{f_\omega}\omega^n = 0$. Hence, the compatibility condition

$$C_1 b_1 + C_2 b_2 = 0 \tag{5.7}$$

holds. Recall that the Futaki character of M vanishes if and only if $b_1 = 0$.

We now divide the whole situation into the following two cases:

Case 1. $b_1 = 0$. In this case, by (5.6) and (5.7), we put $(C_1, C_2) = (1, 0)$, where the compatibility condition (5.7) above reduces to the vanishing $b_1 = 0$ of the Futaki character. Then it is well-known (see for instance [M1]) that the equation for M to admit a Kähler-Einstein metric in an ordinary sense is exactly (5.5).

Case 2. $b_1 \neq 0$. Then M does not admit a Kähler-Einstein metric in an ordinary sense. Let us exclude the case $-n_\infty \le b_2/b_1 \le n_0$, so that either $b_2/b_1 > n_0$ or $b_2/b_1 < -n_\infty$. Now by (5.6) and (5.7), it follows immediately that $(C_1, C_2) = (b_0 b_2/(b_0 b_2 - b_1^2), -b_0 b_1/(b_0 b_2 - b_1^2))$. Then the right-hand side of (5.5) is bounded from below by some positive real number.

We now solve the equation (5.5) with (C_1, C_2) as above. Define a polynomial $A = A(x)$ in $x \in \boldsymbol{R}$ by

$$A(x) := -\int_{-n_\infty}^{x} p(C_1 + C_2\, p)\prod_{i=1}^{k}(1+\mu_i p)dp\,.$$

By the condition (5.7), we have $A(n_0) = A(-n_\infty) = 0$. In view of [M1], the order of zeroes of $A(x)$ at $x = n_0$ (resp. $x = -n_\infty$) is n_0 (resp. n_∞). Note also that both $0 < A(x) \le A(0)$ and $A'(x)/x < 0$ holds for all nonzero x with $-n_\infty < x < n_0$. In particular, the rational function $A'(x)/(xA(x))$ is free from poles and zeroes over the open interval $(-n_\infty, n_0)$, and has a pole of order 1 at both $x = -n_\infty$ and $x = n_0$. Then

$$B(x) := -\int_{-n_\infty}^{x} A'(p)/(pA(p))dp$$

is monotone increasing over the interval $(-n_\infty, n_0)$, and B maps $(-n_\infty, n_0)$ diffeomorphically onto $\boldsymbol{R}$, because in a neighbourhood of $x = n_0$ (resp. $x = -n_\infty$), $B(x)$ is written as $-\log(n_0 - x)$ + real analytic function (resp. $\log(n_\infty + x)$ + real analytic function). Let $B^{-1} : \boldsymbol{R} \to (-n_\infty, n_0)$ be the inverse function of $B : (-n_\infty, n_0) \to \boldsymbol{R}$. We define a smooth function $x = x(r)$ in r by

$$x(r) := B^{-1}(r)\,, \quad r \in \boldsymbol{R}\,.$$

Since $x'(r) = -x(r)A(x(r))/A'(x(r))$, by setting $u(r) := -\log(A(x))$, we obtain $u'(r) = x(r)$. Then by $A'(x(r)) = -x(r)(C_1 + C_2x(r))\prod_{i=1}^{k}(1 + \mu_i x(r))$, we have

$$u''(r)(C_1 + C_2u'(r))\prod_{i=1}^{k}(1+\mu_i u'(r)) = \exp(-u(r))\,,$$

i.e., $y = u(r)$ satisfies (5.5). Moreover, $u''(r) = -x(r)A(x(r))/A'(x(r))$ is a real analytic function in $x(r)$ which is nonvanishing on $(-n_\infty, n_0)$ and has a zero of order 1 at both $x = n_0$ and $x = -n_\infty$.

Let us now assume

(1) The function ρ (resp. ρ^{-1}) on M^0 extends to a smooth function on $M \setminus D_\infty$ (resp. $M \setminus D_0$), and

(2) $\rho^{2n_0}(\mathrm{p}^*\phi)^k\partial r \wedge \bar\partial r$ (resp. $\rho^{-2n_\infty}(\mathrm{p}^*\phi)^k\partial r \wedge \bar\partial r$) extends to a smooth nonvanishing $2(k+1)$-form on $M \setminus D_\infty$ (resp. $M \setminus D_0$),

where these conditions are satisfied, for instance, if $n_0 = n_\infty = 1$. Then by (2), for the solution $y = u(r)$ for (5.5), the Ricci form $\omega = \mathrm{p}^*\phi + u'(r)\mathrm{Ric}(h) + \sqrt{-1}u''(r)\partial r \wedge \bar\partial r$ of $\sqrt{-1}n\, e^{-y}(\mathrm{p}^*\phi)^k\partial r \wedge \bar\partial r = \sqrt{-1}\, n\, e^{-u(r)}\,(\mathrm{p}^*\phi)^k\partial r \wedge \bar\partial r$ extends to a smooth Kähler form on M, which is a Kähler-Einstein form on M in the sense of Section 1.

EXAMPLE 5.8. Let $W = \boldsymbol{P}^k(\boldsymbol{C})$ and $L = \mathcal{O}_{\boldsymbol{P}^k}(1)$, where $n_0 = n_\infty = 1$. Then $M = \boldsymbol{P}_{\cdot}(\mathcal{O}_{\boldsymbol{P}^k} \oplus \mathcal{O}_{\boldsymbol{P}^k}(1))$ admits a Kähler-Einstein metric in the sense of Section 1 as follows. For the standard Hermitian metric h for L, we have $\mu_1 = \mu_2 = \cdots = \mu_k = 1/n$, where

$n = k + 1$. Moreover,

$$b_\alpha = \int_{-1}^{1} p^\alpha (1 + p/n)^k dp > 0\,, \quad \alpha = 0, 1, 2\,,$$

and we can easily check that $b_2 > b_1 > 0$. Now by $b_2/b_1 > 1 = n_0$ and $n_0 = n_\infty = 1$, the above solution of (5.5) applied to this case gives an explicit example of a Kähler-Einstein form in the sense of Section 1. Note also that M has nonvanishing Futaki character by $b_1 \neq 0$.

EXAMPLE 5.9. Let $W = \boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^2(\boldsymbol{C})$ and $L = \mathcal{O}(1, -1)$, where $\mathcal{O}(1, -1)$ denotes the line bundle $p_1^*\mathcal{O}_{\boldsymbol{P}^1}(1) \otimes p_2^*\mathcal{O}_{\boldsymbol{P}^2}(-1)$ on W, and $p_i : \boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^2(\boldsymbol{C}) \to \boldsymbol{P}^i(\boldsymbol{C})$, $i = 1, 2$, are the natural projections. Moreover, let $n_0 = n_\infty = 1$. Then we may choose $(\mu_1, \mu_2, \mu_3) = (1/2, -1/3, -1/3)$, and b_1, b_2 are computed as follows:

$$b_1 = -4/45\,, \quad b_2 = 26/45\,.$$

By $b_2/b_1 = -13/2 < -1 = -n_\infty$, the above solution of (5.5) applied to this case gives again an explicit example of a Kähler-Einstein form in the sense of Section 1. This M also has nonvanishing Futaki character by $b_1 \neq 0$.

6. A Fano manifold M satisfying $\alpha_M > 1$. We here consider exactly the same situation as in the last section except that, in this section, M does not necessarily admit a Kähler-Einstein metric in the sense of Section 1. We in particular keep the same notation as in the last section. For simplicity, we further assume that W is a k-dimensional irreducible Hermitian symmetric space of compact type, where W obviously admits a Kähler-Einstein form ϕ in an ordinary sense such that $\mathrm{Ric}(\phi) = \phi$. As in the last section, we consider a Kähler form $\omega := \mathrm{Ric}(\eta)$ on M, where η is a Kähler form on M in the class $2\pi c_1(M)_{\boldsymbol{R}}$ such that the restriction to M^0 of the volume form η^n is given by (5.2). We write

$$\omega \;=\; \mathrm{p}^*\phi + y'(r)\mathrm{Ric}(h) + \sqrt{-1}\, y''(r)\, \partial r \wedge \bar{\partial} r\,.$$

Recall that h is a Hermitian fibre metric for the line bundle L such that all eigenvalues μ_1, $\mu_2, \dots, \mu_k$ of the curvature form $\mathrm{Ric}(h)$ are constant with respect to ϕ on W. Recall further that $y'(r)$ maps M onto the closed interval $[-n_\infty, n_0]$, and this defines the moment map for the Kähler manifold (M, ω) with respect to the $\boldsymbol{C}^*$-action on M. In particular, $y'(r) \in \mathrm{Ker}_{\boldsymbol{R}}(\tilde{\Box}_\omega + 1)$.

Now, using the notation in Section 2, we consider the finite-dimensional vector subspace $S := \boldsymbol{R} \oplus \tilde{\mathfrak{k}}_\omega$ of the Hilbert space $L^2(M, \omega)_{\boldsymbol{R}}$, where $\boldsymbol{R}$ denotes the space of real constant functions on M. Let $\mathrm{pr}_S : L^2(M, \omega)_{\boldsymbol{R}} (= S \oplus S^\perp) \to S$ be the orthogonal projection. Then

$$y'(r) \in \mathrm{Ker}_{\boldsymbol{R}}(\tilde{\Box}_\omega + 1) \subset S\,. \tag{6.1}$$

Since the extremal Kähler vector field v^ω is in the center of $\mathfrak{k}$, we can find real constants C_1, C_2 such that

$$\mathrm{pr}_S(e^{f_\omega}) \;= C_1 + C_2 y'(r)\,. \tag{6.2}$$

By (6.1), we have $\int_M y'(r)\,\mathrm{pr}_S(e^{f_\omega})\,\omega^n = \int_M y'(r)\,e^{f_\omega}\omega^n = -\int_M (\tilde{\Box}_\omega y'(r))\,e^{f_\omega}\omega^n = 0$, and in view of (6.2), we obtain the equality

$$(6.3) \qquad C_1 b_1 + C_2 b_2 = 0,$$

where b_0, b_1, b_2 are the same as in the last section. Recall the inequality $b_0 b_2 > b_1^2$. By $\int_M \mathrm{pr}_S(e^{f_\omega})\,\omega^n = \int_M e^{f_\omega}\,\omega^n = \int_M \omega^n$, we further obtain

$$(6.4) \qquad C_1 b_0 + C_2 b_1 = b_0.$$

We here consider the case where $b_1 \neq 0$, i.e., the Futaki character is nonvanishing. Then as in Case 2 of the last section, it follows from (6.3) and (6.4) that

$$(C_1, C_2) = (b_0 b_2/(b_0 b_2 - b_1^2), -b_0 b_1/(b_0 b_2 - b_1^2)).$$

Since $\mathrm{pr}(1 - e^{f_\omega}) = \mathrm{pr}_S(1 - e^{f_\omega}) = 1 - C_1 - C_2\,y'(r)$, and since $y'(r)$ maps M onto $[-n_\infty, n_0]$, the invariant $\alpha_M = \max_M \mathrm{pr}(1 - e^{f_\omega})$ is given by the following:

THEOREM 6.5. *In the above situation, we have*

$$\alpha_M = \begin{cases} 1 + (-b_0 b_2 + b_0 b_1 n_0)\cdot(b_0 b_2 - b_1^2)^{-1} & \textit{if } b_1 > 0, \\ 1 + (-b_0 b_2 - b_0 b_1 n_\infty)\cdot(b_0 b_2 - b_1^2)^{-1} & \textit{if } b_1 < 0, \end{cases}$$

and in particular $\alpha_M > 1$ if and only if $-n_\infty < b_2/b_1 < n_0$. Moreover, $\alpha_M = 1$ if and only if $(b_2/b_1 + n_\infty)(b_2/b_1 - n_0) = 0$.

EXAMPLE 6.6. Let $W = \boldsymbol{P}^2(\boldsymbol{C})$ and $L = \mathcal{O}_{\boldsymbol{P}^2}(2)$, where $n_0 = n_\infty = 1$. Then the Fano manifold $M = \boldsymbol{P}_{.}(\mathcal{O}_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(2))$ admits no Kähler-Einstein forms in the sense of Section 1, because the inequality $\alpha_M > 1$ holds as follows. For the standard Hermitian metric h for L, we have $\mu_1 = \mu_2 = 2/3$. Therefore,

$$b_1 = 8/9, \quad b_2 = 38/45,$$

and we can easily check that $-1 = -n_\infty < b_2/b_1 < 1 = n_0$. Thus, $\alpha_M > 1$.

7. The equation for "Kähler-Einstein forms" is a second order PDE. Let M and $\mathcal{K}$ be as in Section 1, and fix a maximal compact subgroup K of $\mathrm{Aut}(M)$. Consider the set $\mathcal{K}^K$ of all K-invariant Kähler forms in $\mathcal{K}$, and let $C^\infty(M)^K_{\boldsymbol{R}}$ denote the space of all K-invariant functions in $C^\infty(M)_{\boldsymbol{R}}$. Take a Kähler form $\omega \in \mathcal{K}^K$. For each $\varphi \in C^\infty(M)^K_{\boldsymbol{R}}$, we put $\omega_\varphi := \omega + \sqrt{-1}\,\partial\bar\partial\varphi$. The equation for $\omega_\varphi \in \mathcal{K}^K$ to be a Kähler-Einstein form in an ordinary sense is known to be

$$(7.1) \qquad \frac{(\omega + \sqrt{-1}\,\partial\bar\partial\varphi)^n}{\omega^n} = e^{-\varphi + f_\omega}.$$

If the Futaki character of M does not vanish, then (7.1) above has no solutions. Let $v^\omega \in \mathfrak{g}$ be the extremal Kähler vector field as in Section 2, and let $\tilde{v}^\omega$ be the corresponding element of $\tilde{\mathfrak{g}}_\omega$ by the isomorphism $\mathfrak{g} \cong \tilde{\mathfrak{g}}_\omega$. Note that v^ω is independent of the choice of ω in $\mathcal{K}^K$, and hence v^ω will be written simply as V. Then by [FM; p. 208],

$$\tilde{v}^\omega + \sqrt{-1}\,V\varphi = \tilde{v}^{\omega_\varphi}.$$

Now, the equation for $\omega_\varphi \in \mathcal{K}$ to belong to $\mathcal{E}$, i.e., to be a Kähler-Einstein form in the sense of Section 1 is

$$(7.2) \qquad \frac{(\omega + \sqrt{-1}\,\partial\bar\partial\varphi)^n}{\omega^n} = \frac{e^{-\varphi+f_\omega}}{1 - \tilde{v}^\omega - \sqrt{-1}\,V\varphi}.$$

Because, taking the $\sqrt{-1}\,\bar\partial\partial\log$ of both sides of (7.2), we obtain $\mathrm{Ric}(\omega_\varphi) - \mathrm{Ric}(\omega) = \sqrt{-1}\,\partial\bar\partial\varphi - \sqrt{-1}\,\partial\bar\partial f_\omega + \sqrt{-1}\,\partial\bar\partial\log(1 - \tilde{v}^{\omega_\varphi})$ and this together with $\mathrm{Ric}(\omega) = \omega + \sqrt{-1}\,\partial\bar\partial f_\omega$ implies $\sqrt{-1}\,\partial\bar\partial f_{\omega_\varphi} = \mathrm{Ric}(\omega_\varphi) - \omega_\varphi = \sqrt{-1}\,\partial\bar\partial\log(1 - \tilde{v}^{\omega_\varphi})$, and hence

$$e^{f_{\omega_\varphi}} = 1 - \tilde{v}^{\omega_\varphi}, \text{ i.e., } \omega_\varphi \in \mathcal{E}.$$

In view of Section 3, we may assume that $\alpha_M < 1$, where α_M is the holomorphic invariant of M as defined in Section 2. Note that the denominator of the right-hand side of (7.2) is $1 - \tilde{v}^{\omega_\varphi}$, and it is bounded from below by a positive real number, since

$$\max_M \tilde{v}^{\omega_\varphi} = \alpha_M < 1.$$

Note also that the equation (7.2) is a second order PDE. Moreover, Kähler-Einstein forms in the sense of Section 1 have various good properties about uniqueness. For instance, the arguments in [BM] go through also for Kähler-Einstein forms in the sense of Section 1, except that some special care must be taken in handling a priori estimates. (This delicate part was pointed out in ICMS by Tian.) These will be treated elsewhere (cf. [M2]).

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