Tohoku Math. J.
58 (2006), 329–357

ON THE RANGE OF PINNED RANDOM WALKS

Dedicated to Professor Tokuzo Shiga on his sixtieth birthday

YUJI HAMANA

(Received September 2, 2004, revised April 20, 2005)

Abstract. The range of random walks means the number of distinct sites visited at least once by the random walk. In two-or-more-dimensional cases, we established the law of large numbers for the range of simple symmetric random walks under the conditional probability given the event that the last point is the origin. Moreover we studied the large deviations in the upward direction and obtained similar results to the original random walk.

Introduction. For a random walk the range at time n implies the number of distinct points entered by the random walk in the first n steps. Dvoretzky and Erdős [4] have investigated the law of large numbers for the range of simple random walks on a two-or-more-dimensional integer lattice. They supplied the asymptotic behavior of the expectation and the variance of the range at time n of the simple random walk to show its weak law of large numbers. They also proved the strong law of large numbers. However, the proof has a gap in the two-dimensional case, which was finally filled by Jain and Pruitt [12] under more general situation.

The condition that the random walk moves in one step to the nearest-neighbor points with the same probability is not necessary. Indeed, the same conclusions as those obtained by Dvoretzky and Erdős can be proved under weaker assumptions (cf. Jain and Pruitt [12, 15], Spitzer [21]). The small deviation results were studied under some suitable assumptions. The central limit theorems are given in [11, 13, 15, 16, 18], the law of the iterated logarithms are given in [2, 14] and almost sure invariance principles are given in [2, 6]. Moreover, several results concerning large deviations are supplied in [3, 7–9].

In this article we study the weak law of large numbers and the large deviations in the upward direction for the range at time $2n$ of the simple random walk, under the conditional probability given the event that the random walk returns to the origin at time $2n$. One might guess that this condition has no effect on the behavior of the range of two-dimensional random walks because of recurrence, while it has much influence for transient random walks. However, the conclusion in this paper shows that the range of the pinned random walk behaves just like the range of the original random walk.

2000 *Mathematics Subject Classification.* 60G50; Secondary 60F05, 60F10.

Key words and phrases. Pinned random walks, law of large numbers, large deviations.

This work was partly supported by the Grant-in-Aid for Scientific Research (C), Japan Society for the Promotion of Science.

The continuous analog of the range of a pinned random walk is the volume of the Wiener sausage for a Brownian bridge. The expectation of the volume of the pinned Wiener sausage was obtained by van den Berg and Bolthausen [1] in the two-dimensional case, and by McGillivray [20] in three-or-more-dimensional cases. They concluded that the leading term of the expectation of the volume of the pinned Wiener sausage is the same as that of the (non-pinned) Wiener sausage. However, the law of large numbers and large deviations are not discussed.

1. Preliminaries and notation. By a random walk $\{S_n\}_{n=0}^{\infty}$ on the d-dimensional integer lattice $\boldsymbol{Z}^d$, we mean a sequence of random variables defined as $S_0 = 0$ and $S_n = X_1 + X_2 + \cdots + X_n$, where $\{X_n\}_{n=1}^{\infty}$ is a sequence of independent identically distributed random variables with values in $\boldsymbol{Z}^d$. The simple random walk means a random walk such that $P[X_1 = x] = 1/2d$ if $x \in \boldsymbol{Z}^d$ is a unit vector and 0 otherwise. Throughout this paper we consider the d-dimensional simple random walk. Let γ_d be the probability that a random walk never returns to the starting point. It is well known that γ_d is strictly positive if $d \geqq 3$ and equal to 0 otherwise.

Since it will be convenient to regard the random walk as a Markov chain, we will use some terminology of general Markov chains. For $x \in \boldsymbol{Z}^d$ let $P_x[\,\cdot\,]$ denote the probability measures of events related to the random walk starting at x. When $x = 0$, we simply write $P[\,\cdot\,]$ instead of $P_0[\,\cdot\,]$. For $n \geqq 0$ and $x, y \in \boldsymbol{Z}^d$ the notation $p^n(x, y)$ means $P_x[S_n = y]$. Note that $p^n(x, y) = p^n(0, y - x)$. There is a positive constant A such that

$$p^n(0, x) \leqq An^{-d/2} \tag{1.1}$$

for all $x \in \boldsymbol{Z}^d$ and $n \geqq 1$ (cf. Spitzer [21]). For $x \in \boldsymbol{Z}^d$ let τ_x be the first hitting time of x; that is, $\tau_x = \inf\{n \geqq 1; S_n = x\}$. If there are no positive integers with $S_n = x$, then $\tau_x = \infty$. The taboo probabilities are defined by

$$p_z^n(x, y) = P_x[S_n = y, \tau_z \geqq n]\,,$$
$$p_{zw}^n(x, y) = P_x[S_n = y, \tau_z \geqq n, \tau_w \geqq n]\,.$$

We will use u_n for $p^n(0, 0)$ and f_n for $p_0^n(0, 0)$. If n is odd, both u_n and f_n are equal to 0. For $x \in \boldsymbol{Z}^d$ let $|x| = |x_1| + \cdots + |x_d|$ and $\|x\| = \sqrt{x_1^2 + \cdots + x_d^2}$, where x_j is the jth component of x. If $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are sequences of real numbers such that $c_n > 0$ for $n \geqq 1$, then $a_n = b_n + O[c_n]$ means that $(a_n - b_n)/c_n$ remains bounded; $a_n = b_n + o[c_n]$ means that $(a_n - b_n)/c_n$ converges to zero as $n \to \infty$; $a_n \sim c_n$ means that a_n/c_n converges to one as $n \to \infty$, respectively.

It is well known that uniformly in $x \in \boldsymbol{Z}^d$

$$p^n(0, x) = 2\left(\frac{d}{2\pi n}\right)^{d/2} \exp\left(-\frac{d\|x\|^2}{2n}\right) + O\left[\frac{1}{n^{1+d/2}}\right] \tag{1.2}$$

if $n + |x|$ is even (cf. Lawler [17]). This is called the local central limit theorem. An immediate conclusion of (1.2) is that

$$u_{2n} = \kappa_d n^{-d/2} + O[n^{-1-d/2}]\,, \tag{1.3}$$

where $\kappa_d = 2(d/4\pi)^{d/2}$, which then implies

$$\frac{1}{u_{2n}} = \frac{n^{d/2}}{\kappa_d} + O[n^{d/2-1}]\,. \tag{1.4}$$

In particular, $1/u_{2n}$ is bounded by a constant multiple of $n^{d/2}$. Another useful formula is that

$$u_m = \sum_{k=1}^{m-1} f_k u_{m-k} + f_m\,. \tag{1.5}$$

Let $r_n = P_0[\tau_0 > n]$ for $n \geqq 1$. It was proved in Dvoretzky and Erdős [4] that

$$r_n = \begin{cases} \dfrac{\pi}{\log n} + O\left[\dfrac{\log\log n}{\log^2 n}\right] & \text{if } d = 2\,, \\ \gamma_d + O[n^{1-d/2}] & \text{if } d \geqq 3\,, \end{cases} \tag{1.6}$$

where $\log^\alpha x$ stands for $(\log x)^\alpha$ for real numbers α and $x > 0$.

Lastly, we give a classification of random walks. Let Σ be the support of X_1. If the smallest subgroup of $\boldsymbol{Z}^d$ generated by Σ coincides with $\boldsymbol{Z}^d$, the random walk is called adapted (aperiodic in the sense of Spitzer [21]) and d is called the dimension of the random walk. The period of the random walk is defined to be the greatest common division of the set of positive integers n such that $P[S_n = 0]$ is positive. The random walk is called aperiodic (strongly aperiodic in the sense of Spitzer [21]) if its period is 1 and is called periodic otherwise. An equivalent criterion for aperiodicity is that the smallest subgroup generated by $x+\Sigma$ coincides with $\boldsymbol{Z}^d$ for any $x \in \boldsymbol{Z}^d$ (see Spitzer [21]). We note that the simple random walk on $\boldsymbol{Z}^d$ is adapted and d-dimensional but periodic, since its period is two.

Throughout this paper, $C_1, C_2, \dots, C_{24}$ will denote suitable positive real constants.

2. Main results. For a positive integer n let

$$R_n = |\{S_1, S_2, \dots, S_n\}|\,,$$

where $|A|$ denotes the cardinality of a set A. We call R_n the range at time n of the random walk or the range of the random walk up to time n. The asymptotic behavior of the expectation of R_n was obtained by Dvoretzky and Erdős [4]. Their result shows that

$$ER_n = \begin{cases} \dfrac{\pi n}{\log n} + O\left[\dfrac{n\log\log n}{\log^2 n}\right] & \text{if } d = 2\,, \\ \gamma_3 n + O[n^{1/2}] & \text{if } d = 3\,, \\ \gamma_4 n + O[\log n] & \text{if } d = 4\,, \\ \gamma_d n + c_d + O[n^{2-d/2}] & \text{if } d \geqq 5\,, \end{cases}$$

for some suitable positive constant c_d. The problem is to obtain the explicit order of the second term for dimension four or lower. In the three- and four-dimensional cases we can give it, however there are no further results in the two-dimensional case.

PROPOSITION 2.1. *If $d = 3$,*

$$ER_n = \gamma_3 n + 2^{5/2}\gamma_3^2\kappa_3 n^{1/2} + O\left[\frac{n^{1/2}}{\log^\delta n}\right]$$

for any given $\delta > 0$ and if $d = 4$,

$$ER_n = \gamma_4 n + 2\gamma_4^2\kappa_4 \log n + O[1]\,.$$

If $\operatorname{Var} R_n = o[(ER_n)^2]$ is proved, we can obtain the weak law of large numbers with the help of the Chebyshev inequality. Indeed, Dvoretzky and Erdős [4] showed that

$$\operatorname{Var} R_n = \begin{cases} O\left[\dfrac{n^2 \log\log n}{\log^3 n}\right] & \text{if } d = 2\,, \\ O[n^{3/2}] & \text{if } d = 3\,, \\ O[n \log n] & \text{if } d = 4\,, \\ O[n] & \text{if } d \geqq 5\,. \end{cases}$$

One of our purposes in this paper is to establish the weak law of large numbers for R_{2n} for pinned simple random walks. For $n \geqq 1$ and $y \in \mathbf{Z}^d$ such that $n + |y|$ is even, let

$$P_{n,y}[\,\cdot\,] = P[\,\cdot\,|S_n = y]\,.$$

For a random variable X we denote by $E_n X$ and $\operatorname{Var}_n X$ the expectation and the variance of X under the probability measure $P_{2n,0}$, respectively. Namely, $E_n X$ means $E[X|S_{2n} = 0]$ and $\operatorname{Var}_n X$ means $E_n(X - E_n X)^2$. We can obtain the asymptotic behavior of the expectation of R_{2n} under $P_{2n,0}$.

THEOREM 2.2. *If $d = 2$,*

$$E_n R_{2n} = \frac{2\pi n}{\log n} + O\left[\frac{n \log\log n}{\log^2 n}\right]. \tag{2.1}$$

If $d = 3$,

$$E_n R_{2n} = 2\gamma_3 n + O\left[\frac{n^{1/2}}{\log^\delta n}\right] \tag{2.2}$$

for any $\delta > 0$. If $d = 4$,

$$E_n R_{2n} = 2\gamma_4 n - 4\gamma_4^2\kappa_4 \log n + O[1]\,. \tag{2.3}$$

If $d \geqq 5$,

$$E_n R_{2n} = 2\gamma_d n + O[1]\,. \tag{2.4}$$

It is remarkable that the second term of $E_n R_{2n}$ is small in comparison with that of ER_n in the three- and four-dimensional cases. Unfortunately, there are no further results concerning the second term for other dimensional cases.

In order to show the weak law of large numbers for R_{2n} under $P_{2n,0}$, it suffices to calculate the variance of R_{2n}.

THEOREM 2.3. *We have*

$$\operatorname{Var}_n R_{2n} = \begin{cases} O\left[\dfrac{n^2 \log\log n}{\log^3 n}\right] & \text{if } d = 2, \\ O[n^{3/2}] & \text{if } d = 3, \\ O[n \log n] & \text{if } d = 4, \\ O[n] & \text{if } d \geqq 5. \end{cases}$$

This theorem immediately leads to the following.

COROLLARY 2.4. *We have*

$$\lim_{n\to\infty} P_{2n,0}\left[\left|\frac{\log n}{2n} R_{2n} - \pi\right| > \rho_n \left(\frac{\log\log n}{\log n}\right)^{1/2}\right] = 0 \quad \text{if } d = 2,$$

$$\lim_{n\to\infty} P_{2n,0}\left[\left|\frac{R_{2n}}{2n} - \gamma_3\right| > \rho_n n^{-1/4}\right] = 0 \quad \text{if } d = 3,$$

$$\lim_{n\to\infty} P_{2n,0}\left[\left|\frac{R_{2n}}{2n} - \gamma_4\right| > \rho_n \left(\frac{\log n}{n}\right)^{1/2}\right] = 0 \quad \text{if } d = 4,$$

$$\lim_{n\to\infty} P_{2n,0}\left[\left|\frac{R_{2n}}{2n} - \gamma_d\right| > \rho_n n^{-1/2}\right] = 0 \quad \text{if } d \geqq 5,$$

whenever the sequence $\{\rho_n\}_{n=1}^{\infty}$ of real numbers satisfies that $\rho_n \to \infty$ as n tends to infinity.

We remark that these asymptotic behaviors of $\operatorname{Var} R_n$ were improved by Jain and Pruitt [13, 15, 16]. For an adapted random walk they proved that there exists a positive constant σ such that $\operatorname{Var} R_n \sim \sigma^2 n$ if $d \geqq 4$ and $\gamma_d < 1$, and that $\operatorname{Var} R_n$ is asymptotically equal to $n\xi(n)$ for some non-decreasing slowly varying function ξ if $d = 3$ and $\gamma_d < 1$. Moreover, they showed that there exists a positive constant ϱ such that $\operatorname{Var} R_n \sim \varrho^2 n^2/\log^4 n$ for the two-dimensional random walk with zero mean and finite variance. However, we have no further result for $\operatorname{Var}_n R_{2n}$ other than Theorem 2.3.

Another purpose of this paper is to show the large deviations for R_{2n} under $P_{2n,0}$ in the upward direction. Hamana and Kesten [8] proved for a two-or-more-dimensional adapted random walk that there exists

$$\psi(x) = -\lim_{n\to\infty} \frac{1}{n} \log P[R_n \geqq xn]$$

for any $x \in \boldsymbol{R}$ (but $\psi(x)$ may equal $+\infty$) and that ψ has the following properties:

(1) $\psi(x) = 0$ for $x \leqq \gamma_d$;

(2) $0 < \psi(x) < \infty$ for $\gamma_d < x \leqq 1$;

(3) $\psi(x) = \infty$ for $x > 1$;
(4) ψ is continuous and convex on $[0, 1]$;
(5) ψ is strictly increasing on $[\gamma_d, 1]$.

For one-dimensional adapted random walks the same result was shown by Hamana and Kesten [9] except for the convexity of ψ. It should be remarked that the definition of R_n is $|\{S_0, \dots, S_{n-1}\}|$ in their paper, which is slightly different from that in this article. However, we cannot find any difference in adopting either definition. Indeed, $|\{S_0, \dots, S_{n-1}\}|$ is equal to $|\{S_n - S_0, \dots, S_n - S_{n-1}\}|$ and thus has the same distribution as $|\{S_1, \dots, S_n\}|$ by considering the time reversed random walk or by relabeling X_j as X_{n-j+1} for $1 \leqq j \leqq n$. The following theorem implies that the range of the pinned simple random walk satisfies the upward large deviations and that the limiting function is the same as that of the original random walk in two-or-more-dimensional cases.

THEOREM 2.5. *If $d \geqq 2$, then*

$$\psi(x) = -\lim_{n\to\infty} \frac{1}{2n} \log P_{2n,0}[R_{2n} \geqq 2xn]\,. \tag{2.5}$$

It is not difficult to derive the following corollary from Theorem 2.5.

COROLLARY 2.6. *Let $\{y_n\}_{n=1}^{\infty}$ be a sequence of points in $\boldsymbol{Z}^d$ such that $n + |y_n|$ is even and that $|y_n| = o[n]$. If $d \geqq 2$, then*

$$\psi(x) = -\lim_{n\to\infty} \frac{1}{n} \log P_{n,y_n}[R_n \geqq xn]\,. \tag{2.6}$$

In the case that $x = 1$, Hammersley [10] has already proved (2.5) and (2.6), and these proofs are also supplied in Madras and Slade [19].

3. Proof of Proposition 2.1 and Theorem 2.2. For a calculation of ER_n and E_nR_{2n}, the estimate of f_n will play an important role. We first give an estimate of f_n for dimension 2 or higher. If $d = 2$, Jain and Pruitt [15] obtained that

$$f_n \sim \frac{2\pi(\det \Xi)}{n \log^2 n} \tag{3.1}$$

for an aperiodic random walk with mean 0 and finite variance, where Ξ is the symmetric and positive definite matrix such that $E(\theta, X_1)^2 = \|\Xi\theta\|^2$ for $\theta \in \boldsymbol{R}^2$ and $(\cdot\,,\,\cdot)$ is the standard inner product on $\boldsymbol{R}^2$. We note that such Ξ is always defined for adapted random walks on $\boldsymbol{Z}^d$ with mean 0 and finite variance, and a simple calculation shows that

$$\Xi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for the two-dimensional simple random walk.

Although the simple random walk is not aperiodic, we can apply (3.1) to obtain the similar limiting behavior in the following fashion. For $k \geqq 1$, let

$$Y_k = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}(X_{2k-1} + X_{2k})\,.$$

Then $\{Y_k\}_{k=1}^{\infty}$ is a sequence of independent and identically distributed random variables taking values in $\boldsymbol{Z}^2$. A new random walk $\{Z_n\}_{n=0}^{\infty}$, defined by $Z_0 = 0$ and $Z_n = Y_1 + \cdots + Y_n$, moves on $\boldsymbol{Z}^2$ and is aperiodic. Note that $S_{2m} = 0$ is equivalent to $Z_m = 0$ for each $m \geqq 1$. Moreover, it follows that

$$f_{2n} = P[Z_n = 0, Z_k \neq 0 \ \text{ for } \ k = 1, \dots, n-1]$$

since S_{2m-1} is never equal to 0 for $m \geqq 1$. In virtue of (3.1), f_{2n} is asymptotically equal to $2\pi(\det \Lambda)/n \log^2 n$, where Λ is the symmetric and positive definite matrix such that $E(\theta, Y_1)^2 = \|\Lambda\theta\|^2$ for $\theta \in \boldsymbol{R}^2$. It is easy to show that $\det \Lambda = 1/2$. Indeed, the equality $\Lambda = \Xi\, {}^t\!A$ follows from the identity

$$E(\theta, Y_1)^2 = E(\theta, AX_1)^2 = \|\Xi\, {}^t\!A\theta\|^2,$$

where

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Therefore, we have

$$f_{2n} \sim \frac{\pi}{n \log^2 n}. \tag{3.2}$$

The argument above only yields the leading term of f_{2n}; however, (3.2) will be used in deriving asymptotic behavior of the second term of f_{2n} in the two-dimensional case.

LEMMA 3.1. *If $d = 2$, then*

$$f_{2n} = \frac{\pi}{n \log^2 n} + O\left[\frac{\log \log n}{n \log^3 n}\right].$$

If $d \geqq 3$, then for any $\delta > 0$

$$f_{2n} = \gamma_d^2 \kappa_d n^{-d/2} + O[n^{-d/2} \log^{-\delta} n].$$

PROOF. We can prove this lemma in an analogous manner to Theorem 4.1 in Jain and Pruitt [15]. Note that

$$f_{2n} = \sum_{x \neq 0} \sum_{y \neq 0} p_0^N(0, x) p_0^{2n-2N}(x, y) p_0^N(y, 0) \tag{3.3}$$

for integers $1 \leqq N \leqq n$. We first consider the effect of replacing $p_0^{2n-2N}(x, y)$ with $p^{2n-2N}(x, y)$ in (3.3), for which the following equality will be useful:

$$\begin{aligned} p^{2m}(x, y) - p_0^{2m}(x, y) &= P[\tau_0 \leqq 2m, S_{2m} = y] \\ &= P[\tau_0 \leqq m, S_{2m} = y] + P[m < \tau_0 \leqq 2m, S_{2m} = y]. \end{aligned} \tag{3.4}$$

We first consider the two-dimensional case. It follows from the second equality in (3.4) that

$$0 \leqq p^{2m}(x, y) - p_0^{2m}(x, y) \leqq \sum_{j=1}^{m} p_0^j(x, 0) p^{2m-j}(0, y) + \sum_{j=m}^{2m-1} p^j(x, 0) p_0^{2m-j}(0, y)$$

for $m \geqq 1$. Applying it to (3.3) yields

$$\left| f_{2n} - \sum_{x\neq 0}\sum_{y\neq 0} p_0^N(0,x)p^{2n-2N}(x,y)p_0^N(y,0)\right| \tag{3.5}$$

$$\leqq \sum_{j=1}^{n-N}\sum_{x\neq 0}\sum_{y\neq 0} p_0^N(0,x)p_0^j(x,0)p^{2n-2N-j}(0,y)p_0^N(y,0) \tag{3.6}$$

$$+ \sum_{j=n-N}^{2n-2N-1}\sum_{x\neq 0}\sum_{y\neq 0} p_0^N(0,x)p^j(x,0)p_0^{2n-2N-j}(0,y)p_0^N(y,0)\,. \tag{3.7}$$

Taking the summation on x and applying (1.1), we see that (3.6) is bounded by

$$A\sum_{j=1}^{n-N}\sum_{y\neq 0} f_{N+j}(2n-2N-j)^{-1}p_0^N(y,0)\,. \tag{3.8}$$

Since $p_0^k(0,z) = p_z^k(0,z)$ for $k \geqq 1$ and $z \in \boldsymbol{Z}^d$, (3.8) is dominated by

$$A(n-N)^{-1}r_N\sum_{j=1}^{n-N} f_{N+j} \leqq \frac{C_1}{(n-N)\log N}\sum_{j=N+1}^{n} f_j\,.$$

Substituting $i = 2n-2N-j$ in the summation on j, we see that (3.7) is equal to

$$\sum_{j=1}^{n-N}\sum_{x\neq 0}\sum_{y\neq 0} p_0^N(0,x)p^{2n-2N-i}(x,0)p_0^i(0,y)p_0^N(y,0)\,,$$

which coincides with (3.6). Therefore, a bound of (3.5) is given by

$$\frac{C_2}{(n-N)\log N}\sum_{j=N+1}^{n}\frac{1}{j\log^2 j}\,. \tag{3.9}$$

Here we have applied (3.2). We take $N = \lfloor n/\log^3 n\rfloor$, where $\lfloor x\rfloor$ means the longest integer which is not grater than x. Then (3.9) and also (3.5) are larger than or equal to a constant multiple of

$$\frac{1}{n\log n}\left(\frac{1}{\log N} - \frac{1}{\log n}\right) = O\left[\frac{\log\log n}{n\log^3 n}\right].$$

We are now going to estimate (3.5) in the three-or-more-dimensional cases. Let $\delta > 0$ be given and take $N = \lfloor n/\log^\delta n\rfloor$. By the first equality in (3.4), we obtain

$$0 \leqq p^{2n-2N}(x,y) - p_0^{2n-2N}(x,y) \leqq \sum_{j=1}^{2n-2N} p^j(x,0)p^{2n-2N-j}(0,y)\,,$$

which immediately implies that (3.5) is bounded by

$$\sum_{j=1}^{2n-2N} u_{N+j}u_{2n-N-j} \leqq A^2\sum_{j=1}^{2n-2N}(N+j)^{-d/2}(2n-N-j)^{-d/2} = O[N^{1-d}]\,,$$

which is, in particular, of order $n^{-d/2}\log^{-\delta} n$. Consequently, we obtain that

$$f_{2n} = \sum_{x\neq 0}\sum_{y\neq 0} p_0^N(0,x)p^{2n-2N}(x,y)p_0^N(y,0) + \begin{cases} O\left[\dfrac{\log\log n}{n\log^3 n}\right] & \text{if } d=2\,. \\ O[n^{-d/2}\log^{-\delta} n] & \text{if } d \geqq 3\,. \end{cases} \tag{3.10}$$

It suffices to calculate the double sum in (3.10). It follows from (1.2) that

$$\begin{aligned} p^{2n-2N}(x,y) &= \kappa_d(n-N)^{-d/2}\exp\left\{-\frac{d\|y-x\|^2}{4(n-N)}\right\} + O[(n-N)^{-1-d/2}] \\ &= \kappa_d(n-N)^{-d/2} + (\|y-x\|^2+1)\times O[(n-N)^{-1-d/2}]\,. \end{aligned}$$

Since $\|y-x\|^2 \leqq 2(\|y\|^2+\|x\|^2)$, we see that (3.10) is

$$\kappa_d(n-N)^{-d/2}\sum_{x\neq 0}\sum_{y\neq 0} p_0^N(0,x)p_0^N(y,0) + O[N(n-N)^{-1-d/2}]\,, \tag{3.11}$$

where we have used

$$\sum_{z\in\mathbf{Z}^d}\|z\|^2 p_0^N(0,z) \leqq \sum_{z\in\mathbf{Z}^d}\|z\|^2 p^N(0,z) = N\,.$$

Since the effect of exchanging $(n-N)^{-d/2}$ for $n^{-d/2}$ is of order $N(n-N)^{-1-d/2}$, it is verified that (3.11) is

$$\kappa_d n^{-d/2} r_N^2 + O[N(n-N)^{-1-d/2}]\,. \tag{3.12}$$

If $d \geqq 3$, (1.6) implies that (3.12) and also (3.10) are

$$\gamma_d^2\kappa_d n^{-d/2} + O[n^{-d/2}\log^{-\delta} n]\,.$$

If $d=2$, (1.6) implies that

$$r_N = \frac{\pi}{\log N} + O\left[\frac{\log\log N}{\log^2 N}\right] = \frac{\pi}{\log n} + O\left[\frac{\log\log n}{\log^2 n}\right],$$

since $N = \lfloor n/\log^3 n\rfloor$. Therefore, (3.12) yields

$$\frac{\pi}{n\log^2 n} + O\left[\frac{\log\log n}{n\log^3 n}\right]. \qquad \square$$

For integers $1 \leqq j < n$ let

$$Z_j^n = \begin{cases} 1 & \text{if } S_j \neq S_\alpha \text{ for all } \alpha \in \{j+1,\dots,n\}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then R_n can be written as

$$R_n = 1 + \sum_{j=1}^{n-1} Z_j^n\,.$$

This identity is very useful in calculating the expectation and the variance of R_n.

We first prove Proposition 2.1. For $m \geqq 1$ we have

$$ER_{2m} = 1 + \sum_{j=1}^{2m-1} r_{2m-j} = 2\gamma_d m + 1 - \gamma_d + \sum_{j=1}^{2m-1} \sum_{i=j+1}^{\infty} f_i \,.$$

Since $f_i = 0$ if i is odd, the double sum in the right-hand side is equal to

$$2 \sum_{k=3}^{m-1} \sum_{l=k+1}^{\infty} f_{2l} + O[1]\,.$$

It follows from Lemma 3.1 that for any $\delta > 0$

$$\begin{aligned} \sum_{l=k+1}^{\infty} f_{2l} &= \gamma_d^2 \kappa_d \sum_{l=k+1}^{\infty} l^{-d/2} + O\left[\sum_{l=k+1}^{\infty} l^{-d/2} \log^{-\delta} l\right] \\ &= \frac{2}{d-2} \gamma_d^2 \kappa_d k^{1-d/2} + O[k^{1-d/2} \log^{-\delta} k] \end{aligned} \tag{3.13}$$

if $d \geqq 3$. It is easy to show that

$$\sum_{k=3}^{m-1} k^{1-d/2} = \begin{cases} 2m^{1/2} + O[1] & \text{if } d = 3\,, \\ \log m + O[1] & \text{if } d = 4\,, \end{cases}$$

which immediately implies that

$$ER_{2m} = 2\gamma_3 m + 4\gamma_3^2 \kappa_3 m^{1/2} + O\left[\sum_{k=3}^{m} k^{-1/2} \log^{-\delta} k\right]$$

if $d = 3$ and that

$$ER_{2m} = 2\gamma_4 m + 2\gamma_4^2 \kappa_4 \log m + O\left[\sum_{k=3}^{m} k^{-1} \log^{-\delta} k\right]$$

if $d = 4$. The error term in the four-dimensional case can be seen as $O[1]$ by taking $\delta = 2$. Estimating the error term in the three-dimensional case is also not difficult. Let $M = \lfloor m/\log^{2\delta} m \rfloor$. Then we obtain

$$\sum_{k=3}^{m} \frac{1}{k^{1/2} \log^{\delta} k} \leqq C_3 \sum_{k=3}^{M} \frac{1}{k^{1/2}} + \frac{1}{\log^{\delta} M} \sum_{k=M}^{m} \frac{1}{k^{1/2}} = O\left[\frac{m^{1/2}}{\log^{\delta} m}\right].$$

Since $R_{2m} \leqq R_{2m+1} \leqq R_{2m} + 1$, the asymptotic behavior of ER_{2m+1} is the same as that of ER_{2m}. Thus, we conclude that

$$ER_n = \gamma_3 n + 4\gamma_3^2 \kappa_3 (\lfloor n/2 \rfloor)^{1/2} + O[n^{1/2}/\log^{\delta} n]$$

if $d = 3$ and that

$$ER_n = \gamma_4 n + 2\gamma_4^2 \kappa_4 \log \lfloor n/2 \rfloor + O[1]$$

if $d = 4$. This proves Proposition 2.1.

We next prove Theorem 2.2. It suffices to calculate the expectation of R_{2n} restricted on the event $\{S_{2n} = 0\}$. Classifying the event by the arrival site at time j, we obtain for $n \geqq 1$ that

$$E[R_{2n}; S_{2n} = 0] = u_{2n} + \sum_{j=1}^{2n-1} \sum_{y \neq 0} p^j(0, y) p_y^{2n-j}(y, 0)$$

by the Markov property. To calculate the second term of the right-hand side, we need the following lemma.

LEMMA 3.2. *For $n \geqq 1$ and $m \geqq 2$ we have*

$$\sum_{y \in \mathbf{Z}^d \setminus \{0\}} p^n(0, y) p_y^m(y, 0) = u_{n+m} - \sum_{k=1}^{m} f_k u_{n+m-k} .$$

PROOF. It can be shown by the Markov property that for $y \neq 0$ and $m \geqq 2$

$$p^m(y, 0) = p_y^m(y, 0) + \sum_{k=1}^{m-1} f_k p^{m-k}(y, 0) .$$

Thus, we have

$$\begin{aligned} \sum_{y \neq 0} p^n(0, y) p_y^m(y, 0) &= \sum_{y \neq 0} p^n(0, y) p^m(y, 0) - \sum_{k=1}^{m-1} \sum_{y \neq 0} f_k p^n(0, y) p^{m-k}(y, 0) \\ &= u_{n+m} - u_n u_m - \sum_{k=1}^{m-1} f_k u_{n+m-k} + u_n \sum_{k=1}^{m-1} f_k u_{m-k} . \end{aligned}$$

Then the lemma follows by applying (1.5) to the last summation on the right-hand side of this identity. □

It is immediate from Lemma 3.2 that $E[R_{2n}; S_{2n} = 0]$ is equal to

$$2nu_{2n} - \sum_{j=1}^{2n-2} \sum_{l=1}^{2n-j} f_l u_{2n-l} = 2nu_{2n} - 2 \sum_{j=1}^{n-1} \sum_{l=1}^{n-j} f_{2l} u_{2(n-l)} .$$

Here we note that $p^1(y, 0) = p_y^1(y, 0)$, which is equal to 0 if $y = 0$, and that both f_m and u_m are valid only when m is even. Changing the order of summations, the double sum on the right-hand side yields

$$\sum_{l=1}^{n-1} (n - l) f_{2l} u_{2(n-l)} , \tag{3.14}$$

which can be expressed by (1.5) as

$$n(u_{2n} - f_{2n}) - \sum_{l=1}^{n-1} l f_{2l} u_{2(n-l)} .$$

Substituting $h = n - l$ in the summation on l, (3.14) yields

$$2\sum_{h=1}^{n-1} hu_{2h} f_{2(n-h)} . \tag{3.15}$$

Therefore, we obtain $E[R_{2n}; S_{2n} = 0]$ in the following two forms:

$$E[R_{2n}; S_{2n} = 0] = \begin{cases} 2\displaystyle\sum_{l=1}^{n-1} lf_{2l}u_{2(n-l)} + 2nf_{2n} , \\ 2nu_{2n} - 2\displaystyle\sum_{h=1}^{n-1} hu_{2h} f_{2(n-h)} . \end{cases} \tag{3.16}$$

The first is useful in the two-dimensional case and the latter in the three-or-more-dimensional cases.

We now consider the five-or-more-dimensional cases. It suffices to calculate (3.15). Let $N = \lfloor n/2 \rfloor$. Then (3.15) is equal to

$$2\sum_{h=1}^{N-1} hu_{2h} f_{2(n-h)} + 2\sum_{h=N}^{n-1} (hu_{2h} - nu_{2n}) f_{2(n-h)} + 2nu_{2n} \sum_{h=N}^{n-1} f_{2(n-h)} . \tag{3.17}$$

By (1.1), the first term of (3.17) is less than or equal to

$$2A^2 \sum_{h=1}^{N-1} h^{1-d/2}(n-h)^{-d/2} \leqq C_4(n-N)^{-d/2} \sum_{h=1}^{n} h^{1-d/2} ,$$

which is of order $n^{-d/2}$ if $d \geqq 5$. It follows from (1.3) that for $1 \leqq h < n$

$$hu_{2h} - nu_{2n} = \kappa_d(h^{1-d/2} - n^{1-d/2}) + O[h^{-d/2}] . \tag{3.18}$$

The mean value theorem implies that

$$|hu_{2h} - nu_{2n}| \leqq C_5(n-h)h^{-d/2} , \tag{3.19}$$

from which the absolute value of the second term of (3.17) is bounded by a constant multiple of

$$\sum_{h=N}^{n-1} h^{-d/2}(n-h)^{1-d/2} \leqq N^{-d/2} \sum_{h=1}^{n-1} (n-h)^{1-d/2} ,$$

which has the same bound as the first term of (3.17). The third term of (3.17) yields

$$2nu_{2n}\{1 - P[\tau_0 > n - N]\} = 2nu_{2n}\{1 - \gamma_d + O[n^{1-d/2}]\} ,$$

where (1.6) has been applied. Consequently, we obtain (2.4) from (1.4).

We next consider the three- and four-dimensional cases. By the second formula of (3.16), we obtain

$$E_n R_{2n} = 2n\left\{1 - \sum_{h=1}^{n-1} f_{2(n-h)}\right\} - \frac{2}{u_{2n}} \sum_{h=1}^{n-1} (hu_{2h} - nu_{2n}) f_{2(n-h)} . \tag{3.20}$$

It follows from (3.19) that if $d \geqq 3$,

$$\left|\sum_{h=1}^{n-1}(hu_{2h} - nu_{2n}) f_{2(n-h)}\right| \leqq C_6 \sum_{h=1}^{n-1} h^{-d/2}(n-h)^{1-d/2} = O[n^{1-d/2}] .$$

Applying (1.4) and (3.18) to the second term of the right-hand side of (3.20), we obtain

$$-2n^{d/2}\sum_{h=1}^{n-1}(h^{1-d/2} - n^{1-d/2}) f_{2(n-h)} + O\left[n^{d/2}\sum_{h=1}^{n-1} h^{-d/2}(n-h)^{-d/2}\right] + O[1] ,$$

which means that if $d \geqq 3$,

$$E_n R_{2n} = 2\gamma_d n + 2n\sum_{h=n}^{\infty} f_{2h} - 2n^{d/2}\sum_{h=1}^{n-1}(h^{1-d/2} - n^{1-d/2}) f_{2(n-h)} + O[1] . \tag{3.21}$$

It follows from (3.13) that the second term of the right-hand side of (3.21) yields

$$4\gamma_3^2\kappa_3 n^{1/2} + O[n^{1/2}\log^{-\delta} n] \tag{3.22}$$

if $d \geqq 3$ and is bounded if $d = 4$. Thus, it suffices to estimate

$$\sum_{h=1}^{n-1}(h^{1-d/2} - n^{1-d/2}) f_{2(n-h)} . \tag{3.23}$$

In virtue of Lemma 3.1, if $d = 4$, (3.23) yields

$$\frac{\gamma_4^2\kappa_4}{n}\sum_{h=1}^{n-1}\frac{1}{h(n-h)} + O\left[\frac{1}{n}\sum_{h=1}^{n-2}\frac{1}{h(n-h)\log^2(n-h)}\right]. \tag{3.24}$$

Recall that $N = \lfloor n/2 \rfloor$. The summation in the second term of (3.24) is bounded by

$$\frac{1}{(n-N)\log^2(n-N)}\sum_{h=1}^{N}\frac{1}{h} + \frac{1}{N}\sum_{h=N}^{n-2}\frac{1}{(n-h)\log^2(n-h)} ,$$

which is of order $1/n$. The summation in the first term of (3.24) gives

$$\frac{1}{n}\sum_{h=1}^{n-1}\left(\frac{1}{h} + \frac{1}{n-h}\right) = \frac{2\log n}{n} + O\left[\frac{1}{n}\right].$$

Therefore, if $d = 4$, (3.23) yields

$$\frac{2\gamma_4^2\kappa_4\log n}{n^2} + O\left[\frac{1}{n^2}\right],$$

which immediately implies (2.3).

By Lemma 3.1 again, (3.23) in the three-dimensional case is given by

$$\frac{\gamma_3^2\kappa_3}{\sqrt{n}}\sum_{h=1}^{n-1}\frac{1}{\sqrt{h(n-h)}(\sqrt{h}+\sqrt{n})} \tag{3.25}$$

$$+\,O\left[\frac{1}{\sqrt{n}}\sum_{h=1}^{n-2}\frac{1}{\sqrt{h(n-h)}(\sqrt{h}+\sqrt{n})\log^{\delta}(n-h)}\right] \tag{3.26}$$

for any $\delta > 0$. Here we have used the identity that

$$\frac{1}{\sqrt{h}}-\frac{1}{\sqrt{n}}=\frac{n-h}{\sqrt{nh}(\sqrt{n}+\sqrt{h})}\,.$$

The summation in (3.26) is bounded by

$$\frac{C_7}{\sqrt{n}}\sum_{h=1}^{n-2}\frac{1}{\sqrt{h(n-h)}\log^{\delta}(n-h)}\,.$$

Dividing the summation into two cases; $h \leqq \lfloor n/\log^{2\delta} n\rfloor$ and $h > \lfloor n/\log^{2\delta} n\rfloor$, we see that it is of order $n/\log^{\delta} n$. Hence, (3.26) is $O[n^{1/2}/\log^{\delta} n]$. For $0 < x < 1$ let

$$f(x)=\frac{1}{\sqrt{x(1-x)}(1+\sqrt{x})}\,.$$

Then (3.25) is expressed as

$$\frac{\gamma_3^2\kappa_3}{n^2}\sum_{h=1}^{n-1} f\left(\frac{h}{n}\right).$$

Note that the basic property of Riemannian integral implies that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{h=1}^{n-1} f\left(\frac{h}{n}\right)=\int_0^1 f(x)\,dx=2\,.$$

Although this is not enough to complete the proof of (2.2), we can improve this limiting behavior by a standard argument of approximating a summation to an integral. Indeed, a simple calculation shows that

$$\frac{1}{n}\sum_{h=1}^{n-1} f\left(\frac{h}{n}\right)=\int_{1/n}^{1-1/n} f(x)\,dx+O\left[\frac{1}{\sqrt{n}}\right]=2+O\left[\frac{1}{\sqrt{n}}\right],$$

which implies that (3.25) is equal to

$$\frac{2\gamma_3^2\kappa_3}{n}+O\left[\frac{1}{n^{3/2}}\right].$$

Therefore, the third term of the right-hand side of (3.21) yields

$$-4\gamma_3^2\kappa_3 n^{1/2}+O\left[\frac{n^{1/2}}{\log^{\delta} n}\right].$$

Combining this with (3.22), we obtain (2.2).

Finally, we prove (2.1). It follows from (3.2) (or Lemma 3.1) and (3.16) that

$$E[R_{2n}; S_{2n} = 0] = 2\sum_{l=2}^{n-1} l f_{2l} u_{2(n-l)} + O\left[\frac{1}{\log^2 n}\right]. \tag{3.27}$$

Applying Lemma 3.1 to the leading term of (3.27) yields

$$2\pi \sum_{l=2}^{n-1} \frac{u_{2(n-l)}}{\log^2 l}, \tag{3.28}$$

while the error term is of order

$$\sum_{l=2}^{n-1} \frac{u_{2(n-l)} \log\log l}{\log^3 l} \leqq A \log\log n \sum_{l=2}^{n-1} \frac{1}{(n-l)\log^3 l}. \tag{3.29}$$

Let $M = \lfloor n/\log^2 n \rfloor$. Then the summation in the right-hand side of (3.29) is dominated by

$$\sum_{l=2}^{M} \frac{1}{(n-l)\log^3 l} + \sum_{l=M}^{n-1} \frac{1}{(n-l)\log^3 l} \leqq \frac{C_8 M}{n-M} + \frac{C_9 \log n}{\log^3 M},$$

both terms of which are of order $1/\log^2 n$. Hence, the right-hand side of (3.29) is bounded by a constant multiple of $\log\log n/\log^2 n$. By (1.3), the leading term of (3.28) yields

$$2\sum_{l=2}^{n-1} \frac{1}{(n-l)\log^2 l}, \tag{3.30}$$

and the error term of (3.28) is of order

$$\sum_{l=2}^{n-1} \frac{1}{(n-l)^2 \log^2 l}.$$

We see, in the same way as for (3.29), that this summation is $O[1/\log^2 n]$. To obtain the upper bound of (3.30), we use $M = \lfloor n/\log^2 n \rfloor$ again. The contribution for $2 \leqq l < M$ in (3.30) is negligible, since

$$\sum_{l=2}^{M-1} \frac{1}{(n-l)\log^2 l} \leqq \frac{C_{10} M}{n-M} = O\left[\frac{1}{\log^2 n}\right].$$

Hence, we concentrate on the summation over $[M, n-1]$ to get

$$\sum_{l=M}^{n-1} \frac{1}{(n-l)\log^2 l} \leqq \frac{1}{\log^2 M} \sum_{l=1}^{n-1} \frac{1}{n-l} = \frac{\log n}{\log^2 M} + O\left[\frac{1}{\log^2 n}\right].$$

Since

$$\frac{1}{\log M} - \frac{1}{\log n} = O\left[\frac{\log\log n}{\log^2 n}\right],$$

the right-hand side yields

$$\frac{1}{\log n} + O\left[\frac{\log\log n}{\log^2 n}\right].$$

The lower bound of (3.30) can be easily obtained. Indeed, it is not less than

$$\frac{2}{\log^2 n}\sum_{l=1}^{n-1}\frac{1}{n-l} - \frac{1}{(n-1)\log^2 n} \geqq \frac{2}{\log n} + O\left[\frac{1}{n\log^2 n}\right].$$

Therefore, (3.27) implies that

$$E_n R_{2n} = \frac{2}{u_{2n}\log n} + O\left[\frac{\log\log n}{u_{2n}\log^2 n}\right].$$

With the help of (1.4), we now conclude (2.1).

4. Proof of Theorem 2.3. In order to prove Theorem 2.3, we need the following two formulas concerning taboo probabilities. A simple calculation shows that for $m \geqq 2$ and $x, y, z \in \boldsymbol{Z}^d$ such that $x \neq y$ and $x \neq z$,

$$p_{xy}^m(x, z) = p_y^m(x, z) - \sum_{k=1}^{m-1} p_{xy}^k(x, x) p_y^{m-k}(x, z), \tag{4.1}$$

$$p_{xy}^m(x, x) = p_x^m(x, x) - \sum_{k=1}^{m-1} p_{xy}^k(x, y) p_x^{m-k}(y, x). \tag{4.2}$$

Here (4.1) can be obtained by classifying the event $\{S_m = z, \tau_y \geqq m\}$ by the value of τ_x, and (4.2) by classifying $\{\tau_x = m\}$ by the value of τ_y.

LEMMA 4.1. *For $n \geqq 2$ we have*

$$\sum_{i=1}^{2n-2}\sum_{j=i+1}^{2n-1} E_n[Z_i^{2n} Z_j^{2n}] \tag{4.3}$$

$$= \frac{1}{u_{2n}}\sum_{i=1}^{2n-2}\sum_{j=i+1}^{2n-1}\sum_{y\neq 0} p^i(0, y) p_y^{2n-i}(y, 0) \tag{4.4}$$

$$- \frac{1}{u_{2n}}\sum_{i=1}^{2n-3}\sum_{j=1}^{2n-i-2}\sum_{l=1}^{2n-i-j-1} f_l u_{2n-l} \tag{4.5}$$

$$+ \frac{1}{u_{2n}}\sum_{i=1}^{2n-3}\sum_{j=1}^{2n-i-2}\sum_{l=1}^{2n-i-j-1}\sum_{h=1}^{2n-i-l} f_l f_h u_{2n-l-h} + \begin{cases} O\left[\dfrac{n^2}{\log^3 n}\right] & \text{if } d = 2, \\ O[n^{3/2}] & \text{if } d = 3, \\ O[n\log n] & \text{if } d = 4, \\ O[n] & \text{if } d \geqq 5. \end{cases} \tag{4.6}$$

PROOF. By the Markov property, (4.3) yields

$$\frac{1}{u_{2n}} \sum_{i=1}^{2n-2} \sum_{j=i+1}^{2n-1} \sum_{\substack{y,z\neq 0\\ z\neq y}} p^i(0,y) p_y^{j-i}(y,z) p_{yz}^{2n-j}(z,0),$$

which is equal to

$$\frac{1}{u_{2n}} \sum_{i=1}^{2n-2} \sum_{j=i+1}^{2n-1} \sum_{\substack{y,z\neq 0\\ z\neq y}} p^i(0,y) p_y^{j-i}(y,z) p_y^{2n-j}(z,0) \tag{4.7}$$

$$-\frac{1}{u_{2n}} \sum_{i=1}^{2n-3} \sum_{j=i+1}^{2n-2} \sum_{l=1}^{2n-j-1} \sum_{\substack{y,z\neq 0\\ z\neq y}} p^i(0,y) p_y^{j-i}(y,z) p_{yz}^{l}(z,z) p_y^{2n-j-l}(z,0) \tag{4.8}$$

by applying (4.1). Regarding the summation on z over $\mathbf{Z}^d \setminus \{0, y\}$ in (4.7) as the sum over $\mathbf{Z}^d \setminus \{y\}$ minus that over $\{0\}$, we obtain

$$\frac{1}{u_{2n}} \sum_{i=1}^{2n-2} \sum_{j=i+1}^{2n-1} \sum_{y\neq 0} p^i(0,y) p_y^{2n-i}(y,0) - \frac{1}{u_{2n}} \sum_{i=1}^{2n-2} \sum_{j=i+1}^{2n-1} \sum_{y\neq 0} p^i(0,y) p_y^{j-i}(y,0) p_y^{2n-j}(0,0).$$

Note that the first term is the same as (4.4), and the absolute value of the second term is not larger than

$$\frac{1}{u_{2n}} \sum_{i=1}^{2n-2} \sum_{j=i+1}^{2n-1} u_j u_{2n-j} = \frac{1}{u_{2n}} \sum_{j=2}^{2n-1} j u_j u_{2n-j},$$

where, by (1.1) and (1.4), the right-hand side is of order

$$n^{d/2} \sum_{j=2}^{2n-1} j^{1-d/2} (2n-j)^{-d/2} \leqq \begin{cases} C_{11} n \log n & \text{if } d = 2, \\ C_{12} n & \text{if } d \geqq 3. \end{cases}$$

Applying (4.2), we see that (4.8) yields

$$-\frac{1}{u_{2n}} \sum_{i=1}^{2n-3} \sum_{j=i+1}^{2n-2} \sum_{l=1}^{2n-j-1} \sum_{y\neq 0} p^i(0,y) f_l p_y^{2n-i-l}(y,0) \tag{4.9}$$

$$+\frac{1}{u_{2n}} \sum_{i=1}^{2n-3} \sum_{j=i+1}^{2n-2} \sum_{l=1}^{2n-j-1} \sum_{y\neq 0} p^i(0,y) p_y^{j-i}(y,0) f_l p_y^{2n-j-l}(0,0) \tag{4.10}$$

$$\begin{aligned} &+\frac{1}{u_{2n}} \sum_{i=1}^{2n-4} \sum_{j=i+1}^{2n-3} \sum_{l=1}^{2n-j-1} \sum_{h=1}^{l-1} \sum_{\substack{y,z\neq 0\\ z\neq y}} p^i(0,y) p_y^{j-i}(y,z) \\ &\qquad \times p_{yz}^{h}(z,y) p_z^{l-h}(y,z) p_y^{2n-j-l}(z,0). \end{aligned} \tag{4.11}$$

Then Lemma 3.2 implies that (4.9) gives (4.5) and (4.6). Also, (4.10) can be easily estimated from (1.5). Indeed, (4.10) is not larger than

$$\frac{1}{u_{2n}} \sum_{i=1}^{2n-3} \sum_{j=i+1}^{2n-2} \sum_{l=1}^{2n-j-1} u_j f_l u_{2n-j-l} \leqq \frac{1}{u_{2n}} \sum_{i=1}^{2n-3} \sum_{j=i+1}^{2n-2} u_j u_{2n-j} .$$

A bound of the right-hand side has been calculated, and thus (4.10) is of order $n \log n$ if $d = 2$ and of order n if $d \geqq 3$.

The estimate of (4.11) is more complicated. We first accomplish it when $d \geqq 3$. Changing the order of summations on l and h, we see that (4.11) is bounded by

$$\frac{1}{u_{2n}} \sum_{i=1}^{2n-4} \sum_{j=i+1}^{2n-3} \sum_{h=1}^{2n-j-2} \sum_{l=h+1}^{2n-j-1} \sum_{y,z \in \mathbf{Z}^d} p^i(0, y) p^{j-i}(y, z) p^h(z, y) p^{l-h}(y, z) p^{2n-j-l}(z, 0) .$$

Regarding the summations on j, l and z as those on $j - i$, $l - h$ and $z - y$, respectively, for fixed i, h and y, we see that

$$\frac{1}{u_{2n}} \sum_{i=1}^{2n-4} \sum_{j=1}^{2n-i-3} \sum_{h=1}^{2n-i-j-2} \sum_{l=1}^{2n-i-j-h-1} \sum_{y,z \in \mathbf{Z}^d} p^i(0, y) p^j(0, z) \times p^h(z, 0) p^l(0, z) p^{2n-i-j-h-l}(z, -y) .$$

We now take summations on y and i in this order, and then apply (1.1). Then (4.11) is not larger than

$$\frac{A^2}{u_{2n}} \sum_{j=1}^{2n-4} \sum_{h=1}^{2n-j-3} \sum_{l=1}^{2n-j-h-2} \sum_{z \in \mathbf{Z}^d} p^j(0, z) p^h(z, 0) l^{-d/2} (2n - j - h - l)^{1-d/2} ,$$

which is bounded by a constant multiple of

$$\frac{1}{u_{2n}} \sum_{j=1}^{2n-4} \sum_{h=1}^{2n-j-3} (2n - j - h)^{1-d/2} (j + h)^{-d/2} .$$

Substituting $k = j + h$ in the sum on h, we see that this is less than or equal to

$$\frac{1}{u_{2n}} \sum_{j=1}^{2n-4} \sum_{k=j+1}^{2n-3} (2n - k)^{1-d/2} k^{-d/2} = \frac{1}{u_{2n}} \sum_{k=1}^{2n-4} (2n - k)^{1-d/2} k^{1-d/2} .$$

It follows from (1.4) that (4.11) is $O[n^{3/2}]$ if $d = 3$, $O[n \log n]$ if $d = 4$ and $O[n]$ if $d \geqq 5$.

We next estimate (4.11) in the two-dimensional case. Similarly to higher-dimensional cases, (4.11) is bounded by

$$\frac{1}{u_{2n}}\sum_{i=1}^{2n-4}\sum_{j=1}^{2n-i-3}\sum_{h=1}^{2n-i-j-2}\sum_{l=1}^{2n-i-j-h-1}\sum_{y,z\neq 0} p^i(0,y)p_0^j(0,z) \times p_z^h(z,0)p_z^l(0,z)p^{2n-i-j-h-l}(z,-y)$$

$$\leqq \frac{A}{u_{2n}}\sum_{j=1}^{2n-4}\sum_{h=1}^{2n-j-3}\sum_{l=1}^{2n-j-h-2}\sum_{z\neq 0} p_0^j(0,z)p_z^h(z,0)p_z^l(0,z)\,.$$

To estimate the summation in the right-hand side, the following inequality given in Jain and Pruitt [15] will be useful:

$$\sum_{k=1}^{m} p_z^k(0,z)r_{m-k}^{\alpha} = \sum_{k=1}^{m} p_0^k(0,z)r_{m-k}^{\alpha} \leqq \sum_{k=1}^{m} p^k(0,z)r_{m-k}^{\alpha+1}$$

for $z \neq 0$. The first equality has been obtained from the fact that

$$p_0^m(0,z) = p_z^m(0,z)$$

for $m \geqq 1$ and $z \in \boldsymbol{Z}^d$. Applying this inequality three times in the above, we accordingly have that (4.11) is not larger than

$$\frac{A}{u_{2n}}\sum_{j=1}^{2n-2}\sum_{h=1}^{2n-j-1}\sum_{l=1}^{2n-j-h}\sum_{z\neq 0} p^j(0,z)p^h(z,0)p^l(0,z)r_{2n-j-h-l}^3\,.$$

We divide the summation into the following two parts: (i) $j+h+l$ is less than or equal to $2n-\alpha$; (ii) $j+h+l$ is larger than $2n-\alpha$, where $\alpha = \lfloor 2n/\log^4 n\rfloor$. Since $r_{2n-j-h-l}^3$ is bounded by r_α^3 if $j+h+l \leqq 2n-\alpha$, the asymptotic behavior (1.6) shows that the contribution for case (i) is not larger than

$$\frac{C_{13}n}{\log^3 n}\sum_{j=1}^{2n}\sum_{h=1}^{2n}\sum_{l=1}^{2n}\sum_{z} p^j(0,z)p^h(z,0)p^l(0,z)\,. \tag{4.12}$$

Applying the same argument used in Jain and Pruitt [15] or Lemma 4.4 in Hamana [5], the summation is of order n. Thus, (4.12) is $O[n^2/\log^3 n]$. The contribution for case (ii) does not exceed a constant multiple of

$$n \sum_{\substack{1\leqq j,h,l\leqq 2n\\ 2n-\alpha<j+h+l\leqq 2n}} (j+h)^{-1}l^{-1} \leqq C_{15}n\log n \sum_{\substack{1\leqq j,h\leqq 2n\\ 2n-\alpha<j+h\leqq 2n}} (j+h)^{-1}$$

$$\leqq C_{15}n\log n \sum_{k=2n-\alpha}^{2n}\sum_{j=1}^{k}\{j+(k-j)\}^{-1}\,,$$

which is $O[n^2/\log^3 n]$. Therefore, (4.11) is $O[n^2/\log^3 n]$ if $d=2$. □

We first prove Theorem 2.3 in the three-or-more-dimensional cases. If $1 \leqq i < j \leqq 2n-1$, then

$$E_n Z_i^{2n} E_n Z_j^{2n} = \frac{1}{u_{2n}^2} \sum_{y \neq 0} p^i(0, y) p_y^{2n-i}(y, 0) \sum_{z \neq 0} p^j(0, z) p_z^{2n-j}(z, 0) . \tag{4.13}$$

If $j \leqq 2n-2$, we can apply Lemma 3.2 to the sum on z, obtaining that (4.13) is

$$\frac{1}{u_{2n}} \sum_{y \neq 0} p^i(0, y) p_y^{2n-i}(y, 0) - \frac{1}{u_{2n}^2} \sum_{y \neq 0} p^i(0, y) p_y^{2n-i}(y, 0) \sum_{l=1}^{2n-j} f_l u_{2n-l} .$$

If $j = 2n-1$, the right-hand side of (4.13) yields

$$\frac{1}{u_{2n}} \sum_{y \neq 0} p^i(0, y) p_y^{2n-i}(y, 0) .$$

Therefore, the summation of (4.13) over $1 \leqq i < j \leqq 2n-1$ is equal to

$$\frac{1}{u_{2n}} \sum_{i=1}^{2n-2} \sum_{j=i+1}^{2n-1} \sum_{y \neq 0} p^i(0, y) p_y^{2n-i}(y, 0) \\ - \frac{1}{u_{2n}^2} \sum_{i=1}^{2n-3} \sum_{j=i+1}^{2n-2} \sum_{l=1}^{2n-j} \sum_{y \neq 0} p^i(0, y) p_y^{2n-i}(y, 0) f_l u_{2n-l} .$$

The first term is the same as (4.4). It follows from Lemma 3.2 again that the second term is a sum of

$$-\frac{1}{u_{2n}} \sum_{i=1}^{2n-3} \sum_{j=i+1}^{2n-2} \sum_{l=1}^{2n-j} f_l u_{2n-l} \tag{4.14}$$

and

$$\frac{1}{u_{2n}^2} \sum_{i=1}^{2n-3} \sum_{j=i+1}^{2n-2} \sum_{l=1}^{2n-j} \sum_{h=1}^{2n-i} f_l f_h u_{2n-l} u_{2n-h} .$$

The difference between (4.5) and (4.14) is

$$\frac{1}{u_{2n}} \sum_{i=1}^{2n-3} \sum_{j=1}^{2n-i-2} f_{2n-i-j} u_{i+j} \leqq \frac{1}{u_{2n}} \sum_{i=1}^{2n-3} \sum_{j=i+1}^{2n-2} u_{2n-j} u_j ,$$

which is of order n if $d \geqq 3$.

For random variables X and Y, let $\mathrm{Cov}_n(X,Y)$ be $E_n[(X - E_nX)(Y - E_nY)]$. These calculations show that

$$\sum_{i=1}^{2n-2}\sum_{j=i+1}^{2n-1} \mathrm{Cov}_n(Z_i^{2n}, Z_j^{2n}) \tag{4.15}$$

$$= \frac{1}{u_{2n}} \sum_{i=1}^{2n-3}\sum_{j=1}^{2n-i-2}\sum_{l=1}^{2n-i-j-1}\sum_{h=1}^{2n-i-l} f_l f_h u_{2n-l-h} \tag{4.16}$$

$$- \frac{1}{u_{2n}^2} \sum_{i=1}^{2n-3}\sum_{j=1}^{2n-i-2}\sum_{l=1}^{2n-i-j}\sum_{h=1}^{2n-i} f_l f_h u_{2n-l} u_{2n-h} + \begin{cases} O[n^{3/2}] & \text{if } d = 3, \\ O[n \log n] & \text{if } d = 4, \\ O[n] & \text{if } d \geqq 5. \end{cases} \tag{4.17}$$

Since $|u_{2n-l-h} - u_{2n-h}| \leqq C_{16} l(2n-l-h)^{-1-d/2}$, which is derived from (1.3) and the mean value theorem, the contribution of replacing u_{2n-l-h} with u_{2n-h} in (4.16) is dominated by a constant multiple of

$$n^{d/2} \sum_{i=1}^{2n-3}\sum_{l=1}^{2n-i-2}\sum_{h=1}^{2n-i-l} (2n-i-l) l^{1-d/2} h^{-d/2} (2n-l-h)^{-1-d/2},$$

which is not larger than

$$n^{d/2} \sum_{i=1}^{2n-2}\sum_{l=1}^{2n-i-1}\sum_{h=1}^{2n-i-l} l^{1-d/2} h^{-d/2} (2n-l-h)^{-d/2}$$
$$+ n^{d/2} \sum_{i=1}^{2n-2}\sum_{l=1}^{2n-i-1}\sum_{h=1}^{2n-i-l} l^{1-d/2} h^{1-d/2} (2n-l-h)^{-1-d/2},$$

since $2n - i - l \leqq (2n-l-h) + h$. Taking summations on i and l in this order, we see that the first term is bounded by

$$n^{d/2} \sum_{h=1}^{2n-2} h^{-d/2} \times \begin{cases} C_{17} & \text{if } d = 3, \\ \dfrac{C_{18} \log(2n-h)}{2n-h} & \text{if } d = 4, \\ C_{19}(2n-h)^{1-d/2} & \text{if } d \geqq 5, \end{cases}$$

which is of order $n^{3/2}$ if $d = 3$, $n \log n$ if $d = 4$ and n if $d \geqq 5$. In a similar way, a bound of the second term is given by

$$C_{20} n^{d/2} \sum_{l=1}^{2n-2} l^{1-d/2} (2n-l)^{1-d/2} = \begin{cases} O[n^{3/2}] & \text{if } d = 3, \\ O[n \log n] & \text{if } d = 4, \\ O[n] & \text{if } d \geqq 5. \end{cases}$$

Therefore, the leading term of (4.16) yields

$$\frac{1}{u_{2n}} \sum_{i=1}^{2n-3}\sum_{j=1}^{2n-i-2}\sum_{l=1}^{2n-i-j-1}\sum_{h=1}^{2n-i-l} f_l f_h u_{2n-h}. \tag{4.18}$$

We also calculate the contribution of exchanging u_{2n-l} for u_{2n} in (4.17). Note that $|u_{2n-l} - u_{2n}| \leqq C_{21} l (2n-l)^{-1-d/2}$. Applying (1.5) to the sum on h, we have that

$$\begin{aligned}
&\frac{1}{u_{2n}^2} \sum_{i=1}^{2n-3} \sum_{j=1}^{2n-i-2} \sum_{l=1}^{2n-i-j} \sum_{h=1}^{2n-i} f_l f_h u_{2n-h} \, |u_{2n-l} - u_{2n}| \\
&\quad \leqq \frac{1}{u_{2n}} \sum_{i=1}^{2n-3} \sum_{j=1}^{2n-i-2} \sum_{l=1}^{2n-i-j} f_l \, |u_{2n-l} - u_{2n}| \\
&\quad \leqq C_{22} n^{d/2} \sum_{i=1}^{2n-2} \sum_{j=1}^{2n-i-1} \sum_{l=1}^{2n-i-j} l^{1-d/2} (2n-l)^{-1-d/2} .
\end{aligned}$$

Taking summations on i and j, we see that the right-hand side is dominated by

$$C_{22} n^{d/2} \sum_{l=1}^{2n-2} l^{1-d/2} (2n-l)^{1-d/2} = \begin{cases} O[n^{3/2}] & \text{if } d = 3 , \\ O[n \log n] & \text{if } d = 4 , \\ O[n] & \text{if } d \geqq 5 . \end{cases}$$

Hence, the leading term of (4.17) yields

$$-\frac{1}{u_{2n}} \sum_{i=1}^{2n-3} \sum_{j=1}^{2n-i-2} \sum_{l=1}^{2n-i-j} \sum_{h=1}^{2n-i} f_l f_h u_{2n-h} . \tag{4.19}$$

Noting that the summands of (4.18) and (4.19) are the same and that the range of summations in (4.19) includes that in (4.18), we see that the sum of (4.18) and (4.19) is negative. Since the sum of $\mathrm{Var}_n Z_j^{2n}$ on j over $[1, 2n-1]$ is not larger than n, the leading term of $\mathrm{Var}_n R_{2n}$ is (4.15). This completes a proof of Theorem 2.3 for the three-or-more-dimensional cases.

The remainder of this section is devoted to the estimate of $\mathrm{Var}_n R_{2n}$ in the two-dimensional case. The proof for higher-dimensional cases is not applicable, since we cannot prove in this case that the contributions from the replacement of u_{2n-l-h} with u_{2n-h} in (4.6) and u_{2n-l} with u_{2n} in (4.7) are negligible. Thus, we proceed as follows. Theorem 2.2 implies that if $d = 2$,

$$(E_n R_{2n})^2 = \frac{4\pi^2 n^2}{\log^2 n} + O\left[\frac{n^2 \log\log n}{\log^3 n}\right].$$

For $n \geqq 2$ we have that

$$E_n R_{2n}^2 = 1 + 3 \sum_{j=1}^{2n-1} E_n Z_j^{2n} + 2 \sum_{i=1}^{2n-2} \sum_{j=i+1}^{2n-1} E_n [Z_i^{2n} Z_j^{2n}] .$$

Since the second term of the right-hand side is not larger than $6n$, it is sufficient to show that if $d = 2$,

$$\sum_{i=1}^{2n-2} \sum_{j=i+1}^{2n-1} E_n [Z_i^{2n} Z_j^{2n}] = \frac{2\pi^2 n^2}{\log^2 n} + O\left[\frac{n^2 \log\log n}{\log^3 n}\right]. \tag{4.20}$$

Note that the left-hand side of (4.20) is the same as (4.3).

From now on, we are going to improve Lemma 4.1. It follows from Lemma 3.2 that (4.4) is equal to

$$\frac{1}{u_{2n}}\sum_{i=1}^{2n-2}\sum_{j=i+1}^{2n-1}\left(u_{2n}-\sum_{l=1}^{2n-i} f_l u_{2n-l}\right)$$
$$=\sum_{i=1}^{2n-2}(2n-i-1)-\frac{1}{u_{2n}}\sum_{i=1}^{2n-2}\sum_{l=1}^{2n-i}(2n-i-1) f_l u_{2n-l}\,.$$

The first term of the right-hand side is $(n-1)(2n-1)$ and the second term is

$$-\frac{4}{u_{2n}}\sum_{i=1}^{n-1}\sum_{l=1}^{n-i}(n-i) f_{2l} u_{2(n-l)}+\frac{1}{u_{2n}}\sum_{i=1}^{n-1}\sum_{l=1}^{n-i} f_{2l} u_{2(n-l)}\,. \tag{4.21}$$

Changing the order of summations in the first term of (4.21) yields

$$-\frac{2}{u_{2n}}\sum_{l=1}^{n-1}(n+l-1)(n-l) f_{2l} u_{2(n-l)}\,.$$

By (1.5), the second term of (4.21) is not larger than n. Hence, it follows that (4.4) is equal to

$$(n-1)(2n-1)-\frac{2}{u_{2n}}\sum_{l=1}^{n-2}(n+l-1)(n-l) f_{2l} u_{2(n-l)}+O[n]\,,$$

since $(n-1) f_{2n-2} u_2/u_{2n} = O[n/\log^2 n]$ which follows from (1.4) and (3.2). Taking summations on j and i in this order, (4.5) yields

$$-\frac{1}{2u_{2n}}\sum_{l=1}^{2n-3}(2n-l-2)(2n-l-1) f_l u_{2n-l}$$
$$=-\frac{2}{u_{2n}}\sum_{l=1}^{n-2}(n-l-1)(n-l) f_{2l} u_{2(n-l)}+\frac{1}{u_{2n}}\sum_{l=1}^{n-2}(n-l-1) f_{2l} u_{2(n-l)}\,.$$

Since each summand in the second term of the right-hand side is bounded by Af_{2l}, the sum of (4.4) and (4.5) is equal to

$$(n-1)(2n-1)-\frac{4(n-1)}{u_{2n}}\sum_{l=1}^{n-2}(n-l) f_{2l} u_{2(n-l)}+O[n]\,.$$

The calculation of (4.6) is similar to (4.4) and (4.5). Taking the summation on j in (4.6) yields

$$\frac{1}{u_{2n}}\sum_{l=1}^{n-2}\sum_{i=1}^{2n-2l-2}\sum_{h=1}^{2n-2l-i}(2n-2l-i-1) f_{2l} f_h u_{2n-2l-h}$$
$$=\frac{4}{u_{2n}}\sum_{l=1}^{n-2}\sum_{i=1}^{n-l-1}\sum_{h=1}^{n-l-i}(n-l-i) f_{2l} f_{2h} u_{2(n-l-h)}+O[n]\,.$$

The first term of the right-hand side is

$$\frac{2}{u_{2n}}\sum_{l=1}^{n-2}\sum_{h=1}^{n-l-1}(n-l+h-1)(n-l-h)f_{2l}f_{2h}u_{2(n-l-h)}$$

$$= \frac{2(n-1)}{u_{2n}}\sum_{l=1}^{n-2}\sum_{h=1}^{n-l-1}(n-l-h)f_{2l}f_{2h}u_{2(n-l-h)} \tag{4.22}$$

$$+ \frac{2}{u_{2n}}\sum_{l=1}^{n-2}\sum_{h=1}^{n-l-1}h(n-l-h)f_{2l}f_{2h}u_{2(n-l-h)} \tag{4.23}$$

$$- \frac{2}{u_{2n}}\sum_{l=1}^{n-2}\sum_{h=1}^{n-l-1}l(n-l-h)f_{2l}f_{2h}u_{2(n-l-h)}\,. \tag{4.24}$$

Exchanging the role of l and h in the double sum in (4.23), we see that the sum in (4.23) is the same as that in (4.24), and thus (4.23) cancel out (4.24). Then the leading term of (4.6) is (4.22) and its remaining term is $O[n]$. It is obvious that (4.22) is equal to

$$\frac{2(n-1)}{u_{2n}}\sum_{l=1}^{n-2}\sum_{h=1}^{n-l-1}(n-l)f_{2l}f_{2h}u_{2(n-l-h)} - \frac{2(n-1)}{u_{2n}}\sum_{l=1}^{n-2}\sum_{h=1}^{n-l-1}hf_{2l}f_{2h}u_{2(n-l-h)}\,.$$

Applying $h = n-(n-h)$ to the second term, we see that (4.22) is

$$\frac{4(n-1)}{u_{2n}}\sum_{l=1}^{n-2}\sum_{h=1}^{n-l-1}(n-l)f_{2l}f_{2h}u_{2(n-l-h)} - \frac{2n(n-1)}{u_{2n}}\sum_{l=1}^{n-2}\sum_{h=1}^{n-l-1}f_{2l}f_{2h}u_{2(n-l-h)}\,.$$

Therefore, we obtain by (1.5) that the sum of (4.4), (4.5) and (4.6) is given by

$$-\frac{4(n-1)}{u_{2n}}\sum_{l=1}^{n-2}(n-l)f_{2l}f_{2(n-l)} + \frac{2n(n-1)}{u_{2n}}f_{2n}$$
$$+ \frac{2n(n-1)}{u_{2n}}f_{2(n-1)}u_2 + \frac{2n(n-1)}{u_{2n}}\sum_{l=1}^{n-2}f_{2l}f_{2(n-l)} + O[n]\,.$$

This implies that if $d=2$, (4.3) yields

$$-\frac{4(n-1)}{u_{2n}}\sum_{l=1}^{n-1}(n-l)f_{2l}f_{2(n-l)} + \frac{2n(n-1)f_{2n}}{u_{2n}} + \frac{2n(n-1)}{u_{2n}}\sum_{l=1}^{n-1}f_{2l}f_{2(n-l)} + O\left[\frac{n^2}{\log^3 n}\right],$$

where we have used that $u_2 = f_2$ and that $(n-1)f_{2(n-1)}f_2 = O[1/\log^2 n]$. Substituting $h = n-l$ in the sum of the first term, we get

$$\sum_{l=1}^{n-1}(n-l)f_{2l}f_{2(n-l)} = \sum_{h=1}^{n-1}hf_{2(n-h)}f_{2h}\,,$$

which implies that

$$n\sum_{l=1}^{n-1} f_{2l} f_{2(n-l)} = 2\sum_{l=1}^{n-1}(n-l) f_{2l} f_{2(n-l)} .$$

Hence, we see that if $d = 2$, the leading term of (4.3) is $2n(n-1) f_{2n}/u_{2n}$ and the remaining term of (4.3) is of order $n^2/\log^3 n$. Since

$$\frac{f_{2n}}{u_{2n}} = \frac{\pi^2}{\log^2 n} + O\left[\frac{\log\log n}{\log^3 n}\right]$$

by (1.4) and Lemma 3.1, we accordingly obtain (4.20). This completes a proof of Theorem 2.3.

5. Large deviations in the upward direction. We give a proof of Theorem 2.5 and its corollary in this section. It is obvious that $P_{2n,0}[R_{2n} \geqq 2xn]$ is equal to 0 for $x > 1$ and is equal to 1 for $x \leqq 0$. Moreover, the assertion of Theorem 2.5 is already established if $x = 1$. Hence, it suffices to prove (2.5) for $0 < x < 1$. It follows from (1.3) that

$$\psi(x) \leqq \liminf_{n\to\infty} -\frac{1}{2n} \log P_{2n,0}[R_{2n} \geqq 2xn] .$$

Therefore, we concentrate on proving that

$$\limsup_{n\to\infty} -\frac{1}{2n} \log P_{2n,0}[R_{2n} \geqq 2xn] \leqq \psi(x)$$

for $0 < x < 1$, which is equivalent to

$$\limsup_{n\to\infty} -\frac{1}{2n} \log P[R_{2n} \geqq 2xn, S_{2n} = 0] \leqq \psi(x) . \tag{5.1}$$

For simplicity we write $\phi(x)$ for the left-hand side of (5.1).

The argument in Lemma 1 in Hamana and Kesten [8] is applicable. Let $\{X'_n\}_{n=0}^{\infty}$ be an independent copy of $\{X_n\}_{n=0}^{\infty}$. We define a new random walk $\{S'_n\}_{n=0}^{\infty}$ moving on $\boldsymbol{Z}^d$ by $S'_0 = 0$ and $S'_n = X'_1 + \cdots + X'_n$ for $n \geqq 1$. Then $\{S'_n\}_{n=0}^{\infty}$ is also the simple random walk on $\boldsymbol{Z}^d$ which is independent of $\{S_n\}_{n=0}^{\infty}$. Let R'_n denote the range at time n of the random walk $\{S'_n\}_{n=0}^{\infty}$. For integers $p, q \geqq 0$ we consider the random walk $\{T_n^{p,q}\}_{n=0}^{\infty}$ defined by

$$T_n^{p,q} = \begin{cases} S_n & \text{if } 0 \leqq n \leqq p+q , \\ S_{p+q} + S'_{n-p-q} & \text{if } n \geqq p+q+1 . \end{cases}$$

Clearly, $\{T_n^{p,q}\}_{n=0}^{\infty}$ has the same distribution as $\{S_n\}_{n=0}^{\infty}$, and the definition of $\{T_n^{p,q}\}_{n=0}^{\infty}$ immediately shows that

$$\begin{aligned} |\{T_1^{p,q}, \ldots, T_{2p+2q}^{p,q}\}| &\geqq |\{T_1^{p,q}, \ldots, T_p^{p,q}\} \cup \{T_{p+q+1}^{p,q}, \ldots, T_{2p+q}^{p,q}\}| \\ &= R_p + R'_p - N_{p,q} \end{aligned} \tag{5.2}$$

for each $p, q \geqq 0$, where

$$N_{p,q} = |\{S_1, \ldots, S_p\} \cap \{S_{p+q} + S'_1, \ldots, S_{p+q} + S'_p\}| .$$

Thus, $N_{p,q}$ counts the number of points which are visited during the time interval $[1, p]$ by the random walk $\{S_n\}_{n=0}^{\infty}$, and also visited during $[1, p]$ by the random walk $\{S'_n\}_{n=0}^{\infty}$ shifted by S_{p+q}.

For a positive integer n let $M = \lceil n^{2/(d+1)} \rceil$ and $m = n - 2dM$. Here $\lceil x \rceil$ denotes the smallest integer which is not less than x. Put

$$\Lambda_M = \left\{ w \in \boldsymbol{Z}^d ;\, w = \sum_{j=1}^{d} k_j e_j ,\, 0 \leqq k_j \leqq M,\, j = 1, 2, \dots, d \right\},$$

where e_j is the unit vector in $\boldsymbol{Z}^d$ of which the jth element is one for each $1 \leqq j \leqq d$. For any $w \in \Lambda_M$ let $h(w) = k_1 + \cdots + k_d$ if $w = k_1 e_1 + \cdots + k_d e_d$. It is obvious that $0 \leqq h(w) \leqq dM$. Let x be a real number such that $0 < x < 1$ and $\varepsilon > 0$ are given. Since ψ is continuous, we can choose $\delta \in (0, 1 - x)$ such that

$$\psi(x + \delta) < \psi(x) + \varepsilon . \tag{5.3}$$

Moreover, we can take a positive integer n_0 such that

$$xn \leqq (x + \delta)m - M$$

for any $n \geqq n_0$. Therefore, we obtain with the help of (5.2) that for each $w \in \Lambda_M$

$$\begin{aligned} &P[R_{2m+2h(w)} \geqq 2xn,\, S_{2m+2h(w)} = 0] \\ &\quad \geqq P[|\{T_1^{m,h(w)}, \dots, T_{2m+2h(w)}^{m,h(w)}\}| \geqq 2(x + \delta)m - 2M,\, T_{2m+2h(w)}^{m,h(w)} = 0] \\ &\quad \geqq P[R_m \geqq (x + \delta)m,\, \tilde{R}_m \geqq (x + \delta)m,\, N_{m,h(w)} \leqq 2M,\, S_{m+h(w)} + \tilde{S}_{m+h(w)} = 0] . \end{aligned}$$

For $w \in \Lambda_M$ let

$$N_m(w) = |\{S_1, \dots, S_m\} \cap \{S_m + w + S'_1, \dots, S_m + w + S'_m\}| .$$

On the event $\{S_{m+h(w)} - S_m = w\}$, we have that $N_m(w)$ coincides with $N_{m,h(w)}$. Then it follows that

$$\begin{aligned} &P[R_{2m+2h(w)} \geqq 2xn,\, S_{2m+2h(w)} = 0] \\ &\quad \geqq P[R_m \geqq (x + \delta)m,\, R'_m \geqq (x + \delta)m,\, N_m(w) \leqq 2M , \\ &\qquad S_{m+h(w)} - S_m = w,\, S'_{m+h(w)} - S'_m = -w,\, S_m + S'_m = 0] . \end{aligned}$$

The event $\{S_{m+h(w)} - S_m = w,\, S'_{m+h(w)} - S'_m = -w\}$ in the last probability depends only on the X_j and X'_j with $m + 1 \leqq j \leqq m + h$, and is therefore independent of the other events. For simplicity we will use ζ for $1/2d$. Note that

$$P[S_{m+h(w)} - S_m = w,\, S'_{m+h(w)} - S'_m = -w] = \{P[S_{m+h(w)} - S_m = w]\}^2 \geqq \zeta^{2h(w)} .$$

Therefore, we obtain for $n \geqq n_0$ and $w \in \Lambda_M$ that

$$\begin{aligned} &P[R_{2m+2h(w)} \geqq 2xn,\, S_{2m+2h(w)} = 0] \\ &\quad \geqq \zeta^{2h(w)} P[R_m \geqq (x + \delta)m,\, R'_m \geqq (x + \delta)m,\, N_m(w) \leqq 2M,\, S_m + S'_m = 0] . \end{aligned}$$

To apply the argument by Hamana and Kesten, the left-hand side must have an upper bound which is independent of w. Indeed, it follows from the monotonicity of R_n with respect to n that

$$\begin{aligned}
&P[R_{2n} \geqq 2xn, S_{2n} = 0] \\
&\quad \geqq P[R_{2m+2h(w)} \geqq 2xn, S_{2m+2h(w)} = 0, S_{2n} = 0] \\
&\quad \geqq \zeta^{2n-2m-2h(w)} P[R_{2m+2h(w)} \geqq 2xn, S_{2m+2h(w)} = 0]\,,
\end{aligned}$$

where we have applied the trivial inequality that $P[S_{2k} = 0] \geqq \zeta^{2k}$ for each $k \geqq 1$. Consequently, we obtain for $n \geqq n_0$ and $w \in \Lambda_M$ that

$$\begin{aligned}
&P[R_{2n} \geqq 2xn, S_{2n} = 0] \\
&\quad \geqq \zeta^{4dM} P[R_m \geqq (x+\delta)m, R'_m \geqq (x+\delta)m, N_m(w) \leqq 2M, S_m + S'_m = 0]\,.
\end{aligned}$$

Since this inequality holds for all $w \in \Lambda_M$ and the left-hand side is independent of w, the same argument used to derive (3.11) in the proof of Lemma 1 in [8] leads to the following inequality:

$$\begin{aligned}
&P[R_{2n} \geqq 2xn, S_{2n} = 0] \\
&\quad \geqq \frac{1}{2}\zeta^{4dM} P[R_m \geqq (x+\delta)m, R'_m \geqq (x+\delta)m, S_m + S'_m = 0]\,.
\end{aligned} \tag{5.4}$$

The calculation is left to the reader (see (2.9) and (2.10) in [8]).

We consider the effect to remove the event $\{S_m + S'_m = 0\}$ from the probability in the right-hand side of (5.4). Applying the Schwarz inequality,

$$\begin{aligned}
\{P[R_m \geqq (x+\delta)m]\}^2 &= \left\{ \sum_{|y| \leqq m} P[R_m \geqq (x+\delta)m, S_m = y] \right\}^2 \\
&\leqq C_{23} m^d \sum_{|y| \leqq m} \{P[R_m \geqq (x+\delta)m, S_m = y]\}^2\,.
\end{aligned}$$

By symmetricity of simple random walks, each summand in the last summation on y is equal to

$$\begin{aligned}
&P[R_m \geqq (x+\delta)m, S_m = y] P[R'_m \geqq (x+\delta)m, S'_m = -y] \\
&\quad = P[R_n \geqq (x+\delta)m, R'_m \geqq (x+\delta)m, S_m = y, S'_m = -y]\,,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{1}{C_{23} m^d} \{P[R_m \geqq (x+\delta)m]\}^2 \\
&\quad \leqq P[R_m \geqq (x+\delta)m, R'_m \geqq (x+\delta)m, S_m + S'_m = 0]\,.
\end{aligned} \tag{5.5}$$

It follows from (5.4) and (5.5) that for $n \geqq n_0$

$$P[R_{2n} \geqq 2xn, S_{2n} = 0] \geqq \frac{C_{24}\zeta^{4dM}}{n^d} \{P[R_{n-2dM} \geqq (x+\delta)(n-2dM)]\}^2\,.$$

Since $M = \lceil n^{2/(d+1)} \rceil$, we can conclude that $\phi(x) \leqq \psi(x+\delta)$. In virtue of (5.3), we have that $\phi(x) \leqq \psi(x) + \varepsilon$ for any given $\varepsilon > 0$, which immediately leads to (5.1). This completes a proof of Theorem 2.5.

The remainder of this section is devoted to showing Corollary 2.6. Similarly to Theorem 2.5, it suffices to prove (2.6) for $0 < x < 1$. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence of points in $\boldsymbol{Z}^d$ satisfying that $n + |y_n|$ is even and that $|y_n| = o[n]$. For simplicity we write L for $|y_n|$. Since

$$P[S_n = y_n] = \kappa_d n^{-d/2} e^{-o[n]} + O[n^{-1-d/2}],$$

which follows from (1.2), it suffices to prove that

$$\psi(x) \geqq \limsup_{n\to\infty} -\frac{1}{n} \log P[R_n \geqq xn, S_n = y_n]$$

for $0 < x < 1$. Recall that δ has been chosen satisfying (5.3) for any given $\varepsilon > 0$. Moreover, since $L = o[n]$, there is an integer $n_1 \geqq 1$ such that $(x+\delta)(n-L) > xn$ for any $n > n_1$. Therefore, we see that for $n > n_1$

$$\begin{aligned} &P[R_n \geqq xn, S_n = y_n] \\ &\quad \geqq P[R_{n-L} \geqq (x+\delta)(n-L), S_{n-L} = 0, S_n - S_{n-L} = y_n] \\ &\quad \geqq \zeta^L P[R_{n-L} \geqq (x+\delta)(n-L), S_{n-L} = 0]. \end{aligned}$$

It follows from Theorem 2.5 that

$$\limsup_{n\to\infty} -\frac{1}{n} \log P[R_n \geqq xn, S_n = y_n] \leqq \psi(x+\delta).$$

This completes the proof by (5.3).

REFERENCES

[1] M. VAN DEN BERG AND E. BOLTHAUSEN, On the expected volume of the Wiener sausage for a Brownian bridge, Math. Z. 224 (1997), 33–48.

[2] R. F. BASS AND T. KUMAGAI, Laws of the iterated logarithm for the range of random walk in two and three dimensions, Ann. Probab. 30 (2002), 1369–1396.

[3] M. D. DONSKER AND S. R. S. VARADHAN, On the number of distinct sites visited by a random walk, Comm. Pure Appl. Math. 32 (1979), 721–747.

[4] A. DVORETZKY AND P. ERDŐS, Some problems on random walk in space, Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability (1950), pp. 353–367, University of California Press, Berkeley and Los Angeles, 1951.

[5] Y. HAMANA, The fluctuation result for the multiple point range of two dimensional recurrent random walks, Ann. Probab. 25 (1997), 598–639.

[6] Y. HAMANA, An almost sure invariance principle for the range of random walks, Stochastic Process. Appl. 78 (1998), 131–143.

[7] Y. HAMANA, Asymptotics of the moment generating function for the range of random walks, J. Theor. Probab. 14 (2001), 189–197.

[8] Y. HAMANA AND H. KESTEN, A large deviation result for the range of random walks and for the Wiener sausage, Probab. Theor. Relat. Fields 120 (2001), 183–208.

[9] Y. HAMANA AND H. KESTEN, Large deviations for the range of an integer valued random walk, Ann. Inst. H. Poincaré Probab. Statist. 38 (2002), 17–58.

[10] J. M. HAMMERSLEY, The number of polygons on a lattice, Proc. Cambridge Philos. Soc. 57 (1961), 516–523.

[11] N. C. JAIN AND S. OREY, On the range of random walk, Israel J. Math. 6 (1968), 373–380.

[12] N. C. JAIN AND W. E. PRUITT, The range of recurrent random walk in the plane, Z. Wahrsch. Verw. Gebiete 16 (1970), 279–292.

[13] N. C. JAIN AND W. E. PRUITT, The range of transient random walk, J. Anal. Math. 24 (1971), 369–393.

[14] N. C. JAIN AND W. E. PRUITT, The law of the iterated logarithm for the range of random walk, Ann. Math. Statist. 43 (1972), 1692–1697.

[15] N. C. JAIN AND W. E. PRUITT, The range of random walk, Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (University of California, Berkeley, Calif., 1970/71), Vol. III, Probability Theory, pp. 31–50, University of California Press, Berkeley, Calif., 1972.

[16] N. C. JAIN AND W. E. PRUITT, Further limit theorems for the range of random walk, J. Anal. Math. 27 (1974), 94–117.

[17] G. F. LAWLER, Intersections of random walks, Probab. Appl., Birkhäuser, Boston, Mass., 1991.

[18] J.-F. LE GALL, Propriétés d'intersection des marches aléatoires I, Convergence vers le temps local d'intersection (French), Comm. Math. Phys. 104 (1986), 471–507.

[19] N. MADRAS AND G. SLADE, The self-avoiding walk, Probab. Appl., Birkhäuser, Boston, Mass., 1993.

[20] I. MCGILLIVRAY, Large time volume of the pinned Wiener sausage, J. Funct. Anal. 170 (2000), 107–140.

[21] F. SPITZER, Principles of random walk, Second edition, Graduate Texts Math., vol. 34, Springer, New York, 1976.

DEPARTMENT OF MATHEMATICS
KUMAMOTO UNIVERSITY
KUROKAMI KUMAMOTO 860–8555
JAPAN

E-mail address: hamana@gpo.kumamoto-u.ac.jp