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EQUIVARIANT COMPLETIONS OF TORIC CONTRACTION MORPHISMS

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Abstract. We treat equivariant completions of toric contraction morphisms as an application of the toric Mori theory. For this purpose, we generalize the toric Mori theory for non-$\boldsymbol{Q}$-factorial toric varieties. So, our theory seems to be quite different from Reid's original combinatorial toric Mori theory. We also explain various examples of non-$\boldsymbol{Q}$-factorial contractions, which imply that the $\boldsymbol{Q}$-factoriality plays an important role in the Minimal Model Program. Thus, this paper completes the foundation of the toric Mori theory and shows us a new aspect of the Minimal Model Program.

1. Introduction. In [FS1], we gave a simple and non-combinatorial proof to the toric Mori theory. As mentioned in [FS1], our method cannot recover combinatorial aspects of [R]. One of the main purposes of this paper is to understand the local behavior of the toric contraction morphisms, which was described in [R, (2.5) Corollary] when the varieties are $\boldsymbol{Q}$-factorial and *complete*. It is obvious that the non-complete fans are much harder to treat than the complete ones. So, we avoid manipulating and subdividing non-complete fans. Our strategy is to compactify the toric contraction morphisms equivariantly and apply Reid's result.

Let $f : X \to Y$ be a projective toric morphism. We would like to compactify $f : X \to Y$ equivariantly, that is,

$$\begin{array}{rccc} \bar{f}: & \bar{X} & \to & \bar{Y} \\ & \cup & & \cup \\ f: & X & \to & Y, \end{array}$$

where $\bar{X}$ (resp. $\bar{Y}$) is an equivariant completion of X (resp. Y). More precisely, we would like to compactify f equivariantly without losing the following properties:

(i) projectivity of the morphism,
(ii) $\boldsymbol{Q}$-factoriality of the source space,
(iii) the relative Picard number is one,

and so on. Note that we do not assume f to be birational. The main results are Theorems 2.11 and 2.12, where we compactify f equivariantly by using the toric Mori theory. The statements are too long to mention here. These theorems guarantee that we can always compactify toric contraction morphisms equivariantly preserving nice properties. However, our proof does not show us how to compactify f even if it is given concretely. As a corollary, we obtain a description of the toric contraction morphisms when the source spaces are $\boldsymbol{Q}$-factorial

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and the relative Picard numbers are one (Theorem 3.2). As mentioned above, it seems to be difficult to obtain a local description of the toric contraction morphism without reducing it to the complete case. This is why we treat the equivariant completions of the toric contraction morphisms.

To carry out our program, we generalize the toric Mori theory for *non-complete* and *non-$\boldsymbol{Q}$-factorial* varieties. It is also the main theme of this paper. We do not need the toric Mori theory for non-$\boldsymbol{Q}$-factorial varieties to construct an equivariant completion of $f : X \to Y$ when X is $\boldsymbol{Q}$-factorial. However, it is natural and interesting to consider the toric Mori theory for non-$\boldsymbol{Q}$-factorial varieties. Without $\boldsymbol{Q}$-factoriality, various new phenomena occur even in the three-dimensional Minimal Model Program (see Section 4 and [FS3]). We believe that this generalized version of the toric Mori theory is not reachable by Reid's combinatorial technique since non-complete and non-simplicial fans are very difficult to manipulate. So, it seems to be reasonable to regard our toric Mori theory to be different from Reid's combinatorial one. The coverage of our theory is much wider than Reid's.

Note that the Minimal Model Program for non-$\boldsymbol{Q}$-factorial varieties may be useful in the study of higher dimensional log flips (see [F3, Section 4]). This paper will open the door to the non-$\boldsymbol{Q}$-factorial world.

This paper mainly treats the conceptual aspects of the toric Mori theory. The appendix [FS3] (see also [FS2]), where we construct an example of global toric 3-dimensional flops, will supplement this paper from the combinatorial viewpoint. Recently, Hiroshi Sato described the combinatorial aspects of the toric contraction morphisms from $\boldsymbol{Q}$-factorial toric varieties by using *extremal primitive relations* and Theorem 2.11. For the details, see [S]. It will help us to understand Theorem 3.2 below.

We summarize the contents of this paper: In Section 2, we prove the existence of equivariant completions of toric contraction morphisms in various settings. For this purpose, we generalize the toric Mori theory for non-$\boldsymbol{Q}$-factorial toric varieties. Section 3 deals with applications of the equivariant completions obtained in Section 2. The final theorem in Section 3 is a slight generalization of the main theorem of [F2]. In Section 4, we will treat various examples of non-$\boldsymbol{Q}$-factorial toric contraction morphisms. They imply that it is difficult to describe the local behavior of the (toric) contraction morphisms without the $\boldsymbol{Q}$-factoriality assumption. This section is independent of the other sections and seems to be valuable for those studying the Minimal Model Program.

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NOTATION. We often use the notation and the results in [FS1]. We will work over an algebraically closed field k throughout this paper.

(i) Let $v_i \in N \simeq \boldsymbol{Z}^n$ for $1 \leq i \leq k$. Then the symbol $\langle v_1, v_2, \ldots, v_k \rangle$ denotes the cone $\boldsymbol{R}_{\geq 0}v_1 + \boldsymbol{R}_{\geq 0}v_2 + \cdots + \boldsymbol{R}_{\geq 0}v_k$ in $N_{\boldsymbol{R}} \simeq \boldsymbol{R}^n$, where $\boldsymbol{R}_{\geq 0}$ is the set of non-negative real numbers.

(ii) A *toric morphism* $f : X \to Y$ means an equivariant morphism f between toric varieties X and Y.

2. Equivariant completions of toric contraction morphisms. Let us start with the following preliminary proposition. Its proof is a warm-up of our toric Mori theory [FS1].

PROPOSITION 2.1. *Let $f : X \to Y$ be a projective toric morphism and $\bar{Y}$ an equivariant completion of Y. Then there exists an equivariant completion of $f : X \to Y$;*

$$\begin{array}{rccc} \bar{f}: & \bar{X} & \to & \bar{Y} \\ & \cup & & \cup \\ f: & X & \to & Y, \end{array}$$

where

(i) *$\bar{X}$ is an equivariant completion of X, and*

(ii) *$\bar{f}$ is a* projective *toric morphism.*

Furthermore,

(1) *if X is $\boldsymbol{Q}$-factorial* (*see* [FS1, Definition 2.3]), *then we can make $\bar{X}$ to be $\boldsymbol{Q}$-factorial, and*

(2) *if X has only* (*$\boldsymbol{Q}$-factorial*) *terminal* (*resp. canonical*) *singularities* (*see* [R, (1.11) Definition] *or* [FS1, Definition 2.9]), *then we can make $\bar{X}$ to have only* (*$\boldsymbol{Q}$-factorial*) *terminal* (*resp. canonical*) *singularities.*

PROOF. By Sumihiro's equivariant embedding theorem, there exists an equivariant completion X_1 of X. Let X_2 be the graph of the rational map $f : X_1 \dashrightarrow \bar{Y}$. Then, we obtain

$$\begin{array}{rccc} f_2: & X_2 & \to & \bar{Y} \\ & \cup & & \cup \\ f: & X & \to & Y. \end{array}$$

Let D be an f-ample Cartier divisor on X and D_2 the closure of D on X_2. By Corollary 5.8 in [FS1], $\bigoplus_{m\geq 0}(f_2)_*\mathcal{O}_{X_2}(mD_2)$ is a finitely generated $\mathcal{O}_{\bar{Y}}$-algebra. We put $\bar{X} := \operatorname{Proj}_{\bar{Y}} \bigoplus_{m\geq 0}(f_2)_*\mathcal{O}_{X_2}(mD_2)$. Then, $\bar{f} : \bar{X} \to \bar{Y}$ has the required properties (i) and (ii) since D_2 is f_2-ample over Y. When X is $\boldsymbol{Q}$-factorial, we replace $\bar{X}$ by its small projective $\boldsymbol{Q}$-factorialization (see [F2, Corollary 5.9]). So, (1) holds. For (2), we apply Proposition 2.3 below. □

The following is the blow-up whose exceptional divisor is the prescribed one.

LEMMA 2.2. *Let $g : Z \to X$ be a projective birational toric morphism. Let E be an irreducible g-exceptional divisor on Z. We put*

$$h : X' := \operatorname{Proj}_X \bigoplus_{m\geq 0} g_*\mathcal{O}_Z(-mE) \to X$$

and let E' be the strict transform of E on X'. Then $-E'$ is h-ample. So, $X' \setminus E' \simeq X \setminus h(E')$.

Furthermore, if X is $\boldsymbol{Q}$-Gorenstein, that is, K_X is $\boldsymbol{Q}$-Cartier, then

$$K_{X'} = h^* K_X + aE',$$

where $a = a(E', X, 0) \in \boldsymbol{Q}$ is the discrepancy of E' with respect to $(X, 0)$ (cf. [KM, Definition 2.5] *and* [FS1, Definition 2.9]).

Sketch of the proof. Run the MMP (see [FS1, 3.1] or 2.9 below) over X with respect to $-E$. In the notation of 2.9 below, X' is the $(-E)$-canonical model over X. □

PROPOSITION 2.3. *Let X be a toric variety and $\bar{X}$ an equivariant completion of X. Assume that X has only terminal (resp. canonical) singularities. Then there exists a projective toric morphism $g : Z \to \bar{X}$ such that Z has only terminal (resp. canonical) singularities and g is isomorphic over X. Moreover, if X is $\boldsymbol{Q}$-factorial, then we can make Z to be $\boldsymbol{Q}$-factorial.*

PROOF. Let $h : V \to \bar{X}$ be a projective toric resolution. We put $g : Z := \mathrm{Proj}_{\bar{X}} \bigoplus_{m \geq 0} h_* \mathcal{O}_V(mK_V) \to \bar{X}$. Then Z has only canonical singularities and K_Z is g-ample. We note that g is isomorphic over X. So, this Z is a required one when X has only canonical singularities. Thus, we may assume that X has only terminal singularities. Since the number of the divisors that are exceptional over Z and whose discrepancies are zero is finite, we can make Z to have only terminal singularities by applying Lemma 2.2 finitely many times.

Furthermore, if X is $\boldsymbol{Q}$-factorial, then we can make Z to be $\boldsymbol{Q}$-factorial by [F2, Corollary 5.9]. □

The next proposition is useful when we treat non-$\boldsymbol{Q}$-factorial toric varieties.

PROPOSITION 2.4. *Let X be a toric variety and D a Weil divisor on X. Then there exists a small projective toric morphism $g : Z \to X$ such that the strict transform D_Z of D on Z is $\boldsymbol{Q}$-Cartier.*

Furthermore, let U be the Zariski open set of X on which D is $\boldsymbol{Q}$-Cartier. Then we can construct $g : Z \to X$ so that D_Z is g-ample and g is isomorphic over U.

PROOF. By Corollary 5.8 in [FS1], $\bigoplus_{m \geq 0} \mathcal{O}_X(mD)$ is a finitely generated $\mathcal{O}_X$-algebra. We put $g : Z := \mathrm{Proj}_X \bigoplus_{m \geq 0} \mathcal{O}_X(mD) \to X$. This $g : Z \to X$ has the required property. See, for example, [KM, Lemma 6.2] or [K$^+$, 4.2 Proposition]. □

COROLLARY 2.5. *Let X be a toric variety. We assume that X is $\boldsymbol{Q}$-Gorenstein, that is, K_X is $\boldsymbol{Q}$-Cartier. Then there exists an equivariant completion $\bar{X}$ of X such that $\bar{X}$ is $\boldsymbol{Q}$-Gorenstein.*

PROOF. Let X' be an equivariant completion of X. We put $\bar{X} := \mathrm{Proj}_{X'} \bigoplus_{m \geq 0} \mathcal{O}_{X'}(mK_{X'})$. This $\bar{X}$ has the required property by Proposition 2.4. □

The following theorem is a generalization of the elementary transformations. We need it for the MMP in 2.9 below.

THEOREM 2.6 (cf. [FS1, Theorem 4.8]). *Let $\varphi : X \to W$ be a projective birational toric morphism and D a $\mathbf{Q}$-Cartier Weil divisor on X such that $-D$ is φ-ample. We put*

$$\varphi^+ : X^+ := \mathrm{Proj}_W \bigoplus_{m\geq 0} \varphi_* \mathcal{O}_X(mD) \to W$$

and let D^+ be the strict transform of D_W on X^+, where $D_W := \varphi_* D$. Then φ^+ is a small projective toric morphism such that D^+ is a φ^+-ample $\mathbf{Q}$-Cartier Weil divisor on X^+.

Let U be the Zariski open set of W over which φ is isomorphic. Then, so is φ^+ over U.

The commutative diagram

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow \quad \swarrow & \\ & W & \end{array}$$

is called the elementary transformation (with respect to D) *if $\varphi : X \to W$ is small* (*cf.* [FS1, Theorem 4.8]).

PROOF. We put $\varphi' : X' := \mathrm{Proj}_W \bigoplus_{m\geq 0} \mathcal{O}_W(mD_W) \to W$ and let D' be the strict transform of D_W on X' as in Proposition 2.4. Then, by the negativity lemma (see Lemma 2.7 below), $\varphi'_* \mathcal{O}_{X'}(mD') \simeq \varphi_* \mathcal{O}_X(mD)$ for every $m \geq 0$. We note that φ' is small. Thus, we obtain

$$\begin{aligned} X' &= \mathrm{Proj}_W \bigoplus_{m\geq 0} \mathcal{O}_W(mD_W) \\ &\simeq \mathrm{Proj}_W \bigoplus_{m\geq 0} \varphi'_* \mathcal{O}_{X'}(mD') \\ &\simeq \mathrm{Proj}_W \bigoplus_{m\geq 0} \varphi_* \mathcal{O}_X(mD) = X^+ . \end{aligned}$$

So, φ^+ and D^+ have the required properties. Note that this X^+ is the D-canonical model over W in the notation of 2.9 below. See also Example 4.3. □

Let us recall the following well-known negativity lemma ([FS1, Lemma 4.10]), which we already used in the proof of Theorem 2.6. The proof can be found in [KM, Lemma 3.38].

LEMMA 2.7 (the Negativity Lemma). *We consider a commutative diagram*

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ U & \dashrightarrow & V \\ \searrow & & \swarrow \\ & W & \end{array}$$

and $\mathbf{Q}$-Cartier divisors D and D' on U and V, respectively, where

(1) *$f : U \to W$ and $g : V \to W$ are proper birational morphisms between normal varieties,*

(2) *$f_* D = g_* D'$,*

(3) *$-D$ is f-ample and D' is g-ample,*

(4) $\mu : Z \to U$, $\nu : Z \to V$ *are common resolutions.*

Then $\mu^*D = \nu^*D' + E$, *where* E *is an effective* $\boldsymbol{Q}$*-divisor and is exceptional over* W. *Moreover, if* f *or* g *is non-trivial, then* $E \neq 0$.

REMARK 2.8. In Theorem 2.6, let us further assume that X is $\boldsymbol{Q}$-factorial and $\rho(X/W) = 1$. If φ contracts a divisor, then W is $\boldsymbol{Q}$-factorial. In particular, D_W is $\boldsymbol{Q}$-Cartier. So, $\varphi^+ : X^+ \to W$ is an isomorphism. If φ is small, then X^+ is $\boldsymbol{Q}$-factorial and $\rho(X^+/W) = 1$. For the non-$\boldsymbol{Q}$-factorial case, see the examples in Section 4.

The following Minimal Model Program (MMP, for short) for toric varieties is a slight generalization of the MMP explained in [FS1, 3.1]. This MMP works without the $\boldsymbol{Q}$-factoriality assumption. See also Remark 2.10 below.

2.9 (Minimal Model Program for Toric Varieties). We start with a projective toric morphism $f : X \to Y$ and a $\boldsymbol{Q}$-Cartier divisor D on X. Let l be a positive integer such that lD is a Weil divisor. We put $X_0 := X$ and $D_0 := D$. The aim is to set up a recursive procedure which creates intermediate $f_i : X_i \to Y$ and D_i on X_i. After finitely many steps, we obtain a finial objects $\tilde{f} : \tilde{X} \to Y$ and $\tilde{D}$. Assume that we already constructed $f_i : X_i \to Y$ and D_i with the following properties:

(i) f_i is projective,

(ii) D_i is a $\boldsymbol{Q}$-Cartier divisor on X_i.

If D_i is f_i-nef, then we set $\tilde{X} := X_i$ and $\tilde{D} := D_i$. Assume that D_i is not f_i-nef. Then we can take an extremal ray R of $NE(X_i/Y)$ such that $R \cdot D_i < 0$. Thus we have a contraction morphism $\varphi_R : X_i \to W_i$ over Y. If $\dim W_i < \dim X_i$, then we set $\tilde{X} := X_i$ and $\tilde{D} := D_i$ and stop the process. If φ_R is birational, then we put

$$X_{i+1} := \operatorname{Proj}_{W_i} \bigoplus_{m \geq 0} \varphi_{R*}\mathcal{O}_{X_i}(mlD_i)$$

and let D_{i+1} be the strict transform of $\varphi_{R*}D_i$ on X_{i+1} (see Theorem 2.6). By counting the number of the torus invariant irreducible divisors, we may assume that $\varphi_R : X_i \to W_i$ is small or $\dim W_i < \dim X_i$ after finitely many steps. By Theorem 4.9 (Termination of Elementary Transformations) in [FS1], there are no infinite sequences of the elementary transformations with respect to D_i (cf. Theorem 2.6). Therefore, this process always terminates and we obtain $\tilde{f} : \tilde{X} \to Y$ and $\tilde{D}$. We note that the relative Picard number may increase in the process (see Example 4.2 below). When $\tilde{D}$ is $\tilde{f}$-nef, $\tilde{X}$ is called a *D-minimal model over Y*. We call this process (*D-*)*Minimal Model Program over Y*, where D is the divisor used in the process. When we apply the Minimal Model Program (MMP, for short), we say that, for example, we *run the MMP over Y with respect to the divisor D*. If $\tilde{X}$ is a D-minimal model over Y, then we put

$$X^\dagger := \operatorname{Proj}_Y \bigoplus_{m \geq 0} \tilde{f}_*\mathcal{O}_{\tilde{X}}(ml\tilde{D}) .$$

It is not difficult to see that $X^\dagger \simeq \operatorname{Proj}_Y \bigoplus_{m \geq 0} f_*\mathcal{O}_X(mlD)$. We call $X^\dagger$ the *D-canonical model over Y*. We note that there exists a toric morphism $\tilde{X} \to X^\dagger$ over Y which corresponds

to $\tilde{f}^*\tilde{f}_*\mathcal{O}_{\tilde{X}}(k\tilde{D}) \to \mathcal{O}_{\tilde{X}}(k\tilde{D}) \to 0$, where k is a sufficiently large and divisible integer (see [FS1, Proposition 4.1]).

REMARK 2.10. (i) When X is $\boldsymbol{Q}$-factorial, this process coincides with the one explained in [FS1, 3.1]. See Remark 2.8.

(ii) If X has only terminal (resp. canonical) singularities and $D = K_X$, then so does X_i for every i. It is an easy consequence of the negativity lemma (see Lemma 2.7).

The following Theorems 2.11 and 2.12 are the main results in this paper. We divide them since Theorem 2.11 is sufficient for various applications and the proof of Theorem 2.12 is complicated.

THEOREM 2.11 (Equivariant completions of toric contraction morphisms). *Let $f : X \to Y$ be a projective toric morphism. Let $\varphi := \varphi_R : X \to W$ be the contraction morphism over Y with respect to an extremal ray R of $NE(X/Y)$. Then there exists an equivariant completion of $\varphi : X \to W$ as follows*;

$$\begin{array}{rccc} \bar{\varphi}: & \bar{X} & \to & \bar{W} \\ & \cup & & \cup \\ \varphi: & X & \to & W, \end{array}$$

where

(i) *$\bar{X}$ and $\bar{W}$ are equivariant completions of X and W, and*

(ii) *$\bar{\varphi}$ is a projective toric morphism with the relative Picard number $\rho(\bar{X}/\bar{W}) = 1$.*

Furthermore,

(1) *if X is $\boldsymbol{Q}$-factorial, then we can make $\bar{X}$ to be $\boldsymbol{Q}$-factorial, and*

(2) *if X has only ($\boldsymbol{Q}$-factorial) terminal (resp. canonical) singularities and $-K_X$ is φ-ample, then we can make $\bar{X}$ to have only ($\boldsymbol{Q}$-factorial) terminal (resp. canonical) singularities.*

Let $\bar{Y}$ be an equivariant completion of Y. Then we can construct $\bar{\varphi}$ with the following property:

(iii) *$\bar{W} \to \bar{Y}$ is an equivariant completion of $W \to Y$ such that $\bar{W} \to \bar{Y}$ is projective.*

PROOF. Let W' be an equivariant completion of W. If $\bar{Y}$ is given, then we can take W' to be projective over $\bar{Y}$ by Proposition 2.1. Let $\varphi' : X' \to W'$ be an equivariant completion of $\varphi : X \to W$. By Proposition 2.1, we may assume that φ' is projective. We may further assume that X' is $\boldsymbol{Q}$-factorial (resp. X' has only ($\boldsymbol{Q}$-factorial) terminal or canonical singularities) when X is $\boldsymbol{Q}$-factorial (resp. X has only ($\boldsymbol{Q}$-factorial) terminal or canonical singularities). Let D be a $\boldsymbol{Q}$-Cartier divisor on X such that $-D$ is φ-ample. Take a $\boldsymbol{Q}$-Cartier divisor D' on X' such that $D'|_X = D$. We note that we can always take such D' by Proposition 2.4 if we modify X' suitably. We put $D' = K_{X'}$ in the case (2). Run the MMP (as explained in 2.9) over W' with respect to D'. If an extremal ray R does not contain the numerical equivalence class of the curves contracted by $\varphi : X \to W$, then the contraction with respect to R occurs outside X. So, we obtain

$$X' =: X'_0 \dashrightarrow X'_1 \dashrightarrow X'_2 \dashrightarrow \cdots \dashrightarrow X'_k =: \bar{X}$$

over W' and a contraction $\bar{\varphi} : \bar{X} \to \bar{W}$ such that $\rho(\bar{X}/\bar{W}) = 1$ and $\bar{\varphi}$ contracts the curves in the fibers of φ. It is easy to see that $\bar{\varphi} : \bar{X} \to \bar{W}$ has the required properties. See also Remarks 2.8 and 2.10. □

THEOREM 2.12. *We use the same notation as in Theorem* 2.11. *We can generalize Theorem* 2.11(2) *as follows*:

(2′) *if X has only* (**Q**-*factorial*) *terminal* (*resp. canonical*) *singularities and $-K_X$ is φ-*nef, *then we can make $\bar{X}$ to have only* (**Q**-*factorial*) *terminal* (*resp. canonical*) *singularities.*

PROOF. By Theorem 2.11 (2), we may assume that $-K_X$ is not φ-ample, or equivalently, K_X is φ-numerically trivial. As in the proof of Theorem 2.11, we run the MMP over W' with respect to $D' = K_{X'}$. In this case, we obtain

$$X' =: X'_0 \dashrightarrow X'_1 \dashrightarrow X'_2 \dashrightarrow \cdots \dashrightarrow X'_k =: \tilde{X}$$

over W', and $\tilde{X}$ is a D'-minimal model over W', that is, $K_{\tilde{X}}$ is nef over W'. It is easy to see that each step occurs outside X. Note that $K_{\tilde{X}}$ is not ample over W' since K_X is φ-numerically trivial.

Let B be the complement of the big tours in X regarded as a reduced divisor. Then it is well-known that $K_X + B \sim 0$. So, B is φ-numerically trivial. Therefore, it is not difficult to see that there exists an effective torus-invariant Cartier divisor E on X such that $-E$ is φ-ample. Let F be the closure of E on $\tilde{X}$. By modifying $\tilde{X}$ birationally outside X (if necessary), we may assume that F is **Q**-Cartier (see Proposition 2.4). Run the MMP over W' with respect to F. For each step, we choose a K-trivial extremal ray R, that is, $K \cdot R = 0$, where K is the canonical divisor. Then we obtain a sequence

$$\tilde{X} =: \tilde{X}_0 \dashrightarrow \tilde{X}_1 \dashrightarrow \tilde{X}_2 \dashrightarrow \cdots \dashrightarrow \tilde{X}_l =: \bar{X}$$

over W' and a contraction $\bar{\varphi} : \bar{X} \to \bar{W}$ such that $\bar{\varphi}$ contracts the curves in the fibers of φ. We note that $(\tilde{X}, \varepsilon F)$ has only terminal singularities for $0 \leq \varepsilon \ll 1$ (resp. $\tilde{X}$ has only canonical singularities) when X' has only terminal (resp. canonical) singularities. So, the pair $(\bar{X}, \varepsilon \bar{F})$, where $\bar{F}$ is the strict transform of F, has only terminal singularities for $0 \leq \varepsilon \ll 1$ (resp. $\bar{X}$ has only canonical singularities) by [KM, Lemma 3.38]. We note that each step of the above MMP does not contract any components of F since it occurs outside X. Therefore, $\bar{\varphi} : \bar{X} \to \bar{W}$ has the desired properties. □

REMARK 2.13. The assumptions on K_X in Theorems 2.11 and 2.12 are useful when we construct global (toric) examples of *flips* and *flops*.

The following is a question of J. Kollár.

QUESTION 2.14. Let $f : X \to Y$ be a projective equivariant morphism between toric varieties with connected fibers. Assume that $\rho(X/Y) = k \geq 2$. Is it possible to compactify f equivariantly preserving $\rho = k$?

3. Applications of equivariant completions. In this section, we treat some applications of Theorem 2.11 and related topics.

3.1. The next theorem is a direct consequence of Theorem 2.11 and Reid's description of the toric contraction morphisms. Theorem 3.2 was obtained by Reid when X is *complete*. For the details, see, for instance, [M, Corollary 14-2-2], where Matsuki corrected minor errors in [R]. See [M, Remark 14-2-3]. Remark 3.3 below is a supplement to [M, Corollary 14-2-2]. For the combinatorial aspects of this theorem, see [S].

THEOREM 3.2 (cf. [R, (2.5) Corollary]). *Let $f : X \to Y$ be a projective toric morphism. Assume that X is $\boldsymbol{Q}$-factorial. Let R be an extremal ray of $NE(X/Y)$ and $\varphi_R : X \to W$ the contraction morphism over Y with respect to R. Let*

$$\begin{array}{rccc} & A & \to & B \\ & \cap & & \cap \\ \varphi_R : & X & \to & W \end{array}$$

be the loci on which φ_R is not an isomorphism; A and B are irreducible, $\varphi^{-1}(P)_{\mathrm{red}}$ is a $\boldsymbol{Q}$-factorial projective toric $(\dim A - \dim B)$*-fold with the Picard number one for every point $P \in B$. More precisely, there exist an open covering $B = \bigcup_{i \in I} U_i$ and a $\boldsymbol{Q}$-factorial projective toric variety F with the Picard number $\rho(F) = 1$ such that*

(i) *U_i is a torus invariant open subvariety of B for every i,*

(ii) *there exists a finite toric morphism $U_i' \to U_i$ such that*

$$(U_i' \times_B A)^{\nu} \simeq U_i' \times F$$

for every $i \in I$, where $(U_i' \times_B A)^{\nu}$ is the normalization of $U_i' \times_B A$.
We note that $-K_F$ is an ample $\boldsymbol{Q}$-Cartier divisor since $\rho(F) = 1$.

PROOF. By Theorem 2.11, we obtain an equivariant completion:

$$\begin{array}{rccc} \bar{\varphi} : & \bar{X} & \to & \bar{W} \\ & \cup & & \cup \\ \varphi_R : & X & \to & W\,. \end{array}$$

We may assume that $\bar{X}$ is $\boldsymbol{Q}$-factorial, $\bar{\varphi}$ is projective, and $\rho(\bar{X}/\bar{W}) = 1$. Let

$$\begin{array}{rccc} & \bar{A} & \to & \bar{B} \\ & \cap & & \cap \\ \bar{\varphi} : & \bar{X} & \to & \bar{W} \end{array}$$

be the loci on which $\bar{\varphi}$ is not an isomorphism. Apply Reid's description: [R, (2.5) Corollary] to $\bar{\varphi}$. For the detailed description of $\bar{\varphi} : \bar{X} \to \bar{W}$, see [M, Corollary 14-2-2] and Remark 3.3 below. □

In the above theorem, the assumption that X is $\boldsymbol{Q}$-factorial plays a crucial role. See Example 4.1 below and [FS3, Example A.1].

REMARK 3.3 (Supplements to the description of contractions of extremal rays by Matsuki). In this remark, we use the same notation as in [M, Chapter 14]. In [M, Corollary 14-2-2], Matsuki claimed that $E \times_F U'_{\tau(w)_Y} \cong G \times U'_{\tau(w)_Y}$. In our notation in Theorem 3.2, he claims that $U_i' \times_B A \simeq U_i' \times F$. However, it is not true in general. We

have to take the normalization of the left hand side. So, the correct statement should be $(E \times_F U'_{\tau(w)_Y})^\nu \cong G \times U'_{\tau(w)_Y}$, where $(E \times_F U'_{\tau(w)_Y})^\nu$ is the normalization of $E \times_F U'_{\tau(w)_Y}$. We note that $(E \times_F U'_{\tau(w)_Y})^\nu$ is irreducible since $E \to F$ has connected fibers. For the details, see [AK, Lemma 5.6]. Therefore, $\varphi_R^{-1}(P)_{\mathrm{red}}$ is not necessarily isomorphic to G. Let $O(\gamma) \subset F$ be the orbit associated to a cone γ. Then $\varphi_R^{-1}(O(\gamma))_{\mathrm{red}} \simeq G_\gamma \times O(\gamma)$ and $\varphi_R^{-1}(O(\gamma))_{\mathrm{red}} \to O(\gamma)$ is isomorphic to the second projection $G_\gamma \times O(\gamma) \to O(\gamma)$, where G_γ is an $(n-\beta)$-dimesnional $\boldsymbol{Q}$-factorial projective toric variety with the Picard number $\rho(G_\gamma) = 1$, Note that G_γ is defined by $n - \beta + 1$ one-dimensional vectors $\{v_{\beta+1}, \ldots, v_{n+1}\}$ for any γ. However, in general, $G_{\gamma_1} \not\simeq G_{\gamma_2}$ for two distinct cones γ_1, γ_2. It is because the lattice group that defines G_γ depends on γ. So, $E \to F$ is not necessarily a fiber bundle but a *quasi*-fiber bundle in Ishida's notation. The following example may help the reader to understand it.

EXAMPLE 3.4 (Extremal Fano contraction). We fix $N = \boldsymbol{Z}^3$ and $N' = \boldsymbol{Z}$. We put

$$v_1 = (0, 0, 1)\,, \qquad v_2 = (-1, 0, 0)\,, \qquad v_3 = (1, 0, -1)\,,$$
$$v_4 = (0, -1, 0)\,, \qquad v_5 = (0, 2, -1)\,.$$

We consider the following fan.

$$\Delta = \left\{ \begin{array}{llll} \langle v_1, v_2, v_4\rangle, & \langle v_1, v_2, v_5\rangle, & \langle v_1, v_3, v_4\rangle, & \langle v_1, v_3, v_5\rangle, \\ \langle v_2, v_3, v_4\rangle, & \langle v_2, v_3, v_5\rangle, & \text{and their faces} & \end{array} \right\}.$$

We define $X = X(\Delta)$. Then X is a $\boldsymbol{Q}$-factorial projective toric 3-fold with $\rho(X) = 2$. Let $N_{\boldsymbol{R}} \simeq \boldsymbol{R}^3 \to N'_{\boldsymbol{R}} \simeq \boldsymbol{R}$ be the projection to the second coordinate. It induces a toric morphism $f : X \to \boldsymbol{P}^1$. Then $f : X \to \boldsymbol{P}^1$ is an extremal contraction. We have the following properties:

(i) $f^{-1}(0)$ is non-reduced since v_5 is mapped onto $2 \in N'$,

(ii) $f^{-1}(0)_{\mathrm{red}}$ is isomorphic to a weighted projective space $\boldsymbol{P}(1, 1, 2)$, and

(iii) $g := f|_{X \setminus f^{-1}(0)} : Y := X \setminus f^{-1}(0) \to Z := \boldsymbol{P}^1 \setminus \{0\}$ is isomorphic to the second projection $\boldsymbol{P}^2 \times \boldsymbol{A}^1 \to \boldsymbol{A}^1$.

We note that $-K_X \cdot C \geq 3$ for any curve C in the fibers $g : Y \to Z$. On the other hand, there exists a torus invariant curve C_0 in the fiber $f^{-1}(0)$ such that $-K_X \cdot C_0 = 3/2$. It can be checked by adjunction and the computations in [F2, Section 2] (cf. Theorem 3.13).

REMARK 3.5. In Theorem 3.2, let Δ be the fan such that $X = X(\Delta)$. Then Δ need not contain n-dimensional cones, where $n = \dim X$.

REMARK 3.6. Let $f : X \to Y$ be a projective equivariant morphism between toric varieties. Let $\varphi_R : X \to W$ be the extremal contraction associated to an extremal ray R of $NE(X/Y)$. Assume that X is $\boldsymbol{Q}$-factorial. If X is complete, then Reid obtained the combinatorial descriptions of φ_R in [R] by using the notion of *walls*. Sato generalized Reid's combinatorial descriptions for non-complete toric varieties in [S] by using Theorem 2.11 and the notion of *extremal primitive relations*. Examples in Section 4 and Example A.1 in [FS3] imply that it is impossible to describe toric extremal contractions combinatorially without $\boldsymbol{Q}$-factoriality.

REMARK 3.7. In [M, Chapter 14], Matsuki left the details of the verifications for the relative case to the reader in various places. In the relative case in [M, Proposition 14-1-2], all we need is the rigidity lemma (see, for example, [KM, Lemma 1.6]). The rest are straightforward. In [M, Chapter 14], $X(\Delta)$ is always assumed to be *complete* even in the relative setting of $\phi : X(\Delta) \to S(\Delta_S)$. So, there are no difficulties to handle the relative setting in [M, Chapter 14]. For the *true* relative setting, that is, $\phi : X(\Delta) \to S(\Delta_S)$ is a projective morphism and $X(\Delta)$ is not necessarily complete, see [FS1].

Here is a general remark on equivariant completions of toric varieties.

REMARK 3.8. Let X be a toric variety corresponding to a fan Δ. It is well-known that compactifying X equivariantly is equivalent to compactifying Δ. We know that to compactify Δ without Sumihiro's theorem is very difficult. Recently, Ewald and Ishida independently succeeded in compactifying (not necessarily rational) fans without using Sumihiro's theorem (see [EI]).

3.9. In [F1, Corollary 4.6], we proved that the target space of a Mori fiber space has at most log terminal singularities. In dimension three, it is conjectured that the target space has only canonical singularities (see, for instance, [P, Conjecture 0.2]). Before we explain an example of Mori fiber spaces, let us recall the definition of the Mori fiber space.

DEFINITION 3.10 (Mori fiber space). A normal projective variety X having only $\boldsymbol{Q}$-factorial terminal singularities with a morphism $\Phi : X \to Y$ is a *Mori fiber space* if (i) Φ is a morphism with connected fibers onto a normal projective variety Y of $\dim Y < \dim X$, (ii) $-K_X$ is Φ-ample, and (iii) $\rho(X/Y) = 1$.

The following is an example of 4-dimensional Mori fiber spaces.

EXAMPLE 3.11 (Mori fiber space whose target space has a bad singularity). Let $\boldsymbol{Z}_4 = \langle\zeta\rangle$ be the cyclic group of fourth roots of unity with $\zeta = \sqrt{-1}$. Let $\boldsymbol{P}^1 \times \boldsymbol{C}^3 \to \boldsymbol{C}^3$ be the second projection. We consider the following actions of $\boldsymbol{Z}_4$ on $\boldsymbol{P}^1 \times \boldsymbol{C}^3$ and $\boldsymbol{C}^3$:

$$([u : v], (x, y, z)) \to ([u : -v], (\zeta x, \zeta y, \zeta z)) ,$$
$$(x, y, z) \to (\zeta x, \zeta y, \zeta z) ,$$

where $[u : v]$ is the homogeneous coordinate of $\boldsymbol{P}^1$. We put $X := (\boldsymbol{P}^1 \times \boldsymbol{C}^3)/\boldsymbol{Z}_4$ and $Y := \boldsymbol{C}^3/\boldsymbol{Z}_4$. Then the induced equivariant morphism $f : X \to Y$ has the following properties:

(i) X has terminal quotient singularities along the central fiber of f,
(ii) Y has a $(1/4)(1, 1, 1)$ quotient singularity, which is not canonical,
(iii) X and Y are $\boldsymbol{Q}$-factorial,
(iv) $\rho(X/Y) = 1$, and
(v) $-K_X$ is f-ample.

By applying Theorem 2.11 (2), we obtain a toric Mori fiber space $\bar{f} : \bar{X} \to \bar{Y}$ that is an equivariant completion of $f : X \to Y$. Note that we can make $\bar{Y}$ projective by Theorem 2.11 (iii). Thus, $\bar{f} : \bar{X} \to \bar{Y}$ is a Mori fiber space such that the target space $\bar{Y}$ has a singularity that is not canonical.

This example shows that our theorem is useful when we construct *global* examples from local ones. For a more combinatorial treatment, see [FS2, Example 5.1].

3.12. The final theorem is a slight generalization of [F2, Theorem 0.1].

THEOREM 3.13 (Length of an extremal ray). *Let $f : X \to Y$ be a projective surjective equivariant morphism between toric varieties. Let $D = \sum_j d_j D_j$ be a $\boldsymbol{Q}$-divisor, where D_j is an irreducible torus invariant divisor and $0 \leq d_j \leq 1$ for every j. Assume that $K_X + D$ is $\boldsymbol{Q}$-Cartier. Then, for each extremal ray R of $NE(X/Y)$, there exists an irreducible curve C such that $[C] \in R$ and*

$$-(K_X + D) \cdot C \leq \dim X + 1 .$$

Moreover, we can choose C in such a way that

$$-(K_X + D) \cdot C \leq \dim X$$

unless $X \simeq \boldsymbol{P}^{\dim X}$ and $\sum_j d_j < 1$. Here, we do not claim that C is a torus invariant curve. We note that R may contain no numerical equivalence classes of torus invariant curves.

Sketch of the proof. If Y is a point, then this is the main theorem of [F2]. So, we may assume that $\dim Y \geq 1$. Since the arguments in Step 2 in the proof of [F2, Theorem 0.1] work with minor modifications, we may further assume that X is $\boldsymbol{Q}$-factorial. Let R be a $(K_X + D)$-negative extremal ray of $NE(X/Y)$. We consider the contraction $\varphi_R : X \to W$ over Y with respect to R. Let U be a quasi-projective torus invariant open subvariety of W such that $X_U := \varphi_R^{-1}(U) \to U$ is not an isomorphism. It is not difficult to see that X_U is $\boldsymbol{Q}$-factorial and $\rho(X_U/U) = 1$ (see [FS3, Example A.1]). We note that $\mathrm{Pic}(X) \otimes \boldsymbol{Q} \to \mathrm{Pic}(X_U) \otimes \boldsymbol{Q}$ is surjective. So, by shrinking W, we may assume that X and W are quasi-projective. By Theorem 2.11, we have an equivariant completion of $\varphi := \varphi_R : X \to W$, that is,

$$\begin{array}{rccc} \varphi : & X & \to & W \\ & \cap & & \cap \\ \bar{\varphi} : & \bar{X} & \to & \bar{W} , \end{array}$$

where $\bar{X}$ and $\bar{W}$ are $\boldsymbol{Q}$-factorial projective toric varieties and $\rho(\bar{X}/\bar{W}) = 1$. Let $\bar{D}$ be the closure of D on $\bar{X}$. Then $-(K_{\bar{X}} + \bar{D})$ is $\bar{\varphi}$-ample. Therefore, $\bar{\varphi}$ is the contraction morphism with respect to a suitable $(K_{\bar{X}} + \bar{D})$-negative extremal ray $Q \subset NE(\bar{X}/\bar{W}) \subset NE(\bar{X})$ (see [R, (1.5)] and [FS2, 3.8]). So, we can apply the arguments in Step 1 in the proof of [F2, Theorem 0.1] to $\bar{\varphi} : \bar{X} \to \bar{W}$. Let

$$\begin{array}{rccc} & A & \to & B \\ & \cap & & \cap \\ \varphi_R : & X & \to & W \end{array}$$

be the loci on which φ_R is not an isomorphism. Let $\bar{A}$ (resp. $\bar{B}$) be the closure of A (resp. B) in $\bar{X}$ (resp. $\bar{W}$). We can calculate $K_{\bar{A}}$ by adjunction (cf. the computation of K_P in [F2, Proof of The Theorem]). Let F be a general fiber of $\bar{A} \to \bar{B}$. Then F is a $\boldsymbol{Q}$-factorial projective toric $(\dim A - \dim B)$-fold with the Picard number one (cf. Theorem 3.2). So, it is sufficient to prove the following claim.

CLAIM. *There exists a curve C in F such that*

$$-(K_X + D) \cdot C \leq -K_{\bar{A}} \cdot C = -K_F \cdot C \leq \dim A - \dim B + 1 \leq \dim X\,.$$

If C is in F, then the first inequality follows from the computations similar to the ones in Step 1 in [F2, Proof of The Theorem]. By adjunction, $K_{\bar{A}}|_F = K_F$. Thus, it is obvious that $-K_{\bar{A}} \cdot C = -K_F \cdot C$ for C in F. The computaitons in [F2, Section 2] imply the existence of C on F such that $-K_F \cdot C \leq \dim F + 1$. □

4. Examples of non-Q-factorial contractions. In this section, we explain various examples of non-$\boldsymbol{Q}$-factorial toric contraction morphisms. All the examples are three-dimensional.

The first one is a beautiful example due to Sato of divisorial contractions. This implies that it is difficult to describe the local behavior of divisorial contractions without the $\boldsymbol{Q}$-factoriality assumption even if the relative Picard number is one.

EXAMPLE 4.1 (Sato's non-$\boldsymbol{Q}$-factorial divisorial contraction). Let e_1, e_2, e_3 form the usual basis of $\boldsymbol{Z}^3$, and let e_4 be given by

$$e_1 + e_2 = e_3 + e_4\,.$$

We put

$$e_5 = e_1 + e_2 = e_3 + e_4$$

and

$$e_6 = e_2 + e_3\,.$$

Let

$$\Delta_Y = \{\langle e_1, e_2, e_3, e_4\rangle, \text{and its faces}\}$$

and $Y := X(\Delta_Y)$. We put

$$\Delta_X = \{\langle e_1, e_4, e_5\rangle, \langle e_1, e_3, e_5, e_6\rangle, \langle e_2, e_4, e_5, e_6\rangle, \text{and their faces}\}\,.$$

We define $X := X(\Delta_X)$. Then $f : X \to Y$ has the following properties:

(i) X has terminal singularities,
(ii) X is not $\boldsymbol{Q}$-factorial,
(iii) f is a projective birational equivariant morphism with $\rho(X/Y) = 1$,
(iv) $-K_X$ is f-ample, and
(v) the exceptional locus contains a reducible divisor.

Figure 1 helps us to understand the above contraction morphism.

We can easily check the following properties:

(1) X_1 and X_2 are non-singular,
(2) φ_1 and φ_2 are blow-ups,
(3) φ_3 and φ_4 are flopping contractions, that is, K_{X_i} is φ_{i+1}-numerically trivial for $i = 2, 3$,
(4) X_3 and X are not $\boldsymbol{Q}$-factorial,
(5) X_3 and X have only terminal singularities,

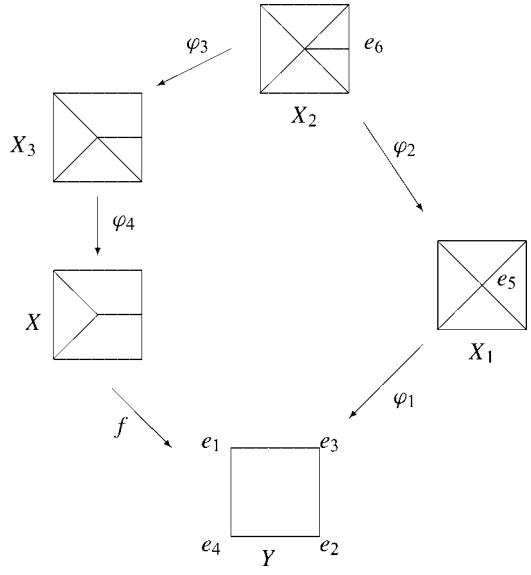

FIGURE 1.

(6) $\rho(X_1/Y) = 2$ and $\rho(X_2/X_1) = \rho(X_2/X_3) = \rho(X_3/X_4) = 1$.

The ampleness of $-K_X$ follows from the convexity of the roofs of the maximal cones in Δ_X (cf. [R, (4.3) Proposition]).

The next is an example of *flips*. In this example, the relative Picard number increases by a flip.

EXAMPLE 4.2 (Non-$\boldsymbol{Q}$-factorial flip). Let e_1, e_2, e_3 form the usual basis of $\boldsymbol{Z}^3$, and let e_4 be given by

$$e_1 + e_2 = e_3 + e_4\,.$$

We put $f_1 = (3, 1, -2)$, $f_2 = (-1, 1, 2) \in \boldsymbol{Z}^3$. We consider the following fans:

$$\Delta_a = \{\langle e_1, e_3, f_1, f_2\rangle, \langle e_2, e_4, f_1, f_2\rangle, \text{and their faces}\}\,,$$
$$\Delta_b = \{\langle e_1, e_4, f_1\rangle, \langle e_2, e_3, f_2\rangle, \langle e_1, e_2, e_3, e_4\rangle, \text{and their faces}\}\,, \quad \text{and}$$
$$\Delta_c = \{\langle e_1, e_2, e_3, e_4, f_1, f_2\rangle, \text{and its faces}\}\,.$$

We put $X := X(\Delta_a)$, $X^+ := X(\Delta_b)$, and $Y := X(\Delta_c)$. Then we have a commutative diagram:

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

such that

(i) $f : X \to Y$ and $f^+ : X^+ \to Y$ are both small projective equivariant morphisms,

(ii) $\rho(X/Y) = 1$ and $\rho(X^+/Y) = 2$,

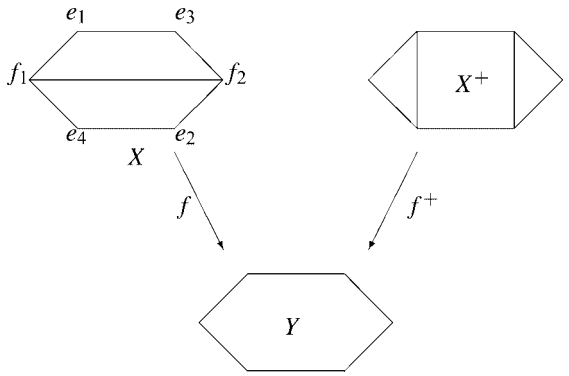

FIGURE 2.

(iii) X and X^+ are not $\boldsymbol{Q}$-factorial, and

(iv) $-K_X$ is f-ample and K_{X^+} is f^+-ample.

Thus, this diagram is a so-called *flip*. Figure 2 helps us to understand this example.

We note that the points e_1, e_3, f_1, and f_2 are on $H_1 = \{(x, y, z) | x + z = 1\}$, e_2, e_4, f_1, and f_2 are on $H_2 = \{(x, y, z) | y = 1\}$. It is obvious that $H_3 = \{(x, y, z) | x + y + z = 2\}$ contains f_1 and f_2, while the points e_1, e_2, e_3, and e_4 are on $H_4 = \{(x, y, z) | x + y + z = 1\}$. We note that the non-trivial lattice points in the *shed* (see [R, p. 414 Definition]) of Δ_a are

$$\frac{1}{4}f_1 + \frac{3}{4}f_2\,, \quad \frac{1}{2}f_1 + \frac{1}{2}f_2 \quad \text{and} \quad \frac{3}{4}f_1 + \frac{1}{4}f_2 \quad \in \boldsymbol{Z}^3\,.$$

We can check the following properties:

(1) The flipping locus is $\boldsymbol{P}^1$ and X has only canonical singularities (see [R, (1.11) Definition]).

(2) The flipping curve is contained in the singular locus of X.

(3) X^+ has only one singular point, which is an ordinary double point. In particular, X^+ has only terminal singularities.

(4) The flipped locus is $\boldsymbol{P}^1 \cup \boldsymbol{P}^1$ and these two $\boldsymbol{P}^1$'s intersect each other at the singular point of X^+.

The ampleness of $-K_X$ (resp. K_{X^+}) follows from the convexity (resp. concavity) of the roofs of the maximal cones (cf. [R, (4.3) Proposition]).

The final example is a non-$\boldsymbol{Q}$-factorial divisorial contraction whose target space is not $\boldsymbol{Q}$-Gorenstein.

EXAMPLE 4.3. We use the same notation as in Example 4.2. We put $f_3 = (0, 1, 1)$. We note that

$$f_3 = e_2 + e_3 = \frac{1}{4}f_1 + \frac{3}{4}f_2\,.$$

We consider the following fans:

$$\Delta_d = \{\langle e_1, e_4, f_3\rangle, \langle e_1, e_3, f_2, f_3\rangle, \langle e_2, e_4, f_2, f_3\rangle, \text{and their faces}\},$$
$$\Delta_e = \{\langle e_1, e_2, e_3, e_4\rangle, \langle e_2, e_3, f_2\rangle, \text{and their faces}\}, \text{ and}$$
$$\Delta_f = \{\langle e_1, e_2, e_3, e_4, f_2\rangle, \text{and its faces}\}.$$

We define $V := X(\Delta_d)$, $V^+ := X(\Delta_e)$, and $W := X(\Delta_f)$. Then we obtain a commutative diagram:

$$\begin{array}{ccc} V & \dashrightarrow & V^+ \\ & \searrow \quad \swarrow & \\ & W & \end{array}$$

such that

(i) $\varphi : V \to W$ is a projective birational equivariant morphism and φ contracts a divisor,

(ii) $\varphi^+ : V^+ \to W$ is a small projective equivariant morphism,

(iii) $-K_X$ is φ-ample and K_{V^+} is φ^+-ample,

(iv) $\rho(V/W) = \rho(V^+/W) = 1$,

(v) V and V^+ have only terminal singularities,

(vi) all V, V^+, and W are not $\boldsymbol{Q}$-factorial, and

(vii) W is not $\boldsymbol{Q}$-Gorenstein.

See Figure 3.

We note that the small morphism $\varphi^+ : V^+ \to W$ is the one given in Theorem 2.6. This operation $V \dashrightarrow V^+$ preserves the relative Picard number over W. Note that the number of the torus invariant divisors decreases.

This example shows that we need to modify W to continue the MMP even if φ contracts a divisor (see 2.9).

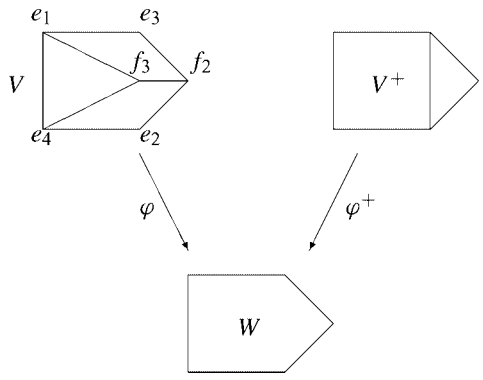

FIGURE 3.

In Examples 4.1, 4.2, and 4.3, the varieties are not complete. To produce global examples, we just compactify them by Theorem 2.11. More concrete global examples can be found in the appendix [FS3].

APPENDIX: AN EXAMPLE OF TORIC FLOPS

OSAMU FUJINO AND HIROSHI SATO

We construct an example of global toric 3-dimensional terminal flops that has interesting properties. We freely use the notation and references in *Equivariant completions of toric contraction morphisms*.

EXAMPLE A.1 (Global toric 3-dimensional terminal flop). We have the following toric flopping diagram;

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ {\scriptstyle\varphi}\searrow & & \swarrow{\scriptstyle\varphi^+} \\ & W & \end{array}$$

such that

(1) X, X^+ and W are all projective toric 3-folds,

(2) $\rho(X/W) = \rho(X^+/W) = 1$, $\rho(X) = 4$, and $\rho(W) = 3$,

(3) K_X (resp. K_{X^+}) is Cartier and φ-numerically trivial (resp. φ^+-numerically trivial), where $\varphi : X \to W$ (resp. $\varphi^+ : X^+ \to W$) is a small toric morphism,

(4) X, X^+ and W have only terminal singularities, and

(5) $\mathrm{Exc}(\varphi) = \boldsymbol{P}^1 \amalg \boldsymbol{P}^1$ and $\mathrm{Exc}(\varphi^+) = \boldsymbol{P}^1 \amalg \boldsymbol{P}^1$.

More precisely,

(6) Both $\mathrm{Sing}X$ and $\mathrm{Sing}X^+$ are only one ordinary double point, where $\mathrm{Sing}X$ (resp. $\mathrm{Sing}X^+$) is the singular locus of X (resp. X^+). In particular, X and X^+ are not $\boldsymbol{Q}$-factorial.

(7) The flop $X \dashrightarrow X^+$ is the union of two *simplest flops*, where the simplest flop means the flop described in [Fl, p. 49–p. 50]. It is sometimes called *Atiyah's flop*. So, W has three ordinary double points.

(8) Let P be the ordinary double point on X. Then $P \cap \mathrm{Exc}(\varphi) = \emptyset$. Thus φ is an isomorphism around P. We put $X^0 := X \setminus P$ and $W^0 := W \setminus \varphi(P)$. Then X^0 is non-singular and $\rho(X^0/W^0) = 2$.

(9) The flop $X \dashrightarrow X^+$ factors as follows:

$$\begin{array}{ccccc} X & \dashrightarrow & Z & \dashrightarrow & X^+ \\ \searrow & & \swarrow\ \searrow & & \swarrow \\ & V_1 & & V_2 & \end{array}$$

Each step is the simplest flop. Every morphism is over W. We note that V_1, V_2 and Z are not projective over W. However, every variety is projective over W^0.

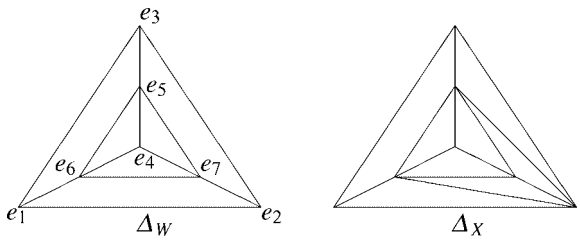

FIGURE 1.

CONSTRUCTION. We fix $N \simeq \boldsymbol{Z}^3$. Let e_1, e_2 and e_3 be the standard basis of $\boldsymbol{Z}^3$. We put

$$\begin{aligned} e_4 &= e_1 + e_2 + e_3 = (1, 1, 1)\,, \\ e_5 &= e_3 + e_4 = (1, 1, 2)\,, \\ e_6 &= e_1 + e_4 = (2, 1, 1)\,, \\ e_7 &= e_2 + e_4 = (1, 2, 1)\,. \end{aligned}$$

We consider the following fans:

$$\Delta_X = \left\{ \begin{array}{ccc} \langle e_1, e_3, e_5, e_6\rangle, & \langle e_2, e_3, e_5\rangle, & \langle e_1, e_2, e_6\rangle, \\ \langle e_2, e_6, e_7\rangle, & \langle e_2, e_5, e_7\rangle, & \langle e_4, e_5, e_6\rangle, \\ \langle e_4, e_6, e_7\rangle, & \langle e_4, e_5, e_7\rangle, & \langle -e_4, e_1, e_2\rangle, \\ \langle -e_4, e_1, e_3\rangle, & \langle -e_4, e_2, e_3\rangle, & \text{and their faces} \end{array} \right\},$$

and

$$\Delta_W = \left\{ \begin{array}{ccc} \langle e_1, e_2, e_6, e_7\rangle, & \langle e_1, e_3, e_5, e_6\rangle, & \langle e_2, e_3, e_5, e_7\rangle, \\ \langle e_4, e_5, e_6\rangle, & \langle e_4, e_6, e_7\rangle, & \langle e_4, e_5, e_7\rangle, \\ \langle -e_4, e_1, e_2\rangle, & \langle -e_4, e_1, e_3\rangle, & \langle -e_4, e_2, e_3\rangle, \\ \text{and their faces} & & \end{array} \right\}.$$

Figure 1 may help us to understand these fans. We put $X = X(\Delta_X)$ and $W = X(\Delta_W)$. To construct X^+, V_1, V_2, and Z and check the properties (1) to (9) are good exercises. The details were carried out in [FS2]. The reader can find many other examples in [FS2].

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REFERENCES

[AK] D. ABRAMOVICH AND K. KARU, Weak semistable reduction in characteristic 0, Invent. Math. 139 (2000), 241–273.

[EI] G. EWALD AND M. ISHIDA, Completion of real fans and Zariski-Riemann spaces, Tohoku Math. J. (2) 58 (2006), 189–218.

[F1] O. FUJINO, Applications of Kawamata's positivity theorem, Proc. Japan Acad. Ser. A Math. Sci. 75 (1999), 75–79.

[F2] O. FUJINO, Notes on toric varieties from Mori theoretic viewpoint, Tohoku Math. J. (2) 55 (2003), 551–564.

[F3] O. FUJINO, Special termination and reduction theorem, to be contained in the book "Flips for 3-folds and 4-folds"prepared by A. Corti et al.

[FS1] O. FUJINO AND H. SATO, Introduction to the toric Mori theory, Michigan Math. J. 52 (2004), 649–665.

[FS2] O. FUJINO AND H. SATO, An example of toric flops, preprint 2003.

[FS3] O. FUJINO AND H. SATO, An example of toric flops, Appendix to: this article.

[Fl] W. FULTON, Introduction to toric varieties, Ann. of Math. Stud. 131, The William H. Roever Lectures in Geometry, Princeton University Press, Princeton, N.J., 1993.

[KM] J. KOLLÁR AND S. MORI, Birational geometry of algebraic varieties, Cambridge Tracts in Math. 134, Cambridge Univ. Press, Cambridge, 1998.

[K^+] J. KOLLÁR et al., Flips and abundance for algebraic threefolds, Astérisque 211, Soc. Math. de. France, Paris, 1992.

[M] K. MATSUKI, Introduction to the Mori program, Universitext, Springer-Verlag, New York, 2002.

[P] YU. G. PROKHOROV, On extremal contractions from threefolds to surfaces: the case of one non-Gorenstein point, Birational algebraic geometry (Baltimore, MD, 1996), 119–141, Contemp. Math. 207, Amer. Math. Soc., Providence, R.I., 1997.

[R] M. REID, Decomposition of toric morphisms, Arithmetic and geometry, Vol. II, 395–418, Progr. Math. 36, Birkhäuser Boston, Boston, M.A., 1983.

[S] H. SATO, Combinatorial descriptions of toric extremal contractions, Nagoya Math. J. 180 (2005), 111–120.

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