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ISOMETRY GROUPS OF RIEMANNIAN MANIFOLDS WITH BOUNDED NORMS

SEONG-HUN PAENG

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Abstract. We generalize several results on the order of the isometry group of a compact manifold with negative Ricci curvature proved by Dai et al. under the assumption of bounded norm and an integral curvature bound. We also show that there exists a bound on the order of the isometry group depending on the weak norm of M.

1. Introduction. Let M be an n-dimensional compact Riemannian manifold. If the Ricci curvature Ric_M of M satisfies $\mathrm{Ric}_M < 0$, then the isometry group $\mathrm{Isom}(M)$ of M is finite by a classical theorem of Bochner (see [10]). Yamaguchi [12] found a bound on the order of isometry groups depending on the volume under the assumption of negative sectional curvatures. In [3], Dai et al. showed that if $\{M_j\}$ is a $C^{1,\alpha}$-convergent sequence of manifolds satisfying $-K \le \mathrm{Ric}_{M_j} \le -k < 0$, the injectivity radius $\mathrm{inj}_{M_j} \ge i_0$ and the volume $\mathrm{vol}(M_j) \le V$, then the order of isometry groups satisfies $\overline{\lim}_{j\to\infty}|\mathrm{Isom}(M_j)| \le |\mathrm{Isom}(M_0)| < \infty$, where $|S|$ denotes the cardinality of a set S. As a corollary, $|\mathrm{Isom}(M)| \le N(n, K, k, i_0, V)$ for a constant N depending only on n, K, k, i_0, V. In [11], Rong proved that a compact manifold with negative Ricci curvature admits no non-trivial invariant F-structure, which implies that if the sectional curvature $|K_M| \le 1$, $\mathrm{Ric}_M \le -K < 0$ and the diameter $\mathrm{diam}(M) \le d$, then $|\mathrm{Isom}(M)| \le N_1(n, K, d)$. Although this theorem has no assumption on the lower bound of the injectivity radius, a lower bound of it is obtained from the main theorem of [11].

Petersen introduced the (weak) norm of a manifold in [10]. The above condition that $-K \le \mathrm{Ric}_M \le -k < 0$ and $\mathrm{inj}_M \ge i_0$ gives a bound on the harmonic $C^{1,\alpha}$-norm. We recall briefly the definition of the (weak) norm of an n-dimensional Riemannian manifold (M, g) on scale $r > 0$, where g is the metric of M in [9, 10].

DEFINITION 1 [9]. The *$C^{k,\alpha}$-norm* of an n-dimensional Riemannian manifold (M, g) on scale $r > 0$, $||(M, g)||_{C^{k,\alpha},r}$, is defined to be the infimum of positive numbers Q such that there exist imbeddings

$$\Phi_\tau : B(0, r) \subset \boldsymbol{R}^n \to U_\tau \subset M$$

with images U_τ, $\tau \in I$ (an index set), with the following properties:

(1) $e^{-2Q}\delta \le \Phi_\tau^*(g) \le e^{2Q}\delta$, where δ is the Euclidean metric;

(2) every metric ball $B(p, re^{-Q}/10)$ for $p \in M$ lies in some U_τ;

(3) $r^{|l|+\alpha}||\partial^l g_{\tau,ij}||_{C^{0,\alpha}} \le Q$ for all multi-indices l with $0 \le |l| \le k$, where $g_\tau = \Phi_\tau^* g$.

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The *weak norm* $||(M,g)||^{\mathrm{W}}_{C^{k,\alpha},r}$ is defined in a similar way except that Φ_τ is assumed to be a local diffeomorphism instead of a diffeomorphism. Then we may regard Φ_τ as a *weak* coordinate. If the (weak) coordinate charts are harmonic, we call such a norm a (*weak*) *harmonic norm*. In [8], it is proved that if the weak $C^{0,\alpha}$-norm is bounded and the diameter is sufficiently small, then M is diffeomorphic to a nilmanifold up to finite cover. Instead of $C^{k,\alpha}$-norm, $L^{k,p}$-norm can be used.

For constants $k \geq 0$, $\alpha \in (0,1]$, $r > 0$ and a positive function $Q(r)$ with $\lim_{r\to 0} Q(r) = 0$, we define the following classes of n-dimensional complete Riemannian manifolds:

$$\begin{aligned} \mathcal{M}^{k,\alpha}(n,Q) &= \{(M,g) \mid \|(M,g)\|^{\mathrm{W}}_{C^{k,\alpha},r} \leq Q(r)\}\,, \\ \tilde{\mathcal{M}}^{k,\alpha}(n,Q) &= \{(M,g) \mid \|(M,g)\|_{C^{k,\alpha},r} \leq Q(r)\}\,, \\ \mathcal{M}^{k,p}_L(n,Q) &= \{(M,g) \mid \|(M,g)\|^{\mathrm{W}}_{L^{k,p},r} \leq Q(r)\}\,, \\ \tilde{\mathcal{M}}^{k,p}_L(n,Q) &= \{(M,g) \mid \|(M,g)\|_{L^{k,p},r} \leq Q(r)\}\,. \end{aligned} \tag{1}$$

Note that $\mathcal{M}^{k+1,p}_L(n,Q) \subset \mathcal{M}^{k,\alpha}(n,Q)$.

For given $Q, n, k, r > 0$, $\tilde{\mathcal{M}}^{k,\alpha}(n,Q)$ is compact in the pointed $C^{k,\alpha'}$-topology for any $\alpha' < \alpha$ (see [11]). Furthermore, Petersen et al. showed that if $M \in \mathcal{M}^{0,\alpha}(n,Q)$ for a harmonic weak coordinate chart, then the metric g of M can be deformed to a metric g' with $|K_{g'}| \leq K$, where $K_{g'}$ is the sectional curvature with respect to the metric g' and K depends on α, Q (see [9]). So the (weak) norm can be considered as a generalization of the curvature.

It is our question if we can generalize the above results on the isometry group of a negatively curved manifold under a bounded (weak) norm. It should be noted that it has not been known that manifolds in $\mathcal{M}^{1,\alpha}(n,Q)$ can be deformed to those with metrics of bounded sectional curvature without harmonicity. So we cannot use the arguments on F-structure in [12] and it is not known whether collapsing could occur.

We will prove the following theorems. We can easily generalize the result in [3] under an integral bound on Ricci curvature as follows: let $h(x)$ be the largest eigenvalue for the Ricci transformation $\mathrm{Ric} : T_xM \to T_xM$. Similarly as in [10], we consider

$$\bar{\mathcal{R}}_M(K) = \frac{1}{\mathrm{vol}(M)} \int_M \max\{h(x)+K, 0\}\, dv\,. \tag{2}$$

The classical Bochner theorem claims that if $\bar{\mathcal{R}}_M(K) = 0$ for some $K > 0$, then $|\mathrm{Isom}(M)| < \infty$ (see [11]). First, as a generalization of [3], we prove the following.

THEOREM 1. *For fixed* K, $p > 0$ *and a positive function* $Q(r)$ *with* $\lim_{r\to 0} Q(r) = 0$, *let* M_j *be a sequence of manifolds in* $\tilde{\mathcal{M}}^{2,p}_L(n,Q)$ *satisfying* $\mathrm{vol}(M_j) \leq V$. *If* M_j *converges to* M *in the* $L^{2,p}$*-topology and* $\bar{\mathcal{R}}_{M_j}(K) \to 0$, *then*:

(1) $|\mathrm{Isom}(M)| < \infty$; *and*
(2) $\overline{\lim}_{i\to\infty} |\mathrm{Isom}(M_j)| \leq |\mathrm{Isom}(M)|$.

COROLLARY 1. *For a given* $K > 0$ *and* $M \in \tilde{\mathcal{M}}^{2,p}_L(n,Q)$ *satisfying* $\mathrm{vol}(M) \leq V$, *there exist constants* $N(n,K,Q,p,V)$ *and* $\varepsilon(n,K,Q,p,V) > 0$ *depending only on*

n, K, Q, p, V such that if $\bar{\mathcal{R}}_M(K) < \varepsilon(n, K, Q, p, V)$, then the order of the isometry group is bounded by $N(n, K, Q, p, V)$.

Note that the class of manifolds satisfying $-k \leq \mathrm{Ric}_M \leq -K$ for some constants $k, K > 0$, $\mathrm{inj}_M \geq i_0$ and $\mathrm{vol}(M) \leq V$ is contained in $\tilde{\mathcal{M}}_L^{2,p}(n, Q)$ for some p, Q, r by [1]. Our proof is much simpler than that of [3]. Corollary 1 can be considered as an integral version of the Bochner theorem.

It is not known whether Theorem 1 can be generalized to the class $\mathcal{M}^{1,\alpha}(n, Q)$ (or $\mathcal{M}_L^{2,p}(n, Q)$) as collapsing could occur. However, our main result is to prove the corollary without non-collapsing conditions (e.g., lower bound on the injectivity radius) as follows.

THEOREM 2. *Let K, Q, α be positive numbers and $Q(r)$ be a positive function satisfying* $\lim_{r\to 0} Q(r) = 0$. *For $M \in \mathcal{M}^{1,\alpha}(n, Q)$ with* $\mathrm{diam}(M) \leq d$, *there exist constants $\varepsilon(n, K, Q, \alpha, d) > 0$ and $N(n, K, Q, \alpha, d)$ depending only on n, K, Q, α, d such that if $\bar{\mathcal{R}}(K) < \varepsilon(n, K, Q, \alpha, d)$, then the order of the isometry group is bounded by $N(n, K, Q, \alpha, d)$.*

It can be considered as a generalization of a result in [11]. In [11], Rong showed that a collapsing does not occur from the main result and used the theorem in [3]. However, in our case, we do not assume that collapsing does not occur, so that we cannot obtain the compactness theorem under a bound of the weak $C^{k,\alpha}$-norm.

2. Proof of Theorem 1. We prove the following theorem which has been proved in [3, Theorem 1.3] under the assumption of bounded Ricci curvature. Once we prove the theorem, the remaining part of the proof of Theorem 1 and Corollary 1 is the same as those in [3], which requires only $L^{2,p}$-convergence of M_j.

THEOREM 3. *Let M be an n-dimensional compact Riemannian manifold in $\tilde{\mathcal{M}}_L^{2,p}(n, Q)$ with* $\lim_{r\to 0} Q(r) = 0$ *and* $\mathrm{vol}(M) \leq V$. *Then there exists $\varepsilon(n, K, Q, p, V) > 0$ such that if $\bar{\mathcal{R}}_M(K) < \varepsilon(n, K, Q, p, V)$ and* $\max\{d(\phi(x), x) \mid x \in M\} \leq \varepsilon(n, K, Q, p, V)$ *for an isometry ϕ, then ϕ is the identity map.*

PROOF. Let $\{M_j \mid j = 1, 2, \ldots\}$ be a sequence of manifolds in $\tilde{\mathcal{M}}_L^{2,p}(n, Q)$ such that $\bar{\mathcal{R}}_{M_j} \to 0$, $\mathrm{vol}(M_j) \leq V$ and $\mathrm{diam}(M_j) \leq D$ for a positive constant $D > 0$ and any $j > 0$. Assume that M_j has an isometry ϕ_j such that $\max\{d(\phi_j(x), x) \mid x \in M_j\} = \varepsilon_j \to 0$. We denote by $(X, p)_g$ a manifold X pointed at p with a metric g. We choose $p_j \in M_j$ such that $d(\phi_j(p_j), p_j) \geq \varepsilon_j/2$. Rescaling the metric g_j of M_j by multiplying ε_j^{-2}, we have the following convergence for $p_j \in M_j$:

$$(M_j, p_j)_{\varepsilon_j^{-2} g_j} \to (\boldsymbol{R}^n, o)_\delta$$

in the $C^{1,\alpha}$-topology and $\phi_j \to \phi \in \mathrm{Isom}(\boldsymbol{R}^n)$ in the C^2-topology from the proof of [2], where $\phi(x) = Ax + b$ for $A \in O(n, \boldsymbol{R}^n)$ and $b \neq 0 \in \boldsymbol{R}^n$ and $d(\phi(x), x) \leq 1$ for all $x \in \boldsymbol{R}^n$. Then it follows that $A = I$ and ϕ_j is almost translational. As $(Rx + t)^k = R^k x + (R^{k-1} + \cdots + I)t$ on $\boldsymbol{R}^n$, if $\phi_{j_*} \to I$, then the order of ϕ_j satisfies that $|\phi_j| \to \infty$.

Considering a subsequence, we may assume that $(M_j, \langle\phi_j\rangle) \to (M_0, Z)$ with respect to the equivalent Hausdorff distance for some $C^{1,\alpha}$-manifold M_0 and an isometry group Z on M_0. There exists a diffeomorphism $F_j : M_0 \to M_j$ such that $F_j^* g_j \to g_0$ in the $C^{1,\alpha}$-topology. First, we show that Z has a non-trivial element. If Z has no non-trivial element, then the diameter of the orbit of ϕ_j $\mathrm{diam}(\{\phi_j^k(x) \mid k \in \boldsymbol{Z}\}) \to 0$ for every $x \in M_j$. Let ε_j be $\sup_{x\in M_j} \mathrm{diam}(\{\phi_j^k(x) \mid k \in \boldsymbol{Z}\})$ and we choose $p_j \in M_j$ such that $\mathrm{diam}(\{\phi_j^k(p_j) \mid k \in \boldsymbol{Z}\}) \geq \varepsilon_j/2$. Rescaling the metric g_j of M_j by multiplying ε_j^{-2}, we have the following convergence for $p_j \in M_j$ and $Z' \subset \mathrm{Isom}(\boldsymbol{R}^n)$:

$$(M_j, p_j, \langle\phi_j\rangle)_{\varepsilon_j^{-2}g_j} \to (\boldsymbol{R}^n, o, Z')_\delta\,,$$

where $\sup\{d(\phi(o), o) \mid \phi \in Z'\} \leq 1$. For the precise definition of pointed Hausdorff approximation, see [4]. It follows from the above arguments that ϕ_j^k converges to a translational isometry as $j \to \infty$ for all k, since $d(\phi_j^k(x), x) \leq \varepsilon_j$. However, this leads to a contradiction as follows: if $d(\phi_j^{k_0}(x), x) \geq 3\sup_k d(\phi_j^k(x), x)/4$, then $d(\phi_j^{2k_0}(x), x) > \sup_k d(\phi_j^k(x), x)$ for sufficiently large j, as $\phi_j^{k_0}$ is almost translational, which is a contradiction.

Now we show that $|Z| = \infty$. For $\gamma \in Z$, let $d(\gamma(p), p) = \beta > 0$ and $\phi_j^{k_j} \to \gamma$. Then there exists $l_j > 0$ such that $\beta/3 < d(\phi_j^{l_j}(p_j), p_j) < \beta/2$, where $p_j \to p$. Considering a subsequence of $\{\phi_j^{l_j}\}$, we can construct an isometry γ_1 with $\beta/3 < d(\gamma_1(p), p) < \beta/2$. Inductively, we can construct isometries γ_m for each positive integer m such that $\beta/(m+2) < d(\gamma_m(p), p) < \beta/(m+1)$. Hence, $|Z| = \infty$.

As $|Z| = \infty$, there exists a non-trivial Killing vector field X on M_0. Let A be $\max_{x\in M_0} \|X(x)\| > 0$. It is well known that if Y is a compact Riemannian manifold, then $\mathrm{Isom}(Y)$ is a Lie group and there exists a Killing vector field when $|\mathrm{Isom}(Y)| = \infty$ (see [7]). Let X_j be a vector field on M_j defined by $F_{j*}X$. Then by Bochner's formula (see [3]), we have

$$\int_{M_j} \{\mathrm{Ric}_{M_j}(X_j, X_j) + (1/2)|L_{X_j} g_j|^2 - |\nabla^{g_j} X_j|^2 - (\mathrm{div}^{g_j} X_j)^2\} dv = 0\,.$$

Note that M_j can be identified with M_0 by F_j as a space with different metrics. Then X_j and X also can be considered as the same vector field on the same space $M_0 = M_j$. In general, we have that

$$\begin{aligned} 2g(\nabla_X^g Y, Z) &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &\quad + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X)\,. \end{aligned} \tag{3}$$

As $F_j^* g_j \to g_0$ in the $C^{1,\alpha}$-topology and $F_{j*}X = X_j$ (i.e., $X_j = X$ if we identify M_j with M_0 by F_j), it follows from (3) that

$$\begin{aligned} (L_{X_j} g_j)(V, W) &= g_j(\nabla_V^{g_j} X_j, W) + g_j(\nabla_W^{g_j} X_j, V) \\ &\to g_0(\nabla_V^{g_0} X, W) + g_0(\nabla_W^{g_0} X, V) = (L_X g_0)(V, W) = 0 \end{aligned} \tag{4}$$

as $j \to \infty$. Let S_j be $\{x \mid h(x) + K > 0\}$. As

$$
\begin{aligned}
&\int_{M_j} \mathrm{Ric}_{M_j}(X_j, X_j)dv + \int_{M_j} K\|X_j\|^2 dv \\
&\quad \le \int_{M_j} (h(x) + K)\|X_j\|^2 dv \\
&\quad \le \int_{S_j} (h(x) + K)\|X_j\|^2 dv \le 2A^2 \bar{\mathcal{R}}_{M_j}(K)\mathrm{vol}(M_j) \to 0
\end{aligned} \tag{5}
$$

and

$$\int_{M_j} \{|\nabla^{g_j} X_j|^2 + (\mathrm{div}^{g_j} X_j)^2\} dv \ge 0$$

as $j \to \infty$, we obtain $X = 0$, which is a contradiction. □

3. Proof of Theorem 2. Under the condition in Theorem 2, we cannot use $C^{1,\alpha}$-compactness theorem, so that we cannot obtain an almost Killing vector field in the limit space with $C^{1,\alpha}$-metric directly. Hence, we first construct local Killing vector fields and then paste them together. We are going to prove Theorem 2 by contradiction. Assume that $\{M_j\}$ is a sequence of manifolds satisfying conditions in Theorem 2 and $\bar{\mathcal{R}}_{M_j}(K) \to 0$ but $|\mathrm{Isom}(M_j)| \to \infty$.

Fix a sufficiently small $r > 0$ such that an r-ball in $M \in \tilde{\mathcal{M}}_L^{2,p}(n, Q)$ can be considered as an almost flat structure [8] and $Q(r) < 1/100$. Let ϕ be an isometry on M such that $d(\phi(p), p) < r/10000$. Then ϕ can be lifted to an isometry $\tilde{\phi} : B(0, r/100)_{\Phi^*(g_j)} \to B(0, r/2)_{\Phi^*(g_j)}$ for a weak coordinate Φ with $\Phi(0) = p$, where $B(0, s)_{\Phi^*(g_j)}$ for $s < r$ is the set $\{(x_1, x_2, \ldots, x_n) \in \boldsymbol{R}^n \mid \sum_i x_i^2 = s^2\}$ with a metric $\Phi^*(g_j)$. A pseudogroup means a set Γ with a product $\alpha\beta \in \Gamma$ defined for some $\alpha, \beta \in \Gamma$ and the local fundamental pseudogroup is the set of geodesic loops based at a given point whose lengths are smaller than a positive constant. (For the precise definition of pseudogroup and local fundamental pseudogroup, see [5].) Let Γ_p be the local fundamental pseudogroup of geodesic loops in $B(p, r/100)$ based at p whose lengths are smaller than $r/100$. Then elements of Γ_p can be considered as isometric embeddings from $B(0, r/100)_{\Phi^*(g_j)}$ into $B(0, r/2)_{\Phi^*(g_j)}$.

We may assume that $M_j \to M_0$ for a compact length space M_0 with respect to the Gromov-Hausdorff distance. From the compactness of M_0, we can find a C_0-covering $\{B(p_k^j, r/100) \mid p_k^j \in M_j,\ k = 1, \ldots, C_0\}$ of M_j such that $(\Phi_k^j)^* g_j \to \boldsymbol{g}^k$ in the $C^{1,\alpha}$-norm, where $\Phi_k^j : B(0, r/100) \subset \boldsymbol{R}^n \to M_j$ are weak coordinates around p_k^j. For each k, we have the convergence

$$(B(0, r/100), \Gamma_{p_k^j})_{(\Phi_k^j)^* g_j} \to (\boldsymbol{B}_k, \boldsymbol{\Gamma}_k)_{\boldsymbol{g}^k},$$

and there exists a diffeomorphism $F_k^j : \boldsymbol{B}_k \to B(0, r/100)$. For convenience, we abbreviate $(\Phi_k^j)^* g_j$ to g_j. We may identify $\boldsymbol{B}_k$ with $B(0, r/100)$ as a space.

We prove Theorem 2 by a contradiction. Assume that there is a sequence of manifolds $\{M_j\}$ in $\mathcal{M}^{1,\alpha}(n, Q)$ such that there exists a sequence of isometries $\phi_j : M_j \to M_j$ with

$\max\{d(\phi_j(x), x) \mid x \in M_j\} \to 0$. Then we have $\tilde{\phi}_j \circ \gamma \circ \tilde{\phi}_j^{-1} \in \Gamma_p$ for $\gamma \in \Gamma_p$ and

$$d(\tilde{x}, \tilde{\phi}_j(\tilde{x})) \leq d(\tilde{x}, \tilde{p}) + d(\tilde{p}, \tilde{\phi}_j(\tilde{p})) + d(\tilde{\phi}_j(\tilde{p}), \tilde{\phi}_j(\tilde{x})) \leq d(\tilde{p}, \tilde{\phi}_j(\tilde{p})) + r/50$$

for all $\tilde{x} \in B(0, r/100)_{\Phi^*(g_j)}$.

For the proof of Theorem 2, we need the following lemma.

LEMMA 1. *If* $|\mathrm{Isom}(M_j)| \to \infty$, *then there exists a sequence of isometries* $\{\phi_j\}$ *such that* $(B(0, r/100), \langle \phi_j \rangle)_{(\Phi_k^j)^* g_j} \to (\boldsymbol{B}_k, \phi_t)_{\boldsymbol{g}^k}$ *for a one-parameter isometry* (*pseudo*)*subgroup* ϕ_t, *which yields a non-trivial Killing vector field* $X_k = d\phi_t/dt$ *on* $\boldsymbol{B}_k$ *for each* k.

In our case, even if there exists a sequence of isometries $\phi_j : M_j \to M_j$ such that $\max_x\{d(\phi_j(x), x)\} \to 0$, it may occur that $\max_x\{d(\tilde{\phi}_j(\tilde{x}), \tilde{x})\}$ does not converge to 0, i.e., rotational parts do not necessarily converge to I if $[\tilde{\phi}_j, \Gamma_{p_j}] \neq 0$, which is the main difference from the case of Theorem 1. (In the proof of Theorem 1, $\max\{d(\phi_j(x), x)\} \to 0$ implies that ϕ_j is almost translational.) We prove this lemma in Section 4. It follows from the proof of Lemma 1 that $\{\|X_k\|_{C^1} \mid k = 1, \ldots\}$ is bounded.

As $L_{X_k}\boldsymbol{g}^k = 0$, we have that $\boldsymbol{g}^k(\nabla_V X_k, W) + \boldsymbol{g}^k(\nabla_W X_k, V) = 0$. Furthermore, on each $\boldsymbol{B}_k$, every X_k can be obtained as a derivative of a one-parameter subgroup of isometries, which is the limit of one cyclic group of isometries on M_j. We identify $\boldsymbol{B}_k$ with $B(0, r/100)$ by the diffeomorphism F_k^j. As X_{k_i} are generated by one cyclic group of isometries on M_j, we have

$$(\Phi_{k_1}^j)_*(X_{k_1}(x_1)) = (\Phi_{k_2}^j)_*(X_{k_2}(x_2)) \tag{6}$$

for some x_1, x_2 such that $\Phi_{k_1}^j(x_1) = \Phi_{k_2}^j(x_2)$.

As $\tilde{\phi}_j \circ \gamma^j \circ \tilde{\phi}_j^{-1} \in \Gamma_{p_k^j}$ for $\gamma^j \in \Gamma_{p_k^j}$, we see that for $\gamma \in \boldsymbol{\Gamma}_k$

$$\gamma_* X_k = X_k\,. \tag{7}$$

As $\gamma^j \to \gamma$ in the C^2-topology for $\gamma^j \in \Gamma_{p_k^j}$ and $\gamma \in \boldsymbol{\Gamma}_k$ from [2], we have

$$\|(\Phi_k^j)_*(X_k(x)) - (\Phi_k^j)_*(X_k(\gamma^j x))\|_{C^1} \to 0 \tag{8}$$

for $\gamma^j \in \Gamma_{p_k^j}$ from (7). Now we construct a global smooth vector field $\mathcal{X}^j$ on M_j which is close to $(\Phi_k^j)_* X_k$ in the C^1-topology. Let A_k^j be the set of pseudogroup defined by $A_k^j = \{\gamma \in \Gamma_{p_k^j} \mid \gamma p_k^j \in B(p_k^j, r/100)\}$. Let $\boldsymbol{X}_k^j$ be a C^1-vector field on $\mathrm{Im}(\Phi_k^j)$ defined as follows:

$$\boldsymbol{X}_k^j(x) = \frac{1}{|A_k^j|} \sum_{\gamma \in A_k^j} (\Phi_k^j)_*(X_k(\gamma \tilde{x}))\,,$$

where $\tilde{x}$ is a point in $(\Phi_k^j)^{-1}(x)$. Then from (8), (3) and $g_j \to g$ in $C^{1,\alpha}$, it follows that

$$\begin{gathered}\|\nabla_V^{g_j} \boldsymbol{X}_k^j - \nabla_V^{g_j} (\Phi_k^j)_*(X_k)\|_{C^0} \to 0\,, \\ \|g_j(\nabla_V^{g_j} (\Phi_k^j)_*(X_k), W) - \boldsymbol{g}^k(\nabla_V^{\boldsymbol{g}^k} X_k, W)\|_{C^0} \to 0\end{gathered} \tag{9}$$

almost everywhere for smooth vector fields V and W on $\mathrm{Im}(\Phi_k^j)$ as $j \to \infty$. Similarly as in (4), we obtain that

$$L_{\boldsymbol{X}_k^j} g_j \to L_{X_k} \boldsymbol{g}^k = 0\,, \tag{10}$$

as $\boldsymbol{X}_k^j \to X_k$ and $g_j \to \boldsymbol{g}^k$ in the C^1-topology, identifying $\boldsymbol{B}_k$ and $\mathrm{Im}(\Phi_k^j)$ locally. Using a partition of unity, we can construct a globally continuous vector field $\bar{\mathcal{X}}^j$ from $\boldsymbol{X}_k^j$. Let $\psi : [0, r) \to [0, 1)$ be a decreasing smooth function such that $\psi(x) = 1$ on $[0, r/100-\varepsilon_0)$ and 0 for $x \geq r/100$ for a fixed $\varepsilon_0 < r/10000$. If $\psi_k(x) = \psi(d(p_k^j, x))$, then $\{\bar{\psi}_k \mid \bar{\psi}_k = \psi_k/(\sum_k \psi_k)\}$ is a partition of unity. Let $\bar{\mathcal{X}}^j$ be $\sum_k \bar{\psi}_k \boldsymbol{X}_k^j$. As $\nabla\bar{\psi}_k$ is uniformly bounded almost everywhere, $\lim_{j\to\infty} |A_k^j| = \infty$ and

$$\|\nabla_V^{g_j} \boldsymbol{X}_{k_1}^j - \nabla_V^{g_j} \boldsymbol{X}_{k_2}^j\|_{C^0} \to 0 \tag{11}$$

by (6) and (9), we have

$$\begin{aligned}
\|\nabla_V^{g_j} \bar{\mathcal{X}}^j - \nabla_V^{g_j} (\Phi_k^j)_*(X_k)\|_{C^0} &= \left\| \nabla_V^{g_j} \sum_k \bar{\psi}_k \boldsymbol{X}_k^j - \nabla_V^{g_j} (\Phi_k^j)_*(X_k) \right\|_{C^0} \\
&\leq \left\| d\bigg(\sum_k \bar{\psi}_k \bigg)(V) \boldsymbol{X}_{k_0}^j + \sum_k d\bar{\psi}_k(V)(\boldsymbol{X}_k^j - \boldsymbol{X}_{k_0}^j) \right\|_{C^0} \\
&\quad + \left\| \sum_k \bar{\psi}_k \nabla_V^{g_j} \boldsymbol{X}_k^j - \nabla_V^{g_j} (\Phi_k^j)_*(X_k) \right\|_{C^0} \to 0
\end{aligned} \tag{12}$$

almost everywhere as $j \to \infty$. Note that $\sum_k \bar{\psi}_k = 1$.

Now we follow the smoothing technique in [6]. Choose ρ_j with $0 < \rho_j < \mathrm{inj}_{M_j}$. Let $d\mu_x$ be the measure on $B(x, \rho_j)$ induced from the Lebesgue measure on $\{v \in T_x M_j \mid \|v\| < 2\rho_j\}$ by exp. We define the smoothing kernel $\Psi_{\rho_j} : M_j \times M_j \to \boldsymbol{R}$ by

$$\Psi_{\rho_j}(x, y) := \frac{\psi(\rho_j^{-1}\, d(x, y))}{\displaystyle\int_{B(x,\rho_j)} \psi(\rho_j^{-1}\, d(x, \cdot))\, d\mu_x}\,.$$

We denote by $P_x^y(V)$ the parallel translation of V from x to y along the minimal geodesic.

Let $\mathcal{X}^j$ be defined as follows:

$$\mathcal{X}^j(x) = \int_{B(x,\rho)} P_y^x(\bar{\mathcal{X}}^j(y)) \Psi_{\rho_j}(x, y) d\mu_x\,. \tag{13}$$

For $u \in T_x M_j$, let γ be the geodesic from x with $\gamma'(0) = u$. Also we let $\gamma_y(t) = \exp_{\gamma(t)}(P_x^{\gamma(t)}\ (\exp_{\gamma(0)}^{-1}(y)))$. Let U be the vector field defined as $U(y) = \gamma_y'(0)$. As $\exp_{\gamma(t)} \circ P_x^{\gamma(t)} \circ \exp_{\gamma(0)}^{-1}$ is measure preserving, we have

$$\mathcal{X}^j(\gamma(t)) = \int_{B(x,\rho)} P_{\gamma_y(t)}^{\gamma(t)}(\bar{\mathcal{X}}^j(\gamma_y(t)) \Psi_{\rho_j}(x, y) d\mu_x\,.$$

Note that

$$\begin{aligned}(14)\qquad &\|P^{\gamma(0)}_{\gamma(t)} \circ P^{\gamma(t)}_{\gamma_y(t)}(X) - P^{\gamma(0)}_{\gamma_y(0)}(X) - P^{\gamma(0)}_{\gamma_y(0)} \circ (P^{\gamma_y(0)}_{\gamma_y(t)}(X) - X)\| \\ &\quad = \|P^{\gamma(0)}_{\gamma(t)} \circ P^{\gamma(t)}_{\gamma_y(t)}(X) - P^{\gamma(0)}_{\gamma_y(0)} \circ P^{\gamma_y(0)}_{\gamma_y(t)}(X)\| \leq K_j t\rho_j \|X\|,\end{aligned}$$

where K_j depending on the sectional curvature K_{M_j} of M_j. Hence,

$$\begin{aligned}&\left\| \frac{d}{dt} P^{\gamma(t)}_{\gamma_y(t)}(\bar{\mathcal{X}}^j(\gamma_y(t)))\big|_{t=0} - P^{\gamma(0)}_{\gamma_y(0)}(\nabla^{g_j}_U \bar{\mathcal{X}}^j) \right\| \\ &\quad = \left\| \lim_{t\to 0} \frac{P^{\gamma(0)}_{\gamma(t)} P^{\gamma(t)}_{\gamma_y(t)}(\bar{\mathcal{X}}^j) - P^{\gamma(0)}_{\gamma_y(0)}(\bar{\mathcal{X}}^j)}{t} - \frac{P^{\gamma(0)}_{\gamma_y(0)} \circ (P^{\gamma_y(0)}_{\gamma_y(t)}(\bar{\mathcal{X}}^j) - \bar{\mathcal{X}}^j)}{t} \right\| \\ &\quad \leq K_j \rho_j \|\bar{\mathcal{X}}^j\|.\end{aligned}$$

Therefore, if $\rho_j K_j \to 0$, then

$$(15)\qquad \left\| \nabla^{g_j}_u \mathcal{X}^j - \int_{B(x,\rho_j)} P^x_y(\nabla^{g_j}_U \bar{\mathcal{X}}^j)\Psi_{\rho_j}(x,y)\, d\mu_x \right\| \leq K_j\rho_j \|\bar{\mathcal{X}}^j\| \to 0,$$

so that we choose very small ρ_j such that $\rho_j K_j \to 0$. From $(\Phi^j_k)^* g_j \to \boldsymbol{g}^k$ in the $C^{1,\alpha}$-topology and (12), (15), we have

$$\begin{aligned}(16)\qquad &\|\nabla^{g_j}_u \mathcal{X}^j - \nabla^{g_j}_u (\Phi^j_k)_*(X_k)\|_{C^0} \\ &\quad \leq \|\nabla^{g_j}_u \mathcal{X}^j - \nabla^{g_j}_u \bar{\mathcal{X}}^j\|_{C^0} + \|\nabla^{g_j}_u \bar{\mathcal{X}}^j - \nabla^{g_j}_u (\Phi^j_k)_*(X_k)\|_{C^0} \to 0.\end{aligned}$$

From (16), $L_{X_k}\boldsymbol{g}^k = 0$ and $(\Phi^j_k)^* g_j \to \boldsymbol{g}^k$,

$$\begin{aligned}(17)\qquad &|g_j(\nabla^{g_j}_V \mathcal{X}^j, W) + g_j(\nabla^{g_j}_W \mathcal{X}^j, V)| \\ &\quad \leq |\boldsymbol{g}^k(\nabla^{\boldsymbol{g}^k}_{\tilde{V}} X_k, \tilde{W}) + \boldsymbol{g}^k(\nabla^{\boldsymbol{g}^k}_{\tilde{W}} X_k, \tilde{V})| \\ &\qquad + |g_j(\nabla^{g_j}_V \mathcal{X}^j, W) - g_j(\nabla^{g_j}_V (\Phi^j_k)_*(X_k), W)| \\ &\qquad + |\boldsymbol{g}^k(\nabla^{\boldsymbol{g}^k}_V X_k, W) - g_j(\nabla^{g_j}_V (\Phi^j_k)_*(X_k), W)| \\ &\qquad + |g_j(\nabla^{g_j}_W \mathcal{X}^j, V) - g_j(\nabla^{g_j}_W (\Phi^j_k)_*(X_k), V)| \\ &\qquad + |\boldsymbol{g}^k(\nabla^{\boldsymbol{g}^k}_W X_k, V) - g_j(\nabla^{g_j}_W (\Phi^j_k)_*(X_k), V)| \to 0.\end{aligned}$$

As $j \to \infty$, we have

$$|L_{\mathcal{X}^j} g_j(V,W)| = |g_j(\nabla^{g_j}_V \mathcal{X}^j, W) + g_j(\nabla^{g_j}_W \mathcal{X}^j, V)| \to 0.$$

Now we use the integral version of Bochner's formula [3]: As $\mathcal{X}^j$ is a C^1-vector field on M_j,

$$\int_{M_j} \{\mathrm{Ric}_{M_j}(\mathcal{X}^j, \mathcal{X}^j) + (1/2)|L_{\mathcal{X}^j} g_j|^2 - |\nabla \mathcal{X}^j|^2 - (\mathrm{div}\mathcal{X}^j)^2\} dv = 0.$$

As $L_{\mathcal{X}^j} g_j \to 0$ and $\|\nabla^{g_j}\mathcal{X}^j\|$ is bounded by virtue of the boundedness of $\|\nabla^{\boldsymbol{g}^k} X_k\|$, we have that

$$
\begin{aligned}
&\frac{1}{\mathrm{vol}(M_j)}\int_{M_j}\{-\mathrm{Ric}_{M_j}(\mathcal{X}^j,\mathcal{X}^j)+|\nabla^{g_j}\mathcal{X}^j|^2+(\mathrm{div}\mathcal{X}^j)^2\}dv\\
&\quad=\frac{1}{\mathrm{vol}(M_j)}\int_{M_j}\{K\|\mathcal{X}^j\|^2+|\nabla^{g_j}\mathcal{X}^j|^2+(\mathrm{div}\mathcal{X}^j)^2\}dv\\
&\quad\quad-\frac{1}{\mathrm{vol}(M_j)}\int_{M_j}\mathrm{Ric}_{M_j}(\mathcal{X}^j,\mathcal{X}^j)+K\|\mathcal{X}^j\|^2dv\\
&\quad=\frac{1}{\mathrm{vol}(M_j)}\int_{M_j}\frac{1}{2}|L_{\mathcal{X}^j}g_j|^2dv\to 0\,.
\end{aligned}
\tag{18}
$$

From (5), $\int_{M_j}\mathrm{Ric}_{M_j}(\mathcal{X}^j,\mathcal{X}^j)+K\|\mathcal{X}^j\|^2dv/\mathrm{vol}(M_j)\le 2A^2\bar{\mathcal{R}}_{M_j}(K)\to 0$. For some coordinate Φ_k^j, $\mathrm{vol}(\mathrm{Im}(\Phi_k^j))\ge \mathrm{vol}(M_j)/C_0$. Taking a subsequence, we have

$$
\frac{1}{\mathrm{vol}(\mathrm{Im}(\Phi_k^j))}\int_{\mathrm{Im}(\Phi_k^j)}\|\mathcal{X}^j\|^2dv\to 0\,,
$$

which implies $\mathcal{X}^j\to 0$ almost everywhere in $\mathrm{Im}(\Phi_k^j)$. As $\|X_k-\mathcal{X}^j\|\to 0$, we obtain that $X_k=0$, which is a contradiction to Lemma 1. Hence, $|\mathrm{Isom}(M_j)|$ is bounded for any sequence $\{M_j\}$, which completes the proof of Theorem 2.

4. Proof of Lemma 1: existence of Killing vector fields. Assume that there exists a sequence of manifolds M_j such that $|\mathrm{Isom}(M_j)|\to\infty$. Let M_j converge to a compact length space M_0 in the Gromov-Hausdorff metric. We may assume that $\{p_k \mid k=1,\dots,N\}$ in M_0 satisfy that M_0 can be covered by $\bigcup_{i=1}^{N(\varepsilon)}B(p_k,\varepsilon)$ and $B(p_k,\varepsilon/2)$ are pairwise disjoint. Let $\{p_k^j \mid k=1,\dots,N(\varepsilon)\}\subset M_j$ converges to $\{p_k \mid k=1,\dots,N(\varepsilon)\}\in M_0$. We define $F(\phi)(i)$ as the smallest k such that $\phi(p_k^j)\in B(p_i^j,\varepsilon)$. Then F is a map from $\mathrm{Isom}(M_j)$ to $S(\varepsilon)^{S(\varepsilon)}=\{f \mid f: S(\varepsilon)\to S(\varepsilon)\}$, where $S(\varepsilon)=\{1,\dots,N(\varepsilon)\}$. As $|S(\varepsilon)^{S(\varepsilon)}|=N(\varepsilon)^{N(\varepsilon)}$ and $|\mathrm{Isom}(M_j)|\to\infty$, there exists $\phi\in\mathrm{Isom}(M_j)$ such that $\max_x\{d(\phi(x),x) \mid x\in M_j\}\le 10\varepsilon$. Furthermore, we obtain that

$$
|\{\phi \mid \max d(\phi(x),x)\le 10\varepsilon\}|\ge \mathrm{Isom}(M_j)/N(\varepsilon)^{N(\varepsilon)}\to\infty
\tag{19}
$$

as $j\to\infty$.

Let $F_j=\{\phi_{j,l} \mid l=1,\dots,n(j)\}$ be a set of isometries of M_j such that $\max_x\{d(\phi_{j,l}(x),x)\}\le\varepsilon_j$. From (19), we can find a sequence ε_j such that $\varepsilon_j\to 0$ and $|F_j|\to\infty$ as $j\to\infty$. If we lift the isometry $\phi_{j,l}\in\mathrm{Isom}(M_j)$ to the isometry $\tilde{\phi}_{j,l}$ on $B(0,r/100)_{(\Phi_k^j)^*g_j}$ and rescale the metric by multiplying ε_j^{-2} as the proof of Theorem 1, we have $\|\tilde{\phi}_{j,l}(x)-(b_{j,l}^k+A_{j,l}^k x)\|_{C^2}\to 0$ as $j\to\infty$ for $A_{j,l}^k\in O(n,\boldsymbol{R})$ by the same reason as in Theorem 1. As $O(n,\boldsymbol{R})^{C_0}=O(n,\boldsymbol{R})\times\cdots\times O(n,\boldsymbol{R})$ is compact, there exist s,t such that $\|A_{j,s}^k-A_{j,t}^k\|\to 0$ as $j\to\infty$ for all k, where C_0 is a number of coverings by $r/100$-ball in Section 3. Then

$|\tilde{\phi}_{j,s}^{-1}\tilde{\phi}_{j,t}| \to \infty$. Hence, there exists $\tilde{\phi}_j$ such that $|\tilde{\phi}_j| \to \infty$ and

$$\|\tilde{\phi}_j(x) - (A_j x + b_j)\|_{C^2} \to 0 \tag{20}$$

on $B(0,r)$ with $A_j \to I$, with respect to the rescaled metrics. We use the same arguments as in Theorem 1. We have the convergence

$$(B(0,r/100), \langle\tilde{\phi}_j\rangle)_{(\Phi_k^j)^* g_j} \to (B(0,r/100), Z)_{\boldsymbol{g}^k}$$

for isometry group Z. By the same reason as in Theorem 1, Z is non-trivial. For non-trivial isometry $\gamma \in Z$, we can construct isometries γ_m such that $d(\gamma(p),p)/(m+2) < d(\gamma_m(p),p) < d(\gamma(p),p)/(m+1)$. Hence, $|Z| = \infty$. Furthermore, there exists a sequence of $\{f_i \in Z\}$ such that f_i converges to a translational isometry in the C^2-topology by (20).

Now we show that there exists a Killing vector field on $B(p,\varepsilon_0/100)$ in a similar way as in [7]. First, we construct a one-parameter sub(pseudo)group of Z as follows. Let $f_i \in Z$ be a sequence such that f_i converge to the identity in the C^2-topology and $\varepsilon_i = \max\{d(f_i(x),x) \mid x \in B(0,r/100)_{(\Phi_k^j)^* g_j}\} \to 0$. If $h_i = [1/\varepsilon_i]$, then $f_i^{h(i)} \to f$ for an isometry f by considering a subsequence. By the same reason as in [7], $\lim_{i\to\infty} f_i^{[rh(i)]}$ exists for every $r \in [0,1]$. We denote $\lim_{i\to\infty} f_i^{[rh(i)]}$ by $f(r)$. Then we have $f(r_1+r_2) = f(r_1)f(r_2)$ if $r_1 + r_2 < r/2$, so that $f(r)$ is a one-parameter pseudo subgroup of Z. As $f(r)$ is an isometry for all r, $\|f(r)\|_{C^2} \le C(r)$ for a function C depending on r (see [2]). It yields a one-parameter (pseudo)subgroup on $B(0,r/2)_{(\Phi_k^j)^* g_j}$ by $f(t)x$. Then we have a Killing vector field V as follows:

$$V(x) = \frac{d}{dt}f(t)x = \lim_{h\to 0}\frac{f(t)\circ f(h)x - f(t)x}{h} = f(t)_* \lim_{h\to 0}\frac{f(h)x - x}{h}.$$

Taking a subsequence if necessary, $\lim_{t\to 0}(f(t)x - x)/t$ exists. Now we show that V is a C^1-vector field. As $f(r)$ is bounded in the C^2-topology,

$$\begin{aligned} \frac{d}{ds}V(\gamma(s)) &= \frac{d}{ds}\left(\frac{d}{dt}f(t)(\gamma(s))\right) = \frac{d}{dt}\left(\frac{d}{ds}f(t)(\gamma(s)) = \frac{d}{dt}(f(t)_*(w))\right) \\ &= \lim_{h\to 0}\frac{f(t+h)_*(w) - f(t)_*(w)}{h} \\ &= f(t)_*\left(\lim_{h\to 0}\frac{f(h)_*(w) - w}{h}\right) \end{aligned} \tag{21}$$

for $\gamma'(0) = w$. Hence, there exists a C^1-Killing vector field on $B(0,r/2)_{(\Phi_k^j)^* g_j}$.

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DEPARTMENT OF MATHEMATICS
KONKUK UNIVERSITY
1 HWAYANG-DONG, GWANGJIN-GU
SEOUL 143–701
KOREA

E-mail address: shpaeng@konkuk.ac.kr