Tohoku Math. J.
58 (2006), 369–391

CHARACTERIZATION OF WAVE FRONT SETS BY WAVELET TRANSFORMS

STEVAN PILIPOVIĆ AND MIRJANA VULETIĆ

(Received October 21,2004, revised November 25, 2005)

Abstract. We consider a special wavelet transform of Moritoh and give new definitions of wave front sets of tempered distributions via that wavelet transform. The major result is that these wave front sets are equal to the wave front sets in the sense of Hörmander in the cases $n = 1, 2, 4, 8$. If $n \in \boldsymbol{N} \setminus \{1, 2, 4, 8\}$, then we combine results for dimensions $n = 1, 2, 4, 8$ and characterize wave front sets in ξ-directions, where ξ are presented as products of non-zero points of $\boldsymbol{R}^{n_1}, \dots, \boldsymbol{R}^{n_s}$, $n_1 + \cdots + n_s = n$, $n_i \in \{1, 2, 4, 8\}$, $i = 1, \dots, s$. In particular, the case $n = 3$ is discussed through the fourth-dimensional wavelet transform.

1. Introduction. In this paper, emphasis is put on the characterization of wave front sets via wavelet transforms. We refer to [2] for the local analysis of functions and distributions through wavelet expansions and wavelet transforms and to [3, 6] for the local and microlocal analysis through wavelet transforms.

The paper is inspired by Moritoh [7], where a wavelet transform of a distribution $f \in \mathcal{S}'(\boldsymbol{R}^n)$ is defined by

$$W_\psi f(x, \xi) = \int_{\boldsymbol{R}^n} f(t)|\xi|^{n/2}\overline{\psi(|\xi|R_\xi(t-x))}\,dt, \quad (x, \xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \setminus \{0\},$$

where ψ is an analyzing wavelet and $R_\xi \in SO(n)$ maps $\xi/|\xi|$ to $e_n = (0, \dots, 0, 1)$. Integral is interpreted in a distributional sense.

This definition of wavelet transform can be obtained from Murenzi's definition [8]. In Murenzi's definition the wavelet transform involves dilatation, translation and rotation as parameters, while Moritoh fixes rotation (with a special choice of R_ξ) and keeps dilatation and translation as parameters.

In [7], the change of variables $\omega = |\xi|^{-1}R_\xi\tau$ satisfies $d\omega/|\omega|^n = d\xi/|\xi|^n$. This is not true for all rotations that were used in [7].

The aim of this paper is twofold. First, for dimensions $1, 2, 4$ and 8, we improve the results of [7] concerning the estimates of wave fronts introducing a parameterized wavelet transform and by an intrinsic analysis of transformations of variables in the frequency domain. The second aim is to extend results to dimensions $n \neq 1, 2, 4, 8$. Actually, we transfer the results for quoted dimensions to general $n \in \boldsymbol{N}$, excluding some directions in the frequency domain.

2000 *Mathematics Subject Classification.* Primary 46F12; Secondary 43A32.
Key words and phrases. Wavelet transform, wave front.

We start in Section 2 by examining the existence of rotations $R_\xi \in O(n)$ such that $R_\xi(\xi) = |\xi|e_n$ holds for every $\xi \in \boldsymbol{R}^n\backslash\{0\}$. It will be shown that if we suppose that the mapping $\xi \mapsto R_\xi$ is continuous, then its existence is limited only to dimensions $n = 1, 2, 4$ and 8. In these cases, explicit constructions of such rotations are given. The proof that no rotations with previously stated property exist for $n \neq 1, 2, 4$ and 8 was given by Wagner [10].

Although it is not stated in [7] that the mapping $\xi \mapsto R_\xi$ has to be continuous, we find that it had been used implicitly. Without continuity it would be hard to follow how some relevant sets, for example conic neighborhoods, are transformed through different changes of variables. Even if we allow discontinuity of $\xi \mapsto R_\xi$ we can construct rotations such that $R_\xi(\xi/|\xi|) = e_n$, but $d\omega/|\omega|^n = d\xi/|\xi|^n$ is not satisfied (at points of continuity). In $\boldsymbol{R}^3$ one such mapping would be a rotation around the line p that passes through the origin and is orthogonal to ξ and e_3.

In Section 3 we introduce an analyzing wavelet ψ and a wavelet transform W_ψ associated with it for the cases $n = 1, 2, 4$ and 8. Wavefront sets WF_ψ and $WF_\psi^{(s)}$ have been introduced by means of this wavelet transform. In these definitions the sets WF_ψ and $WF_\psi^{(s)}$ depend on ψ. It is shown in main theorems that $WF_\psi = WF$ and $WF_\psi^{(s)} = WF^{(s)}$, where WF and $WF^{(s)}$ are defined by Hörmander (see [4, 5]). Moritoh achieved in [7] only lower and upper bounds for (Hörmander's) wave front sets in terms of his wave front sets and the wavelet transforms. In order to obtain an exact description of the wave front sets, we introduce a parameter in the wavelet transform; if the parameter equals one, then the wavelet transform is that defined in [7]. This parameter plays an essential role in achieving independence of WF_ψ and $WF_\psi^{(s)}$ on ψ. We can say that, the construction in [7] tends to overshoot Hörmander's wave front sets by about a conic set obtained from the support of the Moritoh wavelet. By introducing this parameter, this was corrected.

In Section 4 we overcome the fact that our results were limited to cases $n = 1, 2, 4$ and 8. Definitions and theorems from the previous section are extended to the cases with general $n \in \boldsymbol{N}$. In this case some directions had to be omitted in order to have the right characterization of wave front sets. We discuss this question in Remarks 16 and 19. In particular, in the case $n = 3$, we show that all the directions can be analyzed through the analysis of the fourth-dimensional case.

In the proofs of main theorems some properties of rotations R_ξ were used. We give these properties in the form of lemmas in Sections 5 and 6.

2. Rotations. Let $\xi \in \boldsymbol{R}^n\backslash\{0\}$. For each ξ we associate the following matrices for:

- $n = 1$,

$$R_\xi = \frac{1}{|\xi|}[\xi]\,;$$

- $n = 2$,

$$R_\xi = \frac{1}{|\xi|}\begin{bmatrix} \xi_2 & -\xi_1 \\ \xi_1 & \xi_2 \end{bmatrix};$$

- $n = 4$,

$$R_\xi = \frac{1}{|\xi|}\begin{bmatrix} -\xi_4 & -\xi_3 & \xi_2 & \xi_1 \\ -\xi_3 & \xi_4 & \xi_1 & -\xi_2 \\ -\xi_2 & \xi_1 & -\xi_4 & \xi_3 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 \end{bmatrix};$$

- $n = 8$,

$$R_\xi = \frac{1}{|\xi|}\begin{bmatrix} -\xi_8 & -\xi_7 & -\xi_6 & \xi_5 & -\xi_4 & \xi_3 & \xi_2 & \xi_1 \\ -\xi_7 & \xi_8 & -\xi_5 & -\xi_6 & \xi_3 & \xi_4 & \xi_1 & -\xi_2 \\ -\xi_6 & \xi_5 & \xi_8 & \xi_7 & -\xi_2 & \xi_1 & -\xi_4 & -\xi_3 \\ -\xi_5 & -\xi_6 & \xi_7 & -\xi_8 & \xi_1 & \xi_2 & -\xi_3 & \xi_4 \\ -\xi_4 & -\xi_3 & \xi_2 & \xi_1 & \xi_8 & -\xi_7 & \xi_6 & -\xi_5 \\ -\xi_3 & \xi_4 & \xi_1 & -\xi_2 & -\xi_7 & -\xi_8 & \xi_5 & \xi_6 \\ -\xi_2 & \xi_1 & -\xi_4 & \xi_3 & \xi_6 & -\xi_5 & -\xi_8 & \xi_7 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 & \xi_8 \end{bmatrix}.$$

In the cases $n = 2, 4$ and 8 they represent rotations that belong to the groups $SO(2)$, $SO(4)$ and $SO(8)$, i.e., they are orthogonal matrices whose determinant is equal to 1. For $n = 1$ the matrix R_ξ is an orthogonal matrix whose determinant can be either 1 or -1. They satisfy the following.

(i) $\boldsymbol{R}^n \setminus \{0\} \ni \xi \mapsto R_\xi \in O(n)$ is continuous.

(ii) $R_\xi(\xi/|\xi|) = e_n,\ \xi \in \boldsymbol{R}^n \setminus \{0\}$.

(iii) Let $\xi \in \boldsymbol{R}^n \setminus \{0\}$. For every $\tau \in \boldsymbol{R}^n \setminus \{0\}$ there exists $\bar{R}_\tau \in O(n)$ such that

$$\frac{R_\xi(\tau)}{|\tau|} = \frac{\bar{R}_\tau(\xi)}{|\xi|}.$$

(iv) Let $\tau \in \boldsymbol{R}^n \setminus \{0\}$ and $\omega = |\xi|^{-1} R_\xi(\tau), \xi \in \boldsymbol{R}^n \setminus \{0\}$. Then $d\omega/|\omega|^n = d\xi/|\xi|^n$.

These are easy to show and we omit the proof.

We only give a descrription of $\bar{R}_\tau$. Let $n = 1, 2, 4$ or 8 and $\xi, \tau \in \boldsymbol{R}^n \setminus \{0\}$. Then

$$\omega = \frac{1}{|\xi|} R_\xi(\tau) = \frac{1}{|\xi|^2}\begin{bmatrix} \langle \xi^{(1)}, \tau\rangle \\ \langle \xi^{(2)}, \tau\rangle \\ \vdots \\ \langle \xi^{(n)}, \tau\rangle \end{bmatrix},$$

where the vectors $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(n)}$ are row vectors of $|\xi|R_\xi$. Obviously,

$$\xi^{(j)} = (i_{j1}\xi_{\pi^{(j)}(1)}, i_{j2}\xi_{\pi^{(j)}(2)}, \ldots, i_{jn}\xi_{\pi^{(j)}(n)}) ,$$

where $\pi^{(j)}$ is a permutation of $\{1, 2, \ldots, n\}$, and $i_{j1}, \ldots, i_{jn} \in \{1, -1\}$.

If we define $\tau^{(j)} = (i_{j\pi^{(j)}(1)}\tau_{\pi^{(j)}(1)}, i_{j\pi^{(j)}(2)}\tau_{\pi^{(j)}(2)}, \ldots, i_{j\pi^{(j)}(n)}\tau_{\pi^{(j)}(n)})$, then

$$\langle \xi^{(j)}, \tau \rangle = \langle \xi, \tau^{(j)} \rangle$$

and the matrix

$$\bar{R}_\tau = \frac{1}{|\tau|}(\tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(n)})^t$$

satisfies (iii).

We want to emphasize that matrices satisfying properties (i), (ii), (iii) and (iv) are not uniquely determined. These properties remain true if we multiply some of the rows (except the last one) by -1. New matrices obtained in this manner could be used instead of the quoted ones and all the results presented in this paper hold with these matrices.

So, to summarize, for $n = 1, 2, 4$ and 8 there exists a continuous mapping $S : \boldsymbol{R}^n \backslash \{0\} \to Gl(n)$, $\xi \mapsto S_\xi$, such that

$$S_\xi(\xi) \| e_n , \quad \xi \in \boldsymbol{R}^n \backslash \{0\} .$$

One such mapping is given by $S_\xi = R_\xi$.

A natural question arising from the construction of rotations R_ξ is whether it is possible to construct rotations in other dimensions in a similar way. The answer is negative. The proof uses the fact that S^{n-1} is parallelizable which only holds if $n = 2, 4$ or 8 (see [1]). Recall that a differentiable manifold of dimension n is parallelizable if there exist n vector fields that at each point form a basis for the tangent space at that point.

PROPOSITION 1 ([10]). *Let $n \in \boldsymbol{N} \backslash \{1, 2, 4, 8\}$. Then there does not exist continuous mapping $S : S^{n-1} \to Gl(n)$ such that for every $\xi \in S^{n-1}$ the vector $S_\xi(\xi)$ is parallel to e_n (S_ξ stands for $S(\xi)$). The proposition remains true if we substitute e_n with an arbitrary non-zero vector.*

PROOF. Suppose the contrary, i.e., suppose that such S exists.

Let $\xi \in S^{n-1}$. As $S_\xi \in Gl(n)$ it holds that $T_\xi = S_\xi^{-1}$. From $\det T_\xi \neq 0$ it follows that $T_\xi(e_1), T_\xi(e_2), \ldots, T_\xi(e_n)$ are linearly independent vectors. As $T_\xi(e_n)$ is parallel to ξ, it follows that $T_\xi(e_1), T_\xi(e_2), \ldots, \xi$ form a set of linearly independent vectors.

For each $k \in \{1, \ldots, n-1\}$, we define a mapping f_k such that $f_k : S^{n-1} \to \boldsymbol{R}^n$ and

$$f_k(\xi) = T_\xi(e_k) - \langle T_\xi(e_k), \xi \rangle \xi , \quad \xi \in S^{n-1} .$$

This gives us

$$\langle f_k(\xi), \xi \rangle = \langle T_\xi(e_k), \xi \rangle - \langle T_\xi(e_k), \xi \rangle \langle \xi, \xi \rangle = 0 , \quad \xi \in S^{n-1} .$$

As a consequence, $f_1(\xi), f_2(\xi), \ldots, f_{n-1}(\xi)$ are linearly independent and they are all orthogonal to ξ. This means that $\{f_1(\xi), f_2(\xi), \ldots, f_{n-1}(\xi)\}$ is a basis of the tangent space of the sphere S^{n-1} at the point ξ, which is not possible. □

We find that the continuity condition for $\xi \mapsto R_\xi$, $\xi \in \boldsymbol{R}^n \setminus \{0\}$, is essential for our needs. We need to follow how certain sets, such as conic neighborhoods, are transformed by particular changes of variables that involve these rotations. It would be hard to follow this if we would allow discontinuities. For that reason we do not define R_ξ for dimensions different from $n = 1, 2, 4$ or 8.

3. Wavelet transform and wave front set.

3.1. Wavelet transform. By the definition, $\psi \in L^2(\boldsymbol{R}^n)$ is an admissible analyzing wavelet if

$$C_\psi = (2\pi)^n \int_{\boldsymbol{R}^n} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|^n} \, d\xi < \infty . \tag{1}$$

Our analyzing wavelet is similar to those of [7] and is a function $\psi \in \mathcal{S}(\boldsymbol{R}^n)$ satisfying:

(1) $\widehat{\psi} \in C_0^\infty(\boldsymbol{R}^n)$, $\widehat{\psi} \geqslant 0$; and

(2) $\Omega = \operatorname{supp} \widehat{\psi}$ does not contain 0 and $\widehat{\psi}(e_n) \neq 0$, where $e_n = (0, \ldots, 0, 1)$.

It follows from $\widehat{\psi} \in C_0^\infty(\boldsymbol{R}^n)$ that $\psi \in \mathcal{S}(\boldsymbol{R}^n)$, which implies that the microlocal properties are better localized in the frequency domain than in the time domain.

Wavelets satisfying Properties 1 and 2 can be easily constructed in the following way. It suffices to construct a smooth non-negative function ϕ whose support is included in $B_r(e_n)$, where $0 < r < 1$. Then $\psi \in \mathcal{S}(\boldsymbol{R}^n)$, where $\widehat{\psi} = \phi$, is one such wavelet.

We restrict our consideration in this section to dimensions $n = 1, 2, 4$ or 8. For the wavelet ψ, we define a wavelet transform W_ψ of a distribution $f \in \mathcal{S}'(\boldsymbol{R}^n)$ by

$$W_\psi f(x, \xi) = \int_{\boldsymbol{R}^n} f(t) |\xi|^{n/2} \overline{\psi(|\xi| R_\xi (t - x))} \, dt , \quad (x, \xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \setminus \{0\} ,$$

where integration is understood in the distributional sense, as for a fixed (x, ξ) the function $t \mapsto \psi(|\xi| R_\xi (t - x))$ is an element of $\mathcal{S}(\boldsymbol{R}^n)$. Mappings R_ξ are rotations in $O(n)$ that have been described more precisely in Section 2.

It is easy to verify that the Fourier transform of $W_\psi f$ with respect to x is given by

$$\widehat{W_\psi f(x, \xi)}(\tau, \xi) = (2\pi)^{n/2} \widehat{f}(\tau) |\xi|^{-n/2} \widehat{\psi}(|\xi|^{-1} R_\xi(\tau)) , \quad (\tau, \xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \setminus \{0\} ,$$

where τ stands for the variable corresponding, in the frequency domain, to the variable x. In particular,

$$W_\psi f(x, \xi) = \int_{\boldsymbol{R}^n} \widehat{f}(\tau) |\xi|^{-n/2} \widehat{\psi}(|\xi|^{-1} R_\xi(\tau)) e^{i\tau x} \, d\tau , \quad (x, \xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \setminus \{0\} .$$

Here we list several propositions without proofs. They were proven in [7] by making use of Property (iv) in the previous section. As already claimed, this property is not satisfied for all rotations used in [7]. As all R_ξ given in Section 2 satisfy Property (iv), the proofs of [7] are valid in these cases.

PROPOSITION 2 (Parseval's identity). *Let* $f, g \in L^2(\boldsymbol{R}^n)$. *Then* $W_\psi f \in L^2(\boldsymbol{R}^n \times \boldsymbol{R}^n \backslash \{0\})$ *and*

$$\int_{\boldsymbol{R}^n} \int_{\boldsymbol{R}^n} W_\psi f(x, \xi) \overline{W_\psi g(x, \xi)}\, dx d\xi = C_\psi \int_{\boldsymbol{R}^n} f(t) \overline{g(t)}\, dt\,,$$

where C_ψ *is given by* (1).

COROLLARY 3. *Wavelet transform* W_ψ *is an isometric transform of* $L^2(\boldsymbol{R}^n, dt)$ *to* $L^2(\boldsymbol{R}^n \times \boldsymbol{R}^n \backslash \{0\}, C_\psi^{-1}\, dx d\xi)$.

PROPOSITION 4 (Inverse formula). *Let* $f \in L^2(\boldsymbol{R}^n)$. *Then* f *can be expressed via* $W_\psi f$ *as*

$$f(t) = C_\psi^{-1} \int_{\boldsymbol{R}^n} \int_{\boldsymbol{R}^n} W_\psi f(x, \xi) |\xi|^{n/2} \overline{\psi(|\xi| R_\xi(t - x))}\, dx d\xi\,, \quad t \in \boldsymbol{R}^n\,.$$

PROPOSITION 5. *Let* $f \in \mathcal{S}'(\boldsymbol{R}^n)$. *Then* $f \in H^s(\boldsymbol{R}^n)$ *if and only if*

$$\int_{\boldsymbol{R}^n} \int_{\boldsymbol{R}^n} |W_\psi f(x, \xi)|^2 (1 + |\xi|^2)^s\, dx d\xi < \infty\,.$$

3.2. Wavefront set. In the reminder of this section, let ψ be an analyzing wavelet and let $r \in (0, 1)$ be such that supp $\widehat{\psi} \subseteq B_r(e_n)$. For $0 < \lambda < 1$ we define ψ_λ by

$$\psi_\lambda(x) = \lambda^n e^{i x e_n (1-\lambda)} \psi(\lambda x)\,, \quad x \in \boldsymbol{R}^n\,.$$

Put $\Omega_\lambda =$ supp $\widehat{\psi}_\lambda$. It is not hard to see that $\Omega_\lambda \subseteq B_{\lambda r}(e_n)$.

We denote $W_{\psi_\lambda}(f)$ by $W_{\psi,\lambda}(f)$. Let $\xi_0 \in \boldsymbol{R}^n \backslash \{0\}$ and $r_0 > 0$. In the sequel we use $\Gamma(\xi_0)$ to denote an arbitrary conic neighborhood of the point ξ_0 and $\Gamma(\xi_0, r_0)$ for

$$\Gamma(\xi_0, r_0) = \left\{ \xi;\, \left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| < r_0 \right\}.$$

Let $f \in \mathcal{S}'(\boldsymbol{R}^n)$. We define wave front sets $WF_\psi(f)$ and $WF_\psi^s(f)$ through the definition of the complements of these sets, as usual.

3.2.1. *WF*(f) *and* $WF_\psi(f)$. We start with defining our ψ-wave front set $WF_\psi(f)$.

DEFINITION 6. Let $f \in \mathcal{S}'(\boldsymbol{R}^n)$. $WF_\psi(f) \subseteq \boldsymbol{R}^n \times \boldsymbol{R}^n \backslash \{0\}$ is the complement of the set of ψ-microlocally regular points $(x_0, \xi_0) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \backslash \{0\}$, that is, $(x_0, \xi_0) \notin WF_\psi(f)$ if and only if

$$(\exists \phi \in C_0^\infty(\boldsymbol{R}^n)) \quad (\phi(x_0) \neq 0) \quad (\exists \Gamma(\xi_0)) \quad (\exists \lambda \in (0, 1)) \quad (\forall N > 0) \quad (\exists C_N > 0)$$
$$|W_{\psi,\lambda} \widehat{(\phi f)}(x, \xi)(\tau, \xi)| \leqslant C_N |\xi|^{-N}, \quad \tau \in \boldsymbol{R}^n,\ \xi \in \Gamma(\xi_0),\ |\xi| \geqslant 1\,.$$

Note that $(x_0, \xi_0) \notin WF_\psi(f)$ is equivalent to

$$(\exists \phi \in C_0^\infty(\boldsymbol{R}^n)) \quad (\phi(x_0) \neq 0) \quad (\exists \Gamma(\xi_0)) \quad (\exists \lambda \in (0, 1)) \quad (\forall N > 0) \quad (\exists C_N > 0)$$
$$|\widehat{\phi f}(\tau) \widehat{\psi}_\lambda(|\xi|^{-1} R_\xi(\tau))| \leqslant C_N |\xi|^{-N}\,, \quad \tau \in \boldsymbol{R}^n,\ \xi \in \Gamma(\xi_0),\ |\xi| \geqslant 1\,.$$

The next theorem shows that $WF_\psi(f)$ does not depend on ψ. In the proof we use results from Section 5.

THEOREM 7. *Let $f \in \mathcal{S}'(\boldsymbol{R}^n)$. Then $WF_\psi(f) = WF(f)$.*

PROOF. Let $(x_0, \xi_0) \notin WF(f)$. Let $\phi \in C_0^\infty(\boldsymbol{R}^n)$, $\phi(x_0) \neq 0$ and $\Gamma(\xi_0)$ satisfy that for each $N > 0$ there exists $C_N > 0$ such that

$$|\widehat{\phi f}(\xi)| \leqslant C_N |\xi|^{-N}, \quad \xi \in \Gamma(\xi_0),\ |\xi| \geqslant 1\,.$$

To prove that $(x_0, \xi_0) \notin WF_\psi(f)$ we need to find $\tilde{\Gamma}(\xi_0)$ and $\lambda \in (0, 1)$ (ϕ will be the same) such that for each $N > 0$, there exists $C'_N > 0$ satisfying

$$|\widehat{\phi f}(\tau)\widehat{\psi}_\lambda(|\xi|^{-1}R_\xi(\tau))| \leqslant C'_N|\xi|^{-N}, \quad \tau \in \boldsymbol{R}^n,\ \xi \in \tilde{\Gamma}(\xi_0),\ |\xi| \geqslant 1\,.$$

As the mapping $\tau \mapsto \omega = |\xi|^{-1}R_\xi(\tau)$ is a bijection of $\boldsymbol{R}^n$ onto itself, we can reformulate the problem and look for $\tilde{\Gamma}(\xi_0)$ and $\lambda \in (0, 1)$ such that for each $N > 0$ there exists $C'_N > 0$ satisfying

$$|\widehat{\phi f}(|\xi|R_\xi^{-1}(\omega))\widehat{\psi}_\lambda(\omega)| \leqslant C'_N|\xi|^{-N}, \quad \omega \in \boldsymbol{R}^n,\ \xi \in \tilde{\Gamma}(\xi_0),\ |\xi| \geqslant 1\,.$$

With the already introduced notation $\operatorname{supp}\widehat{\psi}_\lambda = \Omega_\lambda$, we further reformulate the problem and look for $\tilde{\Gamma}(\xi_0)$ and $\lambda \in (0, 1)$ such that for every $N > 0$ there exists $C'_N > 0$ satisfying

$$|\widehat{\phi f}(|\xi|R_\xi^{-1}(\omega))\widehat{\psi}_\lambda(\omega)| \leqslant C'_N|\xi|^{-N}, \quad \omega \in \Omega_\lambda,\ \xi \in \tilde{\Gamma}(\xi_0),\ |\xi| \geqslant 1\,. \tag{2}$$

Lemma 23 below implies that for a given $\Gamma(\xi_0)$ there exist $\tilde{\Gamma}(\xi_0)$ and λ such that $|\xi|R_\xi^{-1}(\omega) \in \Gamma(\xi_0)$ for every $\xi \in \tilde{\Gamma}(\xi_0)$ and $\omega \in \Omega_\lambda$. Thus, we conclude that for $\omega \in \Omega_\lambda,\ \xi \in \tilde{\Gamma}(\xi_0),\ |\xi| \geqslant 1$,

$$\begin{aligned} |\widehat{\phi f}(|\xi|R_\xi^{-1}(\omega))\widehat{\psi}_\lambda(\omega)| &\leqslant C|\widehat{\phi f}(|\xi|R_\xi^{-1}(\omega))| \\ &\leqslant CC_N(|\xi||R_\xi^{-1}(\omega)|)^{-N}\,, \end{aligned}$$

where we have used $(x_0, \xi_0) \notin WF(f)$. This continues as

$$\begin{aligned} &= CC_N|\omega|^{-N}|\xi|^{-N} \\ &\leqslant C'_N|\xi|^{-N} \end{aligned}$$

for $\omega \in \Omega_\lambda,\ \xi \in \tilde{\Gamma}(\xi_0),\ |\xi| \geqslant 1$. Thus, $(x_0, \xi_0) \notin WF_\psi(f)$.

Now, let $(x_0, \xi_0) \notin WF_\psi f$. Then there exist $\phi \in C_0^\infty(\boldsymbol{R}^n)$, $\phi(x_0) \neq 0$, $\Gamma(\xi_0, r_0)$ and $\lambda \in (0, 1)$, which satisfy that for each $N > 0$ there exists $C'_N > 0$ such that (2) is satisfied for $\tilde{\Gamma}(\xi_0) = \Gamma(\xi_0, r_0)$. As $e_n \in \Omega_\lambda$ and $\widehat{\psi}_\lambda(e_n) \neq 0$, we get for $\xi \in \Gamma(\xi_0, r_0),\ |\xi| \geqslant 1$,

$$\begin{aligned} |\widehat{\phi f}(\xi)| &= |\widehat{\phi f}(|\xi|R_\xi^{-1}(e_n))| \\ &\leqslant \frac{C'_N}{\widehat{\psi}_\lambda(e_n)}|\xi|^{-N}\,. \end{aligned}$$

Thus, $(x_0, \xi_0) \notin WF(f)$ and the proof is completed. □

3.2.2. *$WF^{(s)}(f)$ and $WF_\psi^{(s)}(f)$.* Now we define our Sobolev ψ-wave front set $WF_\psi^{(s)}(f)$.

DEFINITION 8. Let $f \in \mathcal{S}'(\boldsymbol{R}^n)$. Then $WF_{\psi}^{(s)}(f) \subseteq \boldsymbol{R}^n \times \boldsymbol{R}^n \backslash \{0\}$ is the complement of the set of Sobolev ψ-microlocally regular points $(x_0, \xi_0) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \backslash \{0\}$, that is, $(x_0, \xi_0) \notin WF_{\psi}^{(s)}(f)$ if and only if

$$(\exists \phi \in C_0^{\infty}(\boldsymbol{R}^n)) \quad (\phi(x_0) \neq 0) \quad (\exists \Gamma(\xi_0)) \quad (\exists \lambda \in (0,1))$$
$$\int_{\Gamma(\xi_0)} \int_{\boldsymbol{R}^n} |\widehat{W_{\psi,\lambda}(\phi f)}(\tau, \xi)|^2 (1 + |\xi|^2)^s \, d\tau d\xi < \infty .$$

Now, we show that a theorem equivalent to Theorem 7 holds for the Sobolev ψ-wave front set. In the proof we again use the lemmas in Section 5.

THEOREM 9. *Let $f \in \mathcal{S}'(\boldsymbol{R}^n)$. Then $WF_{\psi}^{(s)}(f) = WF^{(s)}(f)$.*

PROOF. Assume that $(x_0, \xi_0) \notin WF^{(s)}(f)$. Then we have

$$\begin{aligned} I &= \int_{\Gamma(\xi_0, r_0)} \int_{\boldsymbol{R}^n} |\widehat{W_{\psi,\lambda}(\phi f)}(\tau, \xi)|^2 (1 + |\xi|^2)^s \, d\tau d\xi \\ &= (2\pi)^n \int_{\Gamma(\xi_0, r_0)} (1 + |\xi|^2)^s \, d\xi \int_{\boldsymbol{R}^n} |\widehat{\phi f}(\tau)|^2 |\xi|^{-n} \widehat{\psi}_{\lambda}(|\xi|^{-1} R_{\xi}(\tau))^2 \, d\tau \\ &= (2\pi)^n \int_{\Gamma(\xi_0, r_0)} (1 + |\xi|^2)^s \, d\xi \int_{\tilde{\Gamma}(\xi_0)} |\widehat{\phi f}(\tau)|^2 |\xi|^{-n} \widehat{\psi}_{\lambda}(|\xi|^{-1} R_{\xi}(\tau))^2 \, d\tau , \end{aligned}$$

where

$$\tilde{\Gamma}(\xi_0) = \{\tau \in \boldsymbol{R}^n \setminus \{0\} ; \text{ there exists } \xi \in \Gamma(\xi_0, r_0) \text{ such that } |\xi|^{-1} R_{\xi}(\tau) \in \Omega_{\lambda}\}$$

(see Lemma 23 below). So, it follows that

$$I = (2\pi)^n \int_{\tilde{\Gamma}(\xi_0)} |\widehat{\phi f}(\tau)|^2 \, d\tau \int_{\Gamma(\xi_0, r_0)} (1 + |\xi|^2)^s |\xi|^{-n} \widehat{\psi}_{\lambda}(|\xi|^{-1} R_{\xi}(\tau))^2 \, d\xi ,$$

and using the change of variables $\xi \mapsto \omega$, where $\omega = |\xi|^{-1} R_{\xi}(\tau)$, we obtain

$$I = (2\pi)^n \int_{\tilde{\Gamma}(\xi_0)} |\widehat{\phi f}(\tau)|^2 \, d\tau \int_{\Omega'(\tau)} \left(1 + \frac{|\tau|^2}{|\omega|^2}\right)^s |\omega|^{-n} \widehat{\psi}_{\lambda}(\omega)^2 \, d\omega ,$$

where

$$\Omega'(\tau) = \{\omega \in \boldsymbol{R}^n \setminus \{0\} ; \text{ there exists } \xi \in \Gamma(\xi_0, r_0) \text{ such that } \omega = |\xi|^{-1} R_{\xi}(\tau) \in \Omega_{\lambda}\} ,$$

for $\tau \in \tilde{\Gamma}(\xi_0)$ (see Lemmas 21 and 22). There exist positive constants C_1 and C_2 such that

$$C_1 (1 + |\tau|^2)^s \leqslant \frac{1}{|\omega|^n} \left(1 + \frac{|\tau|^2}{|\omega|^2}\right)^s \leqslant C_2 (1 + |\tau|^2)^s, \quad \omega \in \Omega .$$

As $\Omega'(\tau) \subseteq \Omega_{\lambda} \subseteq \Omega$, it follows that

$$C_1 (1 + |\tau|^2)^s \leqslant \frac{1}{|\omega|^n} \left(1 + \frac{|\tau|^2}{|\omega|^2}\right)^s \leqslant C_2 (1 + |\tau|^2)^s, \quad \omega \in \Omega'(\tau) ,$$

which implies that

$$C_1(1+|\tau|^2)^s\int_{\Omega'(\tau)}\widehat{\psi}_\lambda(\omega)^2\,d\omega\leqslant\int_{\Omega'(\tau)}\left(1+\frac{|\tau|^2}{|\omega|^2}\right)^s|\omega|^{-n}\widehat{\psi}_\lambda(\omega)^2\,d\omega\,,$$

$$\int_{\Omega'(\tau)}\left(1+\frac{|\tau|^2}{|\omega|^2}\right)^s|\omega|^{-n}\widehat{\psi}_\lambda(\omega)^2\,d\omega\leqslant C_2(1+|\tau|^2)^s\int_{\Omega'(\tau)}\widehat{\psi}_\lambda(\omega)^2\,d\omega\,.$$

Let $r_0=r/2$ and λ be such that r_{Ω_λ}, the conic radius of Ω_λ (see Section 5), is less than $r/2$. From Lemma 22 we know that $\tilde{\Gamma}(\xi_0)\subseteq\Gamma(\xi_0,r_0+r_{\Omega_\lambda})$, which implies that $\tilde{\Gamma}(\xi_0)\subseteq\Gamma(\xi_0,r)$. Furthermore, we have

$$\begin{aligned}
I&=(2\pi)^n\int_{\tilde{\Gamma}(\xi_0)}|\widehat{\phi f}(\tau)|^2\,d\tau\int_{\Omega'(\tau)}\left(1+\frac{|\tau|^2}{|\omega|^2}\right)^s|\omega|^{-n}\widehat{\psi}_\lambda(\omega)^2\,d\omega\\
&\leqslant C_2(2\pi)^n\int_{\tilde{\Gamma}(\xi_0)}|\widehat{\phi f}(\tau)|^2(1+|\tau|^2)^s\,d\tau\int_{\Omega'(\tau)}\widehat{\psi}_\lambda(\omega)^2\,d\omega\\
&\leqslant C_2(2\pi)^n\int_{\tilde{\Gamma}(\xi_0)}|\widehat{\phi f}(\tau)|^2(1+|\tau|^2)^s\,d\tau\int_{\Omega_\lambda}\widehat{\psi}_\lambda(\omega)^2\,d\omega\\
&\leqslant C\int_{\tilde{\Gamma}(\xi_0)}|\widehat{\phi f}(\tau)|^2(1+|\tau|^2)^s\,d\tau\\
&\leqslant C\int_{\Gamma(\xi_0,r)}|\widehat{\phi f}(\tau)|^2(1+|\tau|^2)^s\,d\tau\,,
\end{aligned}$$

where $C=C_2(2\pi)^n\int_{\Omega_\lambda}\widehat{\psi}_\lambda(\omega)^2\,d\omega$. Thus, the assumption $(x_0,\xi_0)\notin WF^{(s)}(f)$ implies that $(x_0,\xi_0)\notin WF_\psi^{(s)}(f)$.

Let now $(x_0,\xi_0)\notin WF_\psi^{(s)}(f)$. Then by Definition 8 (and $\Gamma(\xi_0,r_0)$ instead of $\Gamma(\xi_0)$)

$$\int_{\Gamma(\xi_0,r_0)}\int_{\boldsymbol{R}^n}|\widehat{W_{\psi,\lambda}(\phi f)}(\tau,\xi)|^2(1+|\xi|^2)^s\,d\tau d\xi<\infty\,.$$

Then, by Lemma 21, there exists Ω' neighborhood of point e_n such that for every $\tau\in\Gamma(\xi_0,r_0/2)$

$$\Omega'\subseteq\Omega'(\tau)\,.$$

Now, from Lemma 22 it follows that $\Gamma(\xi_0,r_0/2)\subseteq\tilde{\Gamma}(\xi_0)$. Then

$$\begin{aligned}
I&=(2\pi)^n\int_{\tilde{\Gamma}(\xi_0)}|\widehat{\phi f}(\tau)|^2\,d\tau\int_{\Omega'(\tau)}\left(1+\frac{|\tau|^2}{|\omega|^2}\right)^s|\omega|^{-n}\widehat{\psi}_\lambda(\omega)^2\,d\omega\\
&\geqslant C_1(2\pi)^n\int_{\tilde{\Gamma}(\xi_0)}|\widehat{\phi f}(\tau)|^2(1+|\tau|^2)^s\,d\tau\int_{\Omega'(\tau)}\widehat{\psi}_\lambda(\omega)^2\,d\omega\\
&\geqslant C_1(2\pi)^n\int_{\Gamma(\xi_0,r_0/2)}|\widehat{\phi f}(\tau)|^2(1+|\tau|^2)^s\,d\tau\int_{\Omega'}\widehat{\psi}_\lambda(\omega)^2\,d\omega\\
&\geqslant C\int_{\Gamma(\xi_0,r_0/2)}|\widehat{\phi f}(\tau)|^2(1+|\tau|^2)^s\,d\tau\,,
\end{aligned}$$

where $C=C_1(2\pi)^n\int_{\Omega'}\widehat{\psi}_\lambda(\omega)^2\,d\omega$. We conclude that $(x_0,\xi_0)\notin WF^{(s)}(f)$. □

4. Wavelet transform and wave front set-generalization. In this section we generalize the definition of wavelet transforms and the characterization of wave front sets via wavelet transforms of Section 3 in dimensions different from $n = 1, 2, 4$ and 8.

Let $n \in \boldsymbol{N}$ and let $n_1, \dots, n_k \in \{1, 2, 4, 8\}$ be such that $n = n_1 + \cdots + n_k$. Let $\psi_j \in \mathcal{S}(\boldsymbol{R}^{n_j})$, $j \in I_k = \{1, \dots, k\}$ satisfy the following:

(1) $\widehat{\psi}_j \in C_0^\infty(\boldsymbol{R}^{n_j})$ and $\widehat{\psi}_j(\xi^j) \geqslant 0$, $\xi^j \in \boldsymbol{R}^{n_j}$ for $j \in I_k$;

(2) $\Omega_j = \operatorname{supp} \widehat{\psi}_j$ does not contain 0 and $\widehat{\psi}_j(e_{n_j}) \neq 0$ for $j \in I_k$.

We will use the following partition of $I_n = \{1, \dots, n\}$. Let $\{p_1^j, \dots, p_{n_j}^j\}$, $j \in I_k$, be disjoint subsets of I_n, $j \in I_k$, such that $p_l^j < p_{l+1}^j$, $1 \le l \le n_j - 1$, $j \in I_k$. Then

$$\psi(x) = \prod_{j=1}^k \psi_j(x_{p_1^j}, \dots, x_{p_{n_j}^j}), \quad x^j = (x_{p_1^j}, \dots, x_{p_{n_j}^j}) \in \boldsymbol{R}^{n_j}, \quad j \in I_k,$$

represents an analyzing wavelet. Clearly,

$$\widehat{\psi}(\xi) = \prod_{j=1}^k \widehat{\psi}_j(\xi_{p_1^j}, \dots, \xi_{p_{n_j}^j}), \quad \xi^j = (\xi_{p_1^j}, \dots, \xi_{p_{n_j}^j}) \in \boldsymbol{R}^{n_j}, \quad j \in I_k.$$

Note that we could choose another wavelet both for a different choice of $n'_1, \dots, n'_{k'}$, different partitions of I_n and thus, different $\psi'_1, \dots, \psi'_{k'}$. The idea is to represent wavelet as a product of wavelets that belong to $\boldsymbol{R}^k$ for some $k = 1, 2, 4$ or 8.

In the sequel, we will assume that

$$x_1 = x_{p_1^1}, \dots, x_{n_1} = x_{p_{n_1}^1}, \dots, x_n = x_{p_{n_k}^k}.$$

Thus, for $x, \xi \in \boldsymbol{R}^n$, we use the notation $x = (x^1, \dots, x^k)$ and $\xi = (\xi^1, \dots, \xi^k)$, where $x^j, \xi^j \in \boldsymbol{R}^{n_j}$, $j \in I_k$. We also use the notation $e^n = (e_{n_1}, \dots, e_{n_k})$, where e_{n_j} are the unit vectors of $\boldsymbol{R}^{n_j}$ for $j \in I_k$.

We define the wavelet transform of a distribution $f \in \mathcal{S}'(\boldsymbol{R}^n)$ by

$$W_\psi f(x, \xi) = \int_{\boldsymbol{R}^n} f(t) \prod_{j=1}^k |\xi^j|^{n_j/2} \overline{\psi_j(|\xi^j| R_{\xi^j}(t^j - x^j))}\, dt,$$

for $(x, \xi) \in \boldsymbol{R}^n \times \prod_{j=1}^k (\boldsymbol{R}^{n_j} \backslash \{0\})$, where integration is understood in the distributional sense. Obviously, for a different factorization of a wavelet into a product of wavelets we obtain different wavelet transforms.

The Fourier transform of $W_\psi f$ with respect to x is given by

$$\widehat{W_\psi f(x, \xi})(\tau, \xi) = (2\pi)^{n/2} \widehat{f}(\tau) \prod_{j=1}^k |\xi^j|^{-n_j/2} \widehat{\psi}_j(|\xi^j|^{-1} R_{\xi^j}(\tau^j)),$$

for $(\tau, \xi) \in \boldsymbol{R}^n \times \prod_{j=1}^k (\boldsymbol{R}^{n_j} \setminus \{0\})$. In other words,

$$W_\psi f(x,\xi) = \int_{\boldsymbol{R}^n} \widehat{f}(\tau) \prod_{j=1}^k |\xi^j|^{-n_j/2} \widehat{\psi}_j(|\xi^j|^{-1} R_{\xi^j}(\tau^j)) e^{i\tau x}\, d\tau\,,$$

for $(x, \xi) \in \boldsymbol{R}^n \times \prod_{j=1}^k (\boldsymbol{R}^{n_j} \setminus \{0\})$.

As f is a tempered distribution, we have that $W_\psi f$ is a tempered distribution with respect to the variable x, while ξ belongs to $\prod_{j=1}^k (\boldsymbol{R}^{n_j} \setminus \{0\})$.

The complement of $\boldsymbol{R}^n \times \prod_{j=1}^k (\boldsymbol{R}^{n_j} \setminus \{0\})$ in $\boldsymbol{R}^n \times \boldsymbol{R}^n$ is of zero measure. Thus, excluding this set in the domain of integration, we can apply the same argument as in the assertions of Section 3.1 for $n = 1, 2, 4, 8$ and obtain the next four assertions.

PROPOSITION 10 (Parseval's identity). *Let $f, g \in L^2(\boldsymbol{R}^n)$. Then $W_\psi f \in L^2(\boldsymbol{R}^n \times \prod_{j=1}^k (\boldsymbol{R}^{n_j} \setminus \{0\}))$ and*

$$\int_{\boldsymbol{R}^n} \int_{\boldsymbol{R}^n} W_\psi f(x,\xi) \overline{W_\psi g(x,\xi)}\, dx d\xi = C_\psi \int_{\boldsymbol{R}^n} f(t) \overline{g(t)}\, dt\,,$$

where

$$C_\psi = (2\pi)^n \int_{\boldsymbol{R}^n} \prod_{j=1}^k \frac{|\widehat{\psi}_j(\xi^j)|^2}{|\xi^j|^{n_j}}\, d\omega\,.$$

COROLLARY 11. *The wavelet transform W_ψ is an isometric mapping from $L^2(\boldsymbol{R}^n, dt)$ to $L^2(\boldsymbol{R}^n \times \prod_{j=1}^k (\boldsymbol{R}^{n_j} \setminus \{0\}), C_\psi^{-1}\, dx d\xi)$.*

PROPOSITION 12 (Inverse formula). *Let $f \in L^2(\boldsymbol{R}^n)$. Then f can be expressed via $W_\psi f$ as*

$$f(t) = C_\psi^{-1} \int_{\boldsymbol{R}^n} \int_{\boldsymbol{R}^n} W_\psi f(x,\xi) \prod_{j=1}^k |\xi^j|^{n_j/2} \overline{\psi_j(|\xi^j| R_{\xi^j}(t^j - x^j))}\, dx d\xi\,.$$

Note that this formula does not hold for polynomials as all the moments of ψ equal zero. We refer to [3, 9] for the (generalized) wavelet transforms of tempered and other classes of distributions.

PROPOSITION 13. *Let $f \in \mathcal{S}'(\boldsymbol{R}^n)$. Then $f \in H^s(\boldsymbol{R}^n)$ if and only if*

$$\int_{\boldsymbol{R}^n} \int_{\boldsymbol{R}^n} |W_\psi f(x,\xi)|^2 (1 + |\xi|^2)^s\, dx d\xi < \infty\,.$$

4.1. Wavefront set. In the remainder of this section we assume that $\operatorname{supp} \widehat{\psi}_j \subseteq B_r(e_{n_j})$, $r \in (0,1)$, $j \in \{1, \dots, k\}$. For $0 < \lambda < 1$, we define $\psi_\lambda(x) = \prod_{j=1}^k \psi_{j,\lambda}(x^j)$, $x \in \boldsymbol{R}^n$, where

$$\psi_{j,\lambda}(x^j) = \lambda^{n_j} e^{ixe_{n_j}(1-\lambda)} \psi_j(\lambda x^j), \quad x^j \in \boldsymbol{R}^{n_j}\,.$$

Put $\Omega_{j,\lambda} = \operatorname{supp} \widehat{\psi}_{j,\lambda}$ and $\Omega_\lambda = \prod_{j=1}^k \Omega_{j,\lambda}$. Clearly, $\Omega_{j,\lambda} \subseteq B_{\lambda r}(e_{n_j})$.

4.1.1. *$WF(f)$ and $WF_\psi(f)$.* We define a ψ-wave front set $WF_\psi(f)$ with respect to the newly introduced wavelet transform as a subset of $\boldsymbol{R}^n \times \prod_{j=1}^k(\boldsymbol{R}^{n_j}\setminus\{0\})$.

DEFINITION 14. Let $f \in \mathcal{S}'(\boldsymbol{R}^n)$. $WF_\psi(f) \subseteq \boldsymbol{R}^n \times \prod_{j=1}^k(\boldsymbol{R}^{n_j}\setminus\{0\})$ is the complement of the set of ψ-microlocally regular points $(x_0,\xi_0) \in \boldsymbol{R}^n \times \prod_{j=1}^k(\boldsymbol{R}^{n_j}\setminus\{0\})$, that is, $(x_0,\xi_0) \notin WF_\psi(f)$ if and only if

$$(\exists\phi \in C_0^\infty(\boldsymbol{R}^n)) \quad (\phi(x_0) \neq 0) \quad \left(\exists\Gamma(\xi_0) \subseteq \prod_{j=1}^k(\boldsymbol{R}^{n_j}\setminus\{0\})\right)$$
$$(\exists\lambda \in (0,1)) \quad (\forall N > 0) \quad (\exists C_N > 0)$$
$$|W_{\psi,\lambda}\widehat{(\phi f)}(x,\xi)(\tau,\xi)| \leqslant C_N|\xi|^{-N}, \quad \tau \in \boldsymbol{R}^n,\ \xi \in \Gamma(\xi_0),\ |\xi| \geqslant 1\,.$$

Note, in this definition the cone $\Gamma(\xi_0)$ is a subset of $\prod_{j=1}^k(\boldsymbol{R}^{n_j}\setminus\{0\})$.

Let $(x_0,\xi_0) \in \boldsymbol{R}^n \times \prod_{j=1}^k(\boldsymbol{R}^{n_j}\setminus\{0\})$. Then $(x_0,\xi_0) \notin WF_\psi(f)$ if and only if

$$(\exists\phi \in C_0^\infty(\boldsymbol{R}^n)) \quad (\phi(x_0) \neq 0) \quad \left(\exists\Gamma(\xi_0) \subseteq \prod_{j=1}^k(\boldsymbol{R}^{n_j}\setminus\{0\})\right)$$
$$(\exists\lambda \in (0,1)) \quad (\forall N > 0) \quad (\exists C_N > 0)$$
$$|\widehat{\phi f}(\tau)| \prod_{j=1}^k |\xi^j|^{-n_j/2}\widehat{\psi}_{j,\lambda}(|\xi^j|^{-1}R_{\xi^j}(\tau^j)) \leqslant C_N|\xi|^{-N}, \quad \tau \in \boldsymbol{R}^n,\ \xi \in \Gamma(\xi_0),\ |\xi| \geqslant 1\,;$$

and, further, if and only if

$$(\exists\phi \in C_0^\infty(\boldsymbol{R}^n)) \quad (\phi(x_0) \neq 0) \quad \left(\exists\Gamma(\xi_0) \subseteq \prod_{j=1}^k(\boldsymbol{R}^{n_j}\setminus\{0\})\right)$$
$$(\exists\lambda \in (0,1)) \quad (\forall N > 0) \quad (\exists C_N > 0)$$
$$|\widehat{\phi f}(\tau)| \prod_{j=1}^k \widehat{\psi}_{j,\lambda}(|\xi^j|^{-1}R_{\xi^j}(\tau^j)) \leqslant C_N|\xi|^{-N}, \quad \tau \in \boldsymbol{R}^n,\ \xi \in \Gamma(\xi_0),\ |\xi| \geqslant 1\,.$$

We are going to give a variant of Theorem 7. We use the lemmas in Section 6.

THEOREM 15. *Let $f \in \mathcal{S}'(\boldsymbol{R}^n)$. Then $WF_\psi(f) = WF(f) \cap \boldsymbol{R}^n \times \prod_{j=1}^k(\boldsymbol{R}^{n_j}\setminus\{0\})$.*

PROOF. It is expected that the proof will be similar to the one of Theorem 7, but one has to take care about the directions in the domain of ξ-variable. In order to achieve completeness, we will give the details of the proof.

Let $(x_0,\xi_0) \notin WF(f)$ and $\xi_0 \in \prod_{j=1}^k(\boldsymbol{R}^{n_j}\setminus\{0\})$. To show that $(x_0,\xi_0) \notin WF_\psi(f)$ we need to find $\tilde{\Gamma}(\xi_0) \subseteq \prod_{j=1}^k(\boldsymbol{R}^{n_j}\setminus\{0\})$ and $\lambda \in (0,1)$ (such that for each $N > 0$ there exists

$C'_N > 0$) such that

$$|\widehat{\phi f}(\tau)| \prod_{j=1}^{k} \widehat{\psi}_{j,\lambda}(|\xi^j|^{-1} R_{\xi^j}(\tau^j)) \leqslant C'_N |\xi|^{-N}, \quad \tau \in \boldsymbol{R}^n, \xi \in \tilde{\Gamma}(\xi_0), |\xi| \geqslant 1 .$$

As for every $\xi \in \prod_{j=1}^{k}(\boldsymbol{R}^{n_j} \setminus \{0\})$ the mapping $\tau \mapsto \omega$, where

$$\omega^j = |\xi^j|^{-1} R_{\xi^j}(\tau^j), \quad \tau \in \boldsymbol{R}^n ,$$

is a bijection of set $\boldsymbol{R}^n$ onto itself, the problem can be reformulated to find $\tilde{\Gamma}(\xi_0) \subseteq \prod_{j=1}^{k}(\boldsymbol{R}^{n_j} \setminus \{0\})$ and $\lambda \in (0,1)$ such that

$$(\forall N > 0) \quad (\exists C'_N > 0) \quad \left(\forall \omega \in \prod_{j=1}^{k} \Omega_{j,\lambda}\right) \quad (\forall \xi \in \tilde{\Gamma}(\xi_0)) \quad (|\xi| \geqslant 1)$$

$$|\widehat{\phi f}(|\xi^1| R_{\xi^1}^{-1}(\omega^1), \ldots, |\xi^k| R_{\xi^k}^{-1}(\omega^k))| \prod_{j=1}^{k} \widehat{\psi}_{j,\lambda}(\omega^j) \leqslant C'_N |\xi|^{-N} . \tag{3}$$

By Lemma 26 there exist a conic neighborhood $\tilde{\Gamma}(\xi_0)$ and $\lambda \in (0,1)$ such that $(|\xi^1| R_{\xi^1}^{-1}(\omega^1), \ldots, |\xi^k| R_{\xi^k}^{-1}(\omega^k)) \in \Gamma(\xi_0)$ for $\omega \in \prod_{j=1}^{k} \Omega_{j,\lambda}$ and $\xi \in \tilde{\Gamma}(\xi_0)$. We conclude that

$$\begin{aligned}
&|\widehat{\phi f}(|\xi^1| R_{\xi^1}^{-1}(\omega^1), \ldots, |\xi^k| R_{\xi^k}^{-1}(\omega^k))| \prod_{j=1}^{k} \widehat{\psi}_{j,\lambda}(\omega^j) \\
&\quad \leqslant C |\widehat{\phi f}(|\xi^1| R_{\xi^1}^{-1}(\omega^1), \ldots, |\xi^k| R_{\xi^k}^{-1}(\omega^k))| \\
&\quad \leqslant C C_N \left(\sqrt{(|\xi^1||R_{\xi^1}^{-1}(\omega^1)|)^2 + \cdots + (|\xi^k||R_{\xi^k}^{-1}(\omega^k)|)^2}\right)^{-N} \\
&\quad \leqslant C C_N \left(\sqrt{(|\xi^1||\omega^1|)^2 + \cdots + (|\xi^k||\omega^k|)^2}\right)^{-N} \\
&\quad \leqslant C C_N (\min\{|\omega^1|, \ldots, |\omega^k|\})^{-N} |\xi|^{-N} \\
&\quad \leqslant C'_N |\xi|^{-N} ,
\end{aligned}$$

where $\omega \in \prod_{j=1}^{k} \Omega_{j,\lambda},\ \xi \in \tilde{\Gamma}(\xi_0),\ |\xi| \geqslant 1$. Thus $(x_0, \xi_0) \notin WF_\psi f$.

Now, let $(x_0, \xi_0) \notin WF_\psi f$. Then there exist $\phi \in C_0^\infty(\boldsymbol{R}^n)$, $\phi(x_0) \neq 0$, $\tilde{\Gamma}(\xi_0)$, a conic neighborhood of ξ_0, and $\lambda \in (0,1)$ such that for each $N > 0$ there exists $C'_N > 0$ such that (3) is satisfied for every $\omega \in \Omega_\lambda = \prod_{j=1}^{k} \Omega_{j,\lambda},\ \xi \in \tilde{\Gamma}(\xi_0),\ |\xi| \geqslant 1$. As $e^n \in \Omega_\lambda$ and $\widehat{\psi}_\lambda(e_n) \neq 0$, we get

$$\begin{aligned}
|\widehat{\phi f}(\xi)| &= |\widehat{\phi f}(|\xi^1| R_{\xi^1}^{-1}(e_{n_1}), \ldots, |\xi^k| R_{\xi^k}^{-1}(e_{n_j}))| \\
&\leqslant \frac{C'_N}{\widehat{\psi}_\lambda(e_n)} |\xi|^{-N} ,
\end{aligned}$$

where $\xi \in \tilde{\Gamma}(\xi_0),\ |\xi| \geqslant 1$. Thus, $(x_0, \xi_0) \notin WF(f)$. □

REMARK 16. For a different choice of factorization of a wavelet ψ into a product of wavelets described in the introduction of Section 4, the equality in Theorem 15 would hold on corresponding subsets of $\boldsymbol{R}^n \times \boldsymbol{R}^n$ due to the fact that W_ψ is defined on these subsets. It is natural to ask whether it would be possible to achieve the characterization of WF_ψ in full for $n \neq 1, 2, 4, 8$. We can start with looking at the set of all factorizations of n, denoted by S, into the sum consisting of addends 1, 2, 4 and 8. Denote by ψ^m, $m \in S$, the corresponding wavelets, as it is described, and denote by $\mathcal{R}_n^m$ the set of directions, where W_{ψ^m} is defined in ξ-direction. Let $\mathcal{R}^n = \bigcup_{m \in S} \mathcal{R}_n^m$. Obviously, we have

$$WF(f) \cap (\boldsymbol{R}^n \times \mathcal{R}^n) = \bigcup_{m \in S} WF_{\psi^m}(f)\,.$$

One can easily check that for $n = 3$ the directions

$$(0, 0, \xi_3)\,, \quad (0, \xi_2, 0)\,, \quad (\xi_1, 0, 0)\,,$$

(different from $0 \in \boldsymbol{R}^3$) are not in $\mathcal{R}^3$.

In order to analyze all directions ξ of $\boldsymbol{R}^3 \setminus \{0\}$, we can proceed as follows. Let $\mathbf{1}_{\boldsymbol{R}}$ be the characteristic function of the real line. Then we see

$$\boldsymbol{R}^n \times \boldsymbol{R}^n \setminus \{0\} \ni (x, \xi) \notin WFf \Longleftrightarrow ((x, 0), (\xi, 0)) \notin WF(f \otimes \mathbf{1}_{\boldsymbol{R}})\,.$$

Thus, for the analysis of directions $(0, 0, \xi_3)$, $(0, \xi_2, 0)$ and $(\xi_1, 0, 0)$, we have to consider $W_\psi(f \otimes \mathbf{1}_{\boldsymbol{R}})$, where $\psi = \psi_4$ is the fourth-dimensional wavelet described in Section 3.1 and to see whether the directions $((x, 0), (0, 0, \xi_3, 0))$, $((x, 0), (0, \xi_2, 0, 0))$ and $((x, 0), (\xi_1, 0, 0, 0))$ are out of $WF_{\psi_4}(f \otimes \mathbf{1}_{\boldsymbol{R}})$.

In the general case, for $n \notin \{1, 2, 4, 8\}$, one has to combine ideas described in the case $n = 3$. Such analysis involves a lot of combinatorics and it will not be given.

4.2. $WF^{(s)}(f)$ and $WF_\psi^{(s)}(f)$. Now we define Sobolev ψ-wave front sets $WF_\psi^{(s)}(f) \subseteq \boldsymbol{R}^n \times \prod_{j=1}^k (\boldsymbol{R}^{n_j} \setminus \{0\})$.

DEFINITION 17. Let $f \in \mathcal{S}'(\boldsymbol{R}^n)$. Then $WF_\psi^{(s)}(f) \subseteq \boldsymbol{R}^n \times \prod_{j=1}^k (\boldsymbol{R}^{n_j} \setminus \{0\})$ is the complement of the set of Sobolev ψ-microlocally regular points $(x_0, \xi_0) \in \boldsymbol{R}^n \times \prod_{j=1}^k (\boldsymbol{R}^{n_j} \setminus \{0\})$, that is, $(x_0, \xi_0) \notin WF_\psi^{(s)}(f)$ if and only if

$$(\exists \phi \in C_0^\infty(\boldsymbol{R}^n)) \quad (\phi(x_0) \neq 0) \quad \left(\exists \Gamma(\xi_0) \subseteq \prod_{j=1}^k (\boldsymbol{R}^{n_j} \setminus \{0\})\right) \quad (\exists \lambda \in (0, 1))$$
$$\int_{\Gamma(\xi_0)} \int_{\boldsymbol{R}^n} |\widehat{W_{\psi,\lambda}(\phi f)}(\tau, \xi)|^2 (1 + |\xi|^2)^s \, d\tau d\xi < \infty\,.$$

Now we give a theorem equivalent to Theorem 9. We use the lemmas from Section 6.

THEOREM 18. *Let $f \in \mathcal{S}'(\boldsymbol{R}^n)$. Then $WF_\psi^{(s)}(f) = WF^{(s)}(f) \cap \boldsymbol{R}^n \times \prod_{j=1}^k (\boldsymbol{R}^{n_j} \setminus \{0\})$.*

PROOF. Let $\xi_0 \in \prod_{j=1}^k \boldsymbol{R}^{n_j} \setminus \{0\}$ and $(x_0, \xi_0) \notin WF^s(f)$. Then

$$I = \int_{\Gamma(\xi_0, r_0)} \int_{\boldsymbol{R}^n} |\widehat{W_{\psi,\lambda}(\phi f)}(\tau, \xi)|^2 (1 + |\xi|^2)^s \, d\tau d\xi$$
$$= (2\pi)^n \int_{\Gamma(\xi_0, r_0)} (1 + |\xi|^2)^s \, d\xi \int_{\tilde{\Gamma}(\xi_0)} |\widehat{\phi f}(\tau)|^2 \prod_{j=1}^k |\xi^j|^{-n_j} \widehat{\psi}_{j,\lambda}(|\xi^j|^{-1} R_{\xi^j}(\tau^j))^2 \, d\tau \,,$$

where $\tilde{\Gamma}(\xi_0) = \{\tau\,;$ there exists $\xi \in \Gamma(\xi_0, r_0)$ such that $|\xi^j|^{-1} R_{\xi^j}(\tau^j) \in \Omega_{j,\lambda}$ for $j = 1, \ldots, k\}$. So, we have

$$I = (2\pi)^n \int_{\tilde{\Gamma}(\xi_0)} |\widehat{\phi f}(\tau)|^2 \, d\tau \int_{\Gamma(\xi_0, r_0)} (1 + |\xi|^2)^s \prod_{j=1}^k |\xi^j|^{-n_j} \widehat{\psi}_{j,\lambda}(|\xi^j|^{-1} R_{\xi^j}(\tau^j))^2 \, d\xi \,,$$

and with the change of variables

$$I = (2\pi)^n \int_{\tilde{\Gamma}(\xi_0)} |\widehat{\phi f}(\tau)|^2 \, d\tau \int_{\Omega'(\tau)} \left(1 + \frac{|\tau^1|^2}{|\omega^1|^2} + \cdots + \frac{|\tau^k|^2}{|\omega^k|^2}\right)^s \prod_{j=1}^k |\omega^j|^{-n_j} \widehat{\psi}_{j,\lambda}(\omega^j)^2 \, d\omega \,,$$

where, for $\tau \in \tilde{\Gamma}(\xi_0)$, $\Omega'(\tau)$ is defined by

$$\left\{\omega\,; \text{ there exists } \xi \in \Gamma(\xi_0, r_0) \text{ such that } (|\xi^1|^{-1} R_{\xi^1}(\tau^1), \ldots, |\xi^k|^{-1} R_{\xi^k}(\tau^k)) \in \prod_{j=1}^k \Omega_{j,\lambda}\right\}.$$

As $\Omega'(\tau) \subseteq \prod_{j=1}^k \Omega_{j,\lambda}$, there exist positive constants C_1 and C_2, independent from τ, such that for every $\omega \in \Omega'(\tau)$

$$C_1(1 + |\tau|^2)^s \leqslant \left(1 + \frac{|\tau^1|^2}{|\omega^1|^2} + \cdots + \frac{|\tau^k|^2}{|\omega^k|^2}\right)^s \prod_{j=1}^k |\omega^j|^{-n_j} \leqslant C_2(1 + |\tau|^2)^s \,,$$

which implies that

$$C_1(1 + |\tau|^2)^s \int_{\Omega'(\tau)} \prod_{j=1}^k \widehat{\psi}_{j,\lambda}(\omega^j)^2 \, d\omega$$
$$\leqslant \int_{\Omega'(\tau)} \left(1 + \frac{|\tau^1|^2}{|\omega^1|^2} + \cdots + \frac{|\tau^k|^2}{|\omega^k|^2}\right)^s \prod_{j=1}^k |\omega^j|^{-n_j} \widehat{\psi}_{j,\lambda}(\omega^j)^2 \, d\omega$$
$$\leqslant C_2(1 + |\tau|^2)^s \int_{\Omega'(\tau)} \prod_{j=1}^k \widehat{\psi}_{j,\lambda}(\omega^j)^2 \, d\omega \,.$$

Lemma 24 implies that we can choose r_0 such that $\tilde{\Gamma}(\xi_0)$ is a subset of $\Gamma(\xi_0)$ and then

$$I = (2\pi)^n \int_{\tilde{\Gamma}(\xi_0)} |\widehat{\phi f}(\tau)|^2 \, d\tau \int_{\Omega'(\tau)} \left(1 + \frac{|\tau^1|^2}{|\omega^1|^2} + \cdots + \frac{|\tau^k|^2}{|\omega^k|^2}\right)^s \prod_{j=1}^k |\omega^j|^{-n_j} \widehat{\psi}_{j,\lambda}(\omega^j)^2 \, d\omega$$

$$\leqslant C_2 (2\pi)^n \int_{\tilde{\Gamma}(\xi_0)} |\widehat{\phi f}(\tau)|^2 (1 + |\tau|^2)^s \, d\tau \int_{\prod \Omega_{j,\lambda}} \prod_{j=1}^k \widehat{\psi}_{j,\lambda}(\omega^j)^2 \, d\omega$$

$$\leqslant C \int_{\Gamma(\xi_0)} |\widehat{\phi f}(\tau)|^2 (1 + |\tau|^2)^s \, d\tau \, ,$$

where $C = C_2(2\pi)^n \int_{\prod \Omega_{j,\lambda}} \prod_{j=1}^k \widehat{\psi}_{j,\lambda}(\omega^j)^2 \, d\omega$. This implies that $(x_0, \xi_0) \notin WF_\psi^{(s)}(f)$.

Let now $(x_0, \xi_0) \notin WF_\psi^{(s)}(f)$. Using Lemma 25, we see that there exists a conic neighborhood $\tilde{\Gamma}'(\xi_0)$ such that $\tilde{\Gamma}'(\xi_0) \subseteq \tilde{\Gamma}(\xi_0)$ and that for every $\tau \in \tilde{\Gamma}'(\xi_0)$ it holds that $\Omega'(\tau)$ contains the same neighborhood of e^n, which we call $\prod_{j=1}^k \Omega'_j$. Then we have

$$I = (2\pi)^n \int_{\tilde{\Gamma}(\xi_0)} |\widehat{\phi f}(\tau)|^2 \, d\tau \left(1 + \frac{|\tau^1|^2}{|\omega^1|^2} + \cdots + \frac{|\tau^k|^2}{|\omega^k|^2}\right)^s \prod_{j=1}^k |\omega^j|^{-n_j} \widehat{\psi}_{j,\lambda}(\omega^j)^2 d\omega$$

$$\geqslant C_1 (2\pi)^n \int_{\tilde{\Gamma}(\xi_0)} |\widehat{\phi f}(\tau)|^2 (1 + |\tau|^2)^s \, d\tau \int_{\Omega'(\tau)} \prod_{j=1}^k \widehat{\psi}_{j,\lambda}(\omega_1)^2 \, d\omega$$

$$\geqslant C \int_{\tilde{\Gamma}'(\xi_0)} |\widehat{\phi f}(\tau)|^2 (1 + |\tau|^2)^s \, d\tau \, ,$$

where $C = C_1(2\pi)^n \int_{\prod \Omega'_j} \prod_{j=1}^k \widehat{\psi}_{j,\lambda}(\omega_1)^2 \, d\omega$. This implies that $(x_0, \xi_0) \notin WF^{(s)}(f)$. □

REMARK 19. The same conclusions concerning the directions laying out of $\mathcal{R}^n$ given in Remark 16 hold for $WF^s(f)$.

5. Auxiliary lemmas A. We assume that $n = 1, 2, 4$ or 8.

LEMMA 20. *Let $\tau \in \boldsymbol{R}^n \backslash \{0\}$. Then the following hold.*

(1) *The mapping $\xi \mapsto \omega$, where $\omega = |\xi|^{-1} R_\xi(\tau)$, is bijective from $\boldsymbol{R}^n \backslash \{0\}$ onto itself. The inverse mapping $\omega \mapsto \xi$ is given by $\xi = (|\tau|/|\omega|^2)[\bar{R}_\tau]^{-1}(\omega)$, where $\omega \in \boldsymbol{R}^n \backslash \{0\}$.*

(2) *For a given $\xi \in \boldsymbol{R}^n \backslash \{0\}$ and every $\omega \in \boldsymbol{R}^n \backslash \{0\}$ there exists $R'_\omega \in O(n)$ such that $\xi = |\omega|^{-1} R'_\omega(\tau)$.*

(3) *Let $r_0 > 0$. The above mapping $\xi \mapsto \omega$ is bijective from $\Gamma(\tau, r_0)$ onto $\Gamma(e_n, r_0)$.*

PROOF. (1) For every $\xi_1, \xi_2 \in \boldsymbol{R}^n \backslash \{0\}$, we see that

$$|\xi_1|^{-1} R_{\xi_1}(\tau) = |\xi_2|^{-1} R_{\xi_2}(\tau) \Rightarrow |\xi_1| = |\xi_2| \Rightarrow R_{\xi_1}(\tau) = R_{\xi_2}(\tau) \, .$$

First, implication is a consequence of the fact that any rotation preserves the norm. Furthermore, we have

$$\frac{R_{\xi_1}(\tau)}{|\tau|} = \frac{R_{\xi_2}(\tau)}{|\tau|} \Rightarrow \frac{\bar{R}_\tau(\xi_1)}{|\xi_1|} = \frac{\bar{R}_\tau(\xi_2)}{|\xi_2|} \Rightarrow \bar{R}_\tau(\xi_1) = \bar{R}_\tau(\xi_2) \Rightarrow \xi_1 = \xi_2 ,$$

and this proves that $\xi \mapsto \omega$ is injective.

To prove that $\xi \mapsto \omega$ is surjective we need to show that for every $\omega \in \boldsymbol{R}^n\backslash\{0\}$ there exists $\xi \in \boldsymbol{R}^n\backslash\{0\}$ such that $|\xi|^{-1}R_\xi(\tau) = \omega$. We can easily verify that ξ given by

$$\xi = \frac{|\tau|}{|\omega|^2}[\bar{R}_\tau]^{-1}(\omega)$$

satisfies this.

(2) As we did before when taking $\bar{R}_\tau \in O(n)$ for each R_ξ such that

$$\frac{R_\xi(\tau)}{|\tau|} = \frac{\bar{R}_\tau(\xi)}{|\xi|} ,$$

(see Section 2), we can take $R'_\omega \in O(n)$ for $[\bar{R}_\tau]^{-1}$ such that

$$\frac{[\bar{R}_\tau]^{-1}(\omega)}{|\omega|} = \frac{R'_\omega(\tau)}{|\tau|} .$$

Then it is clear that

$$\xi = |\omega|^{-1}R'_\omega(\tau) .$$

(3) First, we note that $\xi \in \Gamma(\tau, r_0)$ implies $|\xi|^{-1}R_\xi(\tau) \in \Gamma(e_n, r_0)$, which follows from

$$\left|\frac{|\xi|^{-1}R_\xi(\tau)}{|\xi|^{-1}|\tau|} - e_n\right| = \left|\frac{R_\xi(\tau)}{|\tau|} - \frac{R_\tau(\tau)}{|\tau|}\right| = \left|\frac{\bar{R}_\tau(\xi)}{|\xi|} - \frac{\bar{R}_\tau(\tau)}{|\tau|}\right|$$
$$= \left|\frac{\xi}{|\xi|} - \frac{\tau}{|\tau|}\right| < r_0 .$$

Second, for every $\omega \in \Gamma(e_n, r_0)$ there exists $\xi \in \boldsymbol{R}^n\backslash\{0\}$ such that $\omega = |\xi|^{-1}R_\xi(\tau)$. Indeed, such ξ is given by $\xi = |\tau||\omega|^{-2}[\bar{R}_\tau]^{-1}(\omega)$. We only need to show that it belongs to $\Gamma(\tau, r_0)$. This follows from

$$\left|\frac{\frac{|\tau|}{|\omega|^2}[\bar{R}_\tau]^{-1}(\omega)}{\frac{|\tau|}{|\omega|^2}|\omega|} - \frac{\tau}{|\tau|}\right| = \left|\frac{[\bar{R}_\tau]^{-1}(\omega)}{|\omega|} - [\bar{R}_\tau]^{-1}(e_n)\right|$$
$$= \left|\frac{\omega}{|\omega|} - e_n\right| < r_0 . \qquad \square$$

LEMMA 21. *Let* $\xi_0 \in \boldsymbol{R}^n\backslash\{0\}$ *and* $r_0 > 0$.

(1) *Let* $\tau \in \Gamma(\xi_0, r_0/2)$. *Then*

$$\Gamma(e_n, r_0/2) \subseteq \{\omega\,;\ \text{there exists } \xi \in \Gamma(\xi_0, r_0) \text{ such that } \omega = |\xi|^{-1}R_\xi(\tau)\} ,$$

or equivalently for every $\tau \in \Gamma(\xi_0, r_0/2)$, *it follows that* $\Gamma(e_n, r_0/2)$ *is contained in the image of* $\Gamma(\xi_0, r_0)$ *under the mapping* $\xi \mapsto \omega = |\xi|^{-1} R_\xi(\tau)$.

(2) *Let* Ω *be a neighborhood of* e_n. *Let*

$$\Omega'(\tau) = \{\omega \in \boldsymbol{R}^n \setminus \{0\}\,;\ \text{there exists } \xi \in \Gamma(\xi_0, r_0) \text{ such that } \omega = |\xi|^{-1} R_\xi(\tau) \in \Omega\},$$

where $\tau \in \boldsymbol{R}^n \setminus \{0\}$. *Then there exists a neighborhood* Ω' *of* e_n *such that for every* $\tau \in \Gamma(\xi_0, r_0/2)$,

$$\Omega' \subseteq \bigcap_{\tau \in \Gamma(\xi_0, r_0/2)} \Omega'(\tau)\,.$$

(3) *Let* $\Omega \subseteq B_{r_0}(e_n) \subseteq \boldsymbol{R}^n$, $r_0 \in (0,1)$ *and*

$$r_\Omega = \sup\{|\omega/|\omega| - e_n|\,;\ \omega \in \Omega\}\,.$$

Then r_Ω *is the smallest non-negative real number such that* $\Omega \subseteq \overline{\Gamma(e_n, r_\Omega)}$. *Moreover,* $r_\Omega \leqslant \sqrt{2} r_0/(1-r_0)$.

(4) Ω' *defined in the above can be chosen in such way that* $r_{\Omega'} \leqslant r_0/2$.

PROOF. (1) Let $\tau \in \Gamma(\xi_0, r_0/2)$. Then $\Gamma(\tau, r_0/2) \subseteq \Gamma(\xi_0, r_0)$ because for every $\xi \in \Gamma(\tau, r_0/2)$, we have

$$\left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| \leqslant \left|\frac{\xi}{|\xi|} - \frac{\tau}{|\tau|}\right| + \left|\frac{\tau}{|\tau|} - \frac{\xi_0}{|\xi_0|}\right| < \frac{r_0}{2} + \frac{r_0}{2} = r_0\,.$$

From Lemma 20 we know that $\Gamma(\tau, r_0/2)$ is mapped onto $\Gamma(e_n, r_0/2)$ by $\xi \mapsto \omega = |\xi|^{-1} R_\xi(\tau)$. So, for every $\tau \in \Gamma(\xi_0, r_0/2)$

$$\begin{aligned}\Gamma(e_n, r_0/2) &= \{\omega \in \boldsymbol{R}^n \setminus \{0\}\,;\ \text{there exists } \xi \in \Gamma(\tau, r_0/2) \text{ such that } \omega = |\xi|^{-1} R_\xi(\tau)\}\\ &\subseteq \{\omega \in \boldsymbol{R}^n \setminus \{0\}\,;\ \text{there exists } \xi \in \Gamma(\xi_0, r_0) \text{ such that } \omega = |\xi|^{-1} R_\xi(\tau)\}\,,\end{aligned}$$

which is what we had to prove.

(2) This assertion is a direct corollary of part (1). The claim is satisfied for $\Omega' = \Omega \cap \Gamma(e_n, r_0/2)$.

(3) Clearly, $\Omega \subset \overline{\Gamma(e_n, r_\Omega)}$ and r_Ω is the smallest non-negative number such that this inclusion holds. If we show that $\Omega \subseteq \overline{\Gamma(e_n, \sqrt{2} r_0/(1-r_0))}$, then it follows that $r_\Omega \leq \sqrt{2} r_0/(1-r_0)$. We have $|\omega - e_n| \leqslant r_0,\ \omega \in \Omega$, i.e.,

$$|\omega|^2 \leqslant r_0^2 + 2\omega_n - 1\,, \quad \omega \in \Omega\,.$$

Also, $\omega \in \Omega$ implies that $|\omega| \geqslant |e_n| - |e_n - \omega| \geqslant 1 - r_0 > 0$ and $\omega_n > 0$. Furthermore, we have

$$\begin{aligned}\left|\frac{\omega}{|\omega|} - e_n\right|^2 &= \frac{2|\omega|^2 - 2\omega_n|\omega|}{|\omega|^2} \leqslant \frac{2(r_0^2 + 2\omega_n - 1) - 2\omega_n^2}{|\omega|^2}\\ &\leq \frac{2r_0^2}{|\omega|^2} \leq \frac{2r_0^2}{(1-r_0)^2}\,,\end{aligned}$$

which implies that $\Omega \subseteq \overline{\Gamma(e_n, \sqrt{2} r_0/(1-r_0))}$.

(4) This is a consequence of part (2). □

We call r_Ω the conic radius of $\Omega \subseteq B_{r_0}(e_n) \subseteq \boldsymbol{R}^n$, $r_0 \in (0,1)$. Note that for $n = 1$ the conic radius of any set Ω (contained in $(-r_0, r_0)$) is equal to 0.

LEMMA 22. *Let $\xi_0 \in \boldsymbol{R}^n \backslash \{0\}$ and $e_n \in \Omega \subseteq B_{r_0}(e_n)$, where $r_0 < 1$. Define*

$$\tilde{\Gamma}(\xi_0) = \{\tau \in \boldsymbol{R}^n \setminus \{0\}\,;\ \textit{there exists } \xi \in \Gamma(\xi_0, r_0) \textit{ such that } |\xi|^{-1} R_\xi(\tau) \in \Omega\}\,.$$

Then the following hold.

(1) *$\tilde{\Gamma}(\xi_0)$ is a conic neighborhood of ξ_0, in particular, $\Gamma(\xi_0, r_0) \subseteq \tilde{\Gamma}(\xi_0)$.*

(2) *$\tilde{\Gamma}(\xi_0) \subseteq \Gamma(\xi_0, r_0 + r_\Omega)$.*

PROOF. (1) By definition, if $\tau \in \tilde{\Gamma}(\xi_0)$, then there exists $\xi \in \Gamma(\xi_0, r_0)$ such that $|\xi|^{-1} R_\xi(\tau) \in \Omega$. Let $a > 0$ and $\xi \in \Gamma(\xi_0, r_0)$. Then $a\xi \in \Gamma(\xi_0, r_0)$. As $R_\xi(\tau) = R_{a\xi}(\tau)$ and $aR_\xi(\tau) = R_\xi(a\tau)$, we get

$$|a\xi|^{-1} R_{a\xi}(a\tau) = |\xi|^{-1} R_\xi(\tau) \in \Omega\,.$$

Hence $\tilde{\Gamma}(\xi_0)$ is a conic set. As e_n belongs to Ω and $|\xi|^{-1} R_\xi(\xi) = e_n$, it follows that $\Gamma(\xi_0, r_0) \subseteq \tilde{\Gamma}(\xi_0)$. Now, we conclude that $\tilde{\Gamma}(\xi_0)$ is a conic neighborhood of ξ_0.

(2) To prove that $\tilde{\Gamma}(\xi_0) \subseteq \Gamma(\xi_0, r_0 + r_\Omega)$ we need to show that for every $\tau \in \tilde{\Gamma}(\xi_0)$

$$\left| \frac{\tau}{|\tau|} - \frac{\xi_0}{|\xi_0|} \right| < r_0 + r_\Omega\,.$$

For $\tau \in \tilde{\Gamma}(\xi_0)$ there exist $\xi \in \Gamma(\xi_0, r_0)$ and $\omega \in \Omega$ such that $|\xi|^{-1} R_\xi(\tau) = \omega$, i.e., $\tau = |\xi| R_\xi^{-1}(\omega)$. So, we have

$$\begin{aligned} \left| \frac{|\xi| R_\xi^{-1}(\omega)}{|\xi||\omega|} - \frac{\xi_0}{|\xi_0|} \right| &\leqslant \left| \frac{R_\xi^{-1}(\omega)}{|\omega|} - R_\xi^{-1}(e_n) \right| + \left| R_\xi^{-1}(e_n) - \frac{\xi_0}{|\xi_0|} \right| \\ &= \left| \frac{\omega}{|\omega|} - e_n \right| + \left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| < r_\Omega + r_0\,. \end{aligned}$$ □

LEMMA 23. *Let $\Gamma(\xi_0)$ be a conic neighborhood of $\xi_0 \in \boldsymbol{R}^n \backslash \{0\}$. For every $\lambda \in (0,1)$ let $\Omega_\lambda \subseteq B_{\lambda r}(e_n)$. Then there exist a conic neighborhood $\tilde{\Gamma}(\xi_0)$ of ξ_0 and $\lambda \in (0,1)$ such that*

$$\xi \in \tilde{\Gamma}(\xi_0)\,, \quad \omega \in \Omega_\lambda \Rightarrow |\xi| R_\xi^{-1}(\omega) \in \Gamma(\xi_0)\,. \tag{4}$$

PROOF. The mapping given by $(\boldsymbol{R}^n \setminus \{0\}) \times \boldsymbol{R}^n \ni (\xi, \omega) \mapsto \tau = |\xi| R_\xi^{-1}(\omega)$ is continuous at (ξ_0, e_n), which is mapped to ξ_0. So, there exist neighborhoods $U(\xi_0)$ of ξ_0 and $U(e_n)$ of e_n such that

$$|\xi| R_\xi^{-1}(\omega) \in \Gamma(\xi_0)\,, \quad \xi \in U(\xi_0)\,, \ \ \omega \in U(e_n)\,.$$

Furthermore, there exists $\lambda \in (0,1)$ such that $\Omega_\lambda \subseteq U(e_n)$, which means that

$$|\xi| R_\xi^{-1}(\omega) \in \Gamma(\xi_0)\,, \quad \xi \in U(\xi_0)\,, \ \ \omega \in \Omega_\lambda\,.$$

Let $\tilde{\Gamma}(\xi_0)$ be a conic set formed by $U(\xi_0)$, that is defined by

$$\tilde{\Gamma}(\xi_0) = \{a\xi\ ;\ \xi \in U(\xi_0), a > 0\}\,.$$

For every $\xi \in U(\xi_0)$ and $\omega \in \Omega_\lambda$ we have that $|\xi| R_\xi^{-1}(\omega) \in \Gamma(\xi_0)$. Thus, for $a > 0$, we obtain

$$|a\xi| R_{a\xi}^{-1}(\omega) = a|\xi| R_\xi^{-1}(\omega) \in \Gamma(\xi_0)\,,$$

because $\Gamma(\xi_0)$ is a conic set. So, λ and $\tilde{\Gamma}(\xi_0)$ satisfy property (4). □

6. Auxiliary lemmas B. We have already introduced the multidimensional notation (see the very beginning of Section 4).

LEMMA 24. *Let* $\xi_0 \in \prod_{j=1}^k (\boldsymbol{R}^{n_j} \backslash \{0\})$ *and* $r \in (0,1)$. *For every* $\lambda \in (0,1)$ *let* $\Omega_{j,\lambda} \subseteq B_{\lambda r}(e_{n_j})$, $j = 1,\dots,k$. *For every* $r_0 \in \boldsymbol{R}^+$ *and* $\lambda \in (0,1)$ *denote*

$$\tilde{\Gamma}_{r_0,\lambda}(\xi_0) = \{\tau\ ;\ \text{there exists } \xi \in \Gamma(\xi_0, r_0) \text{ such that } |\xi^j|^{-1} R_{\xi^j}(\tau^j) \in \Omega_{j,\lambda} \text{ for } j = 1,\dots,k\}\,.$$

(1) *If for some* $\lambda \in (0,1)$, $e_{n_j} \in \Omega_{j,\lambda}$ *for* $j = 1,\dots,k$, *then for every* $r_0 \in \boldsymbol{R}^+$ *the set* $\tilde{\Gamma}_{r_0,\lambda}(\xi_0)$ *is a conic neighborhood of* ξ_0. *In particular,* $\Gamma(\xi_0, r_0) \subseteq \tilde{\Gamma}_{r_0,\lambda}(\xi_0)$ (τ *are points in* $\prod_{j=1}^k (\boldsymbol{R}^{n_j} \backslash \{0\})$.)

(2) *There exist* $r_0 \in \boldsymbol{R}^+$ *and* $\lambda \in (0,1)$ *such that* $\tilde{\Gamma}_{r_0,\lambda}(\xi_0) \subseteq \Gamma(\xi_0, r)$.

PROOF. (1) If $\tau = (\tau^1,\dots,\tau^k) \in \tilde{\Gamma}_{r_0,\lambda}(\xi_0)$, then there exist $\xi = (\xi^1,\dots,\xi^k) \in \Gamma(\xi_0, r_0)$ and $\omega = (\omega^1,\dots,\omega^j) \in \Omega_\lambda = \Omega_{1,\lambda} \times \cdots \times \Omega_{k,\lambda}$ such that

$$|\xi^j|^{-1} R_{\xi^j}(\tau^j) = \omega^j\,, \quad j = 1,\dots,k\,. \tag{5}$$

Then, for $a > 0$, it follows that $a\xi \in \Gamma(\xi_0, r_0)$ and

$$|a\xi^j|^{-1} R_{a\xi^j}(a\tau^j) = \omega^j\,, \quad j = 1,\dots,k\,.$$

Thus, $a\tau \in \tilde{\Gamma}_{r_0,\lambda}(\xi_0)$. That $\Gamma(\xi_0, r_0) \subseteq \tilde{\Gamma}_{r_0,\lambda}(\xi_0)$ is a consequence of the fact that $e_{n_j} \in \Omega_{j,\lambda}$ and $|\xi^j|^{-1} R_{\xi^j}(\xi^j) = e_{n_j}$ for $j = 1,\dots,k$.

(2) It is necessary to find $r_0 > 0$ and $\lambda \in (0,1)$ such that for every $\tau \in \tilde{\Gamma}_{r_0,\lambda}(\xi_0)$

$$\left| \frac{\tau}{|\tau|} - \frac{\xi_0}{|\xi_0|} \right| < r\,. \tag{6}$$

Let

$$I(\xi,\omega) = \frac{(|\xi^1| R_{\xi^1}^{-1}(\omega^1),\dots,|\xi^k| R_{\xi^k}^{-1}(\omega^k))}{|(|\xi^1| R_{\xi^1}^{-1}(\omega^1),\dots,|\xi^k| R_{\xi^k}^{-1}(\omega^k))|}\,.$$

For every $\tau \in \tilde{\Gamma}_{r_0,\lambda}(\xi_0)$ there exist $\xi \in \Gamma(\xi_0, r_0)$ and $\omega^j \in \Omega_{j,\lambda}$, $j = 1,\dots,k$, such that (5) holds. So, (6) is equivalent to

$$|I(\xi,\omega) - I(\xi_0, e^n)| < r\,.$$

As the mapping

$$\prod_{j=1}^{k}(\boldsymbol{R}^{n_j}\backslash\{0\}) \times \boldsymbol{R}^{n}\backslash\{0\} \ni (\xi, \omega) \mapsto I(\xi, \omega) \tag{7}$$

is continuous at (ξ_0, e^n), there exist neighborhood $U(\xi_0)$ of ξ_0 and $\lambda \in (0, 1)$ such that

$$|I(\xi, \omega) - I(\xi_0, e^n)| < r\,, \quad \xi \in U(\xi_0),\ \omega \in \Omega_\lambda\,.$$

Now, it remains to construct a conic set from $U(\xi_0)$ and to show that for every ξ from this conic set and every $\omega \in \Omega_\lambda$ the same inequality holds. If the given inequality holds for $\xi \in U(\xi_0)$ it will hold for $a\xi$, $a > 0$, because the mapping in (7) maps (ξ, ω) and $(a\xi, \omega)$ to the same point. Finally, let $r_0 > 0$ be such that

$$\Gamma(\xi_0, r_0) \subseteq \{a\xi\ ;\ \xi \in U(\xi_0) \text{ and } a > 0\}\,.$$

This proves the second part of the lemma. □

LEMMA 25. *Let* $\xi_0 \in \prod_{j=1}^{k}(\boldsymbol{R}^{n_j}\backslash\{0\})$ *and* $r_0 \in \boldsymbol{R}^+$. *Let* $\Omega = \Omega_1 \times \cdots \times \Omega_k$ *such that* $\Omega_j \subseteq \boldsymbol{R}^{n_j}$ *are neighborhoods of* e_{n_j} *for* $j = 1, \ldots, k$. *Let*

$$\tilde{\Gamma}(\xi_0) = \{\tau\ ;\ \text{there exists } \xi \in \Gamma(\xi_0, r_0) \text{ such that } |\xi^j|^{-1}R_{\xi^j}(\tau^j) \in \Omega_j \text{ for } j = 1, \ldots, k\}\,.$$

For every $\tau \in \tilde{\Gamma}(\xi_0)$, *we define* $\Omega'(\tau)$ *by*

$$\Omega'(\tau) = \left\{\omega\ ;\ \begin{matrix}\text{there exists } \xi \in \Gamma(\xi_0, r_0) \text{ such that } \omega^j = |\xi^j|^{-1}R_{\xi^j}(\tau^j) \in \Omega_j \\ \text{for } j = 1, \ldots, k\end{matrix}\right\}.$$

(τ' and ω above are points in $\prod_{j=1}^{k}(\boldsymbol{R}^{n_j}\backslash\{0\})$.)

Then there exist a conic neighborhood of ξ_0, denoted by $\widetilde{\Gamma}'(\xi_0)$, and a neighborhood of e^n, called $\prod_{j=1}^{k}\Omega'_j$, such that $\widetilde{\Gamma}'(\xi_0) \subseteq \tilde{\Gamma}(\xi_0)$ and that for every $\tau \in \widetilde{\Gamma}'(\xi_0)$ it holds that $\Omega'(\tau)$ contains $\prod_{j=1}^{k}\Omega'_j$.

PROOF. Let $\tau \in \boldsymbol{R}^n$ and $r > 0$. Denote by $U(\tau, r)$ the product of balls $B_r(\tau^j)$, $j = 1, \ldots, k$. As $\Gamma(\xi_0, r_0)$ is a neighborhood of ξ_0, there exists $r > 0$ such that $U(\xi_0, r) \subseteq \Gamma(\xi_0, r_0)$. We can take $r \leqslant 2\min_j\{|\xi_0^j|\}$. This requirement will become clear later.

First, we want to show that for every $\tau \in U(\xi_0, r/2)$

$$U(e^n, r/(8|\xi_0|)) \subset \bigcup_{\xi \in U(\tau, r/2)} \{(|\xi^1|^{-1}R_{\xi^1}(\tau^1), \ldots, |\xi^k|^{-1}R_{\xi^k}(\tau^k))\}\,.$$

Note that, if $\tau \in U(\xi_0, r/2)$, then $\tau \in \prod_{j=1}^{k}(\boldsymbol{R}^{n_j}\backslash\{0\})$ because for $j = 1, \ldots, k$,

$$|\tau^j| \geqslant |\xi_0^j| - |\xi_0^j - \tau^j| > |\xi_0^j| - r/2 \geqslant 0\,.$$

Also, $\omega \in \prod_{j=1}^{k}(\boldsymbol{R}^{n_j}\backslash\{0\})$ if $\omega \in U(e^n, r/(8|\xi_0|))$, because for $j = 1, \ldots, k$,

$$|\omega^j| \geqslant 1 - |e_{n_j} - \omega^j| > 1 - r/(8|\xi_0|) > 1 - r/(4|\xi_0^j|) \geqslant 1/2\,.$$

Then from Lemma 20 we know there exists $\xi \in \prod_{j=1}^{k}(\boldsymbol{R}^{n_j}\backslash\{0\})$ such that

$$\omega^j = |\xi^j|^{-1}R_{\xi^j}(\tau^j)\,, \quad j = 1, \ldots, k\,.$$

We need to show that this ξ belongs to $U(\tau, r/2)$. We know that

$$|\omega^j - e_{n_j}| = \left|\frac{R_{\xi^j}(\tau^j)}{|\xi^j|} - \frac{R_{\xi^j}(\xi^j)}{|\xi^j|}\right|$$
$$= \frac{1}{|\xi^j|}|\tau^j - \xi^j| = \frac{|\omega^j|}{|\tau^j|}|\tau^j - \xi^j|, \quad j = 1, \dots, k.$$

So, $|\tau^j - \xi^j| = (|\tau^j|/|\omega^j|)|\omega^j - e_{n_j}|$ for $j = 1, \dots, k$. As

$$|\tau^j| \leqslant |\xi_0^j| + |\tau^j - \xi_0^j| \leqslant |\xi_0| + r/2 \leqslant 2|\xi_0|,$$

we conclude that

$$|\tau^j - \xi^j| < \frac{2|\xi_0|}{1/2}\frac{r}{8|\xi_0|} = \frac{r}{2}, \quad j = 1, \dots, k.$$

This implies that $\xi \in U(\tau, r/2)$. Furthermore, $U(\tau, r/2) \subseteq U(\xi_0, r)$ for every $\tau \in U(\xi_0, r/2)$. As $U(\tau, r/2) \subseteq U(\xi_0, r) \subseteq \Gamma(\xi_0, r_0)$, it follows that

$$U(e^n, r/(8|\xi_0|)) \subset \bigcup_{\xi \in \Gamma(\xi_0, r_0)} \{(|\xi^1|^{-1} R_{\xi^1}(\tau^1), \dots, |\xi^k|^{-1} R_{\xi^k}(\tau^k))\} \tag{8}$$

for every $\tau \in U(\xi_0, r/2)$. We construct a conic set from $U(\xi_0, r/2)$ and denote it by $\tilde{\Gamma}'(\xi_0)$. It is easy to verify that (8) holds for every $\tau \in \tilde{\Gamma}'(\xi_0)$. As $U(\xi_0, r/2) \subseteq U(\xi_0, r) \subseteq \Gamma(\xi_0, r_0) \subseteq \tilde{\Gamma}(\xi_0)$ (see Lemma 24), it follows that $\tilde{\Gamma}'(\xi_0) \subseteq \tilde{\Gamma}(\xi_0)$. Then we choose Ω'_j such that $\prod_{j=1}^k \Omega'_j$ is a neighborhood of e^n and $\prod_{j=1}^k \Omega'_j \subseteq U(e^n, r/(8|\xi_0|)) \cap \Omega$. □

LEMMA 26. *Let $\Gamma(\xi_0)$ be a conic neighborhood of $\xi_0 \in \prod_{j=1}^k (\boldsymbol{R}^{n_j} \backslash \{0\})$ and $r \in (0, 1)$. For every $\lambda \in (0, 1)$ let $\Omega_\lambda = \prod_{j=1}^k \Omega_{j,\lambda}$ satisfy $\Omega_{j,\lambda} \subseteq B_{\lambda r}(e_{n_j})$ for $j = 1, \dots, k$. Then there exist $\tilde{\Gamma}(\xi_0) \subseteq \prod_{j=1}^k (\boldsymbol{R}^{n_j} \backslash \{0\})$, a conic neighborhood of ξ_0 and $\lambda \in (0, 1)$ such that*

$$(|\xi^1| R_{\xi^1}^{-1}(\omega^1), \dots, |\xi^k| R_{\xi^k}^{-1}(\omega^k)) \in \Gamma(\xi_0), \quad \xi \in \tilde{\Gamma}(\xi_0), \ \omega \in \Omega_\lambda. \tag{9}$$

PROOF. The mapping $\prod_{j=1}^k (\boldsymbol{R}^{n_j} \backslash \{0\}) \times \boldsymbol{R}^n \ni (\xi, \omega) \mapsto \tau$, where

$$\tau^j = |\xi^j| R_{\xi^j}^{-1}(\omega^j), \quad j = 1, \dots, k,$$

is continuous at (ξ_0, e^n), which is mapped to ξ_0. So, there exist neighborhoods $U(\xi_0) \subseteq \prod_{j=1}^k (\boldsymbol{R}^{n_j} \backslash \{0\})$ of ξ_0 and $U(e^n)$ of e^n such that

$$(|\xi^1| R_{\xi^1}^{-1}(\omega^1), \dots, |\xi^k| R_{\xi^k}^{-1}(\omega^k)) \in \Gamma(\xi_0), \quad \xi \in U(\xi_0), \ \omega \in U(e^n).$$

Furthermore, there exists $\lambda \in (0, 1)$ such that $\Omega_\lambda \subseteq U(e^n)$. Then we have

$$(|\xi^1| R_{\xi^1}^{-1}(\omega^1), \dots, |\xi^k| R_{\xi^k}^{-1}(\omega^k)) \in \Gamma(\xi_0), \quad \xi \in U(\xi_0), \ \omega \in \Omega_\lambda.$$

Let $\tilde{\Gamma}(\xi_0)$ be a conic set constructed from $U(\xi_0)$. Obviously, $\tilde{\Gamma}(\xi_0) \subseteq \prod_{j=1}^k (\boldsymbol{R}^{n_j} \backslash \{0\})$. These λ and $\tilde{\Gamma}(\xi_0)$ satisfy property (9). □

Acknowledgments. We would like to thank G. Hörmann for the discussion and for the preliminary unpublished notes: G. Hörmann and S. Pilipović, *Wavelet Transforms, Wave Packets, and Wave Front Sets of Generalized Functions* (2002). These notes were used as the basic material for this paper.

The paper is partially supported by the Serbian Ministry of Science, Project 144016 and by the Start Project of Austrian Science Fund 'Nonlinear Distributional Geometry'.

REFERENCES

[1] R. BOTT AND J. MILNOR, On the parallelizability of the spheres, Bull. Amer. Math. Soc. 64 (1958), 87–89.

[2] I. DAUBECHIES, Ten lectures on wavelets CBMS-NSF Regional Conf. Ser. in Appl. Math. 61, SIAM, Philadelphia, Penn., 1992.

[3] M. HOLSCHNEIDER, Wavelets, An analysis tool, Oxford Math. Monogr., Clarendon Press, Oxford Univ. Press, New York, 1995.

[4] L. HÖRMANDER, The analysis of linear partial differential operators I–IV, Springer, Berlin, 1983–1985.

[5] L. HÖRMANDER, Lectures on nonlinear hyperbolic differential equations, Math. Appl. (Berlin) 26, Springer, Berlin, 1997.

[6] Y. MEYER, Wavelets and operators, Cambridge Stud. Adv. Math. 37, Cambridge Univ. Press, Cambridge, 1992.

[7] S. MORITOH, Wavelet transforms in Euclidean spaces—their relation with wave front sets and Besov, Triebel-Lizorkin spaces, Tôhoku Math. J. (2) 47 (1995), 555–565.

[8] R. MURENZI, Wavelet transforms associated to the n-dimensional Euclidean group with dilations: signal in more than one dimension, Wavelets (Marseille, 1987), 239–246, Inverse Probl. Theoret. Imaging, Springer, Berlin, 1989.

[9] R. S. PATHAK, The wavelet transform of distributions, Tohoku Math. J. (2) 56 (2004), 411–421.

[10] P. WAGNER. Private communication, 2003.

DEPARTMENT OF MATHEMATICS AND INFORMATICS
UNIVERSITY OF NOVI SAD
TRG D. OBRADOVIĆA 4
21000 NOVI SAD
SERBIA AND MONTENEGRO

E-mail addresses: mirjana@im.ns.ac.yu
pilipovic@im.ns.ac.yu