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COMBINATORIAL DUALITY AND INTERSECTION PRODUCT: A DIRECT APPROACH

GOTTFRIED BARTHEL, JEAN-PAUL BRASSELET,
KARL-HEINZ FIESELER AND LUDGER KAUP

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Abstract. The proof of the Combinatorial Hard Lefschetz Theorem for the "virtual" intersection cohomology of a not necessarily rational polytopal fan as presented by Karu completely establishes Stanley's conjectures for the generalized h-vector of an arbitrary polytope. The main ingredients, Poincaré Duality and the Hard Lefschetz Theorem, rely on an intersection product. In its original constructions, given independently by Bressler and Lunts on the one hand, and by the authors of the present article on the other, there remained an apparent ambiguity. The recent solution of this problem by Bressler and Lunts uses the formalism of derived categories. The present article instead gives a straightforward approach to combinatorial duality and a natural intersection product, completely within the framework of elementary sheaf theory and commutative algebra, thus avoiding derived categories.

Introduction. In [St], Stanley introduced the generalized h-vector for arbitrary polytopes. For rational polytopes, this new combinatorial invariant agrees with the vector of even (middle perversity) intersection cohomology Betti numbers of a projective toric variety associated with the polytope and thus, it enjoys the same properties. Stanley proved that the Dehn-Sommerville equalities (i.e., Poincaré duality) remain valid in the general case, and he conjectured that non-negativity and unimodality also should continue to hold. In the rational case, the unimodality property follows from the "Hard Lefschetz Theorem" for the intersection cohomology of a projective variety.

This conjecture motivated the search for a purely combinatorial approach to the intersection cohomology of toric varieties that would allow to drop any rationality assumption. A suitable framework has been developed independently in [BBFK2] and by Bressler and Lunts in [BreLu1]. The common basic idea is to imitate the construction of the equivariant intersection cohomology sheaf and the transition to the "usual" intersection cohomology entirely in fan-theoretic terms: To view a (not necessarily rational) fan as a finite topological space with the subfans as non-trivial open sets, naturally endowed with a sheaf $\mathcal{A}$ of polynomial rings, and to study the properties of a certain sheaf of modules $\mathcal{E}$ on that "fan space" that agrees with

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the equivariant intersection cohomology sheaf for the associated toric variety in the rational case. This approach then yields a “virtual” intersection cohomology theory for the class of “quasi-convex” fans that includes all complete and hence, all polytopal fans.

At the time when these articles were written, a purely combinatorial version of the Hard Lefschetz Theorem, as stated in Section 4, was still lacking. This was the only missing piece to prove that the vector of even “virtual” intersection cohomology Betti numbers of a polytopal fan agrees with the generalized h-vector of the polytope, and thus, to fully establish Stanley’s conjecture. As another problem, in the construction of the intersection product on the virtual equivariant intersection cohomology sheaf, apparently non-canonical choices entered.

In the meantime, a proof of the combinatorial Hard Lefschetz Theorem has been presented by Karu in [Ka]. Since that result essentially relies on the Hodge-Riemann bilinear relations for the “primitive” (virtual) intersection cohomology, its proof is based on the study of the intersection product. The apparent ambiguity in the definition of that product, however, makes the argumentation quite involved, since one has to carefully keep track of the choices made.

A first simplified version has recently been presented by Bressler and Lunts in [BreLu2], using the framework of derived categories. In particular, they verify by a detailed analysis that none of the possible choices affects the definition of the pairing.

Our goal is to go one step further, namely, to give a short, direct, and elementary approach to duality and the intersection product in the “geometrical” spirit of [BBFK2], following ideas of [Bri], the only prerequisites being sheaf theory and commutative algebra. For the convenience of the reader, we give here a complete presentation, and we always give references to the corresponding statements in [BreLu1, 2].

Though we are mainly interested in the “virtual equivariant intersection cohomology” sheaf $\mathcal{E}$, it is appropriate to work in the class of “*pure sheaves*” as defined in Section 2. Within this class, we present a direct and explicit construction of the dual, noted $\mathcal{DF}$, of a sheaf $\mathcal{F}$ (see Section 3). (In [BreLu1, 2], the duality functor is defined as an endofunctor of a vast derived category containing the pure sheaves as invariant subcategory.) To give an idea, we summarize the most important results, using this notation, explained more systematically in Section 1: Let Δ be a quasi-convex fan in a vector space V of real dimension n with a fixed volume form, and $\partial\Delta$, its boundary fan. The global sections of the equivariant intersection cohomology sheaf $\mathcal{E}$ on Δ and on $(\Delta, \partial\Delta)$, respectively, are (graded) modules over the (graded) symmetric algebra $A := S(V^*)$ of polynomial functions on V.

DUALITY THEOREM. *The dual $\mathcal{DF}$ of a pure sheaf $\mathcal{F}$ is again pure, and on an oriented fan, there is a natural* biduality isomorphism

$$\mathcal{F} \xrightarrow{\cong} \mathcal{D}(\mathcal{DF})\,.$$

For a pair $(\mathcal{F}, \mathcal{G})$ of mutually dual pure sheaves on an oriented quasi-convex fan Δ, there is a global duality relation

$$\mathcal{F}(\Delta) \cong \operatorname{Hom}(\mathcal{G}(\Delta, \partial\Delta), A[-2n])$$

and hence, a pairing

$$\mathcal{F}(\Delta) \times \mathcal{G}(\Delta, \partial\Delta) \to A[-2n]$$

on the level of global sections.

Here the "relative" sections are those vanishing on the boundary subfan, they replace the sections with compact support in ordinary sheaf theory.

The crucial step in the construction of $\mathcal{DF}$ is the definition of the restriction homomorphisms from a cone to a facet; the image of a section can be interpreted as a kind of "residue" along the facet. While keeping every single step easily accessible, the structure of the proof of Poincaré duality and the naturality of the intersection product is the 'classical' one, as in [BreLu1, 2].

An intersection product on the (graded) sheaf $\mathcal{E}$ corresponds to a sheaf homomorphism $\vartheta : \mathcal{E} \to \mathcal{DE}$ of degree zero; in fact, we check that $\mathcal{E}$ is self-dual in a natural way.

The main applications, namely, the "Poincaré Duality Theorem" and the "Compatibility Theorem" (cf. [BreLu2, 3.16 and 7.2]) as stated below, fit into the inductive proof of the Hard Lefschetz Theorem (HLT) as given in [Ka] (see also [BreLu2]): Assuming HLT for polytopal fans in dimension $d < n$, the "Poincaré Duality Theorem" yields a natural intersection product on every fan in dimension n. In [Ka], it is shown that the Hodge-Riemann bilinear relations (HRR) for simplicial fans in any dimension — which hold by [Mc] — together with HRR for arbitrary fans in dimensions $d < n$ imply HRR and thus, HLT in dimension n. In that induction step, it is most helpful to work with a canonical pairing, and to apply the "Compatibility Theorem".

"EQUIVARIANT" POINCARÉ DUALITY THEOREM. *In the above situation, let us assume that the Hard Lefschetz Theorem holds in all dimensions below n. Then there is a natural intersection product*

$$\mathcal{E}(\Delta) \times \mathcal{E}(\Delta, \partial\Delta) \to A[-2n]\,, \tag{1}$$

giving rise to a dual pairing of finitely generated free A-modules.

For the following supplement, let $\hat{\mathcal{E}}$ be the equivariant intersection cohomology sheaf of a refinement $\hat{\Delta}$ of Δ with refinement map $\iota\colon \hat{\Delta} \to \Delta$.

COMPATIBILITY THEOREM. *Let $\mathcal{E} \to \iota_*(\hat{\mathcal{E}})$ be a homomorphism of graded sheaves extending the identity $\mathcal{E}_o = \boldsymbol{R} = \iota_*(\hat{\mathcal{E}})_o$ at the zero cone o. Then the "global" intersection products are compatible, i.e., the following diagram is commutative*:

$$\begin{tikzcd} \mathcal{E}(\Delta) \times \mathcal{E}(\Delta, \partial\Delta) \arrow[rr] \arrow[dr] & & \hat{\mathcal{E}}(\hat{\Delta}) \times \hat{\mathcal{E}}(\hat{\Delta}, \partial\hat{\Delta}) \arrow[dl] \\ & A[-2n]\,. & \end{tikzcd} \tag{2}$$

The present article is a complete version of the results announced in [Fi]. We want to thank Tom Braden for useful comments and remarks.

1. Preliminaries. For the convenience of the reader, we recall some basic notions, notation and constructions to be used in the sequel.

1.1. Basic algebra. Let V be a real vector space of dimension n, and $A := S(V^*)$, the symmetric algebra on the dual vector space V^*, i.e., the algebra of real valued polynomial functions on V. We endow A with the even grading given by $A^2 = V^*$, a convention motivated by equivariant cohomology, and we let $\mathfrak{m} := A^{>0}$ be the homogeneous maximal ideal of A. For a graded A-module M, its reduction modulo $\mathfrak{m}$,

$$\bar{M} := (A/\mathfrak{m}) \otimes_A M ,$$

is a graded real vector space.

For a strictly convex polyhedral cone $\sigma \subset V$, we let $V_\sigma \subset V$ denote its linear span. In analogy to the definition of A, we consider the graded algebra

$$A_\sigma := S(V_\sigma^*) .$$

We usually identify its elements with polynomial functions on the cone σ.

To avoid cumbersome notation, we admit graded homomorphisms even if they are not of degree zero.

1.2. Fan topology and sheaves. Motivated by the coarse "toric topology" on a toric variety given by torus-invariant open sets, we consider a fan Δ (which need not be rational) in V as a finite topological space with the subfans as open subsets. The "affine" fans

$$\langle\sigma\rangle := \{\sigma\} \cup \partial\sigma \preceq \Delta \quad \text{with boundary fan} \quad \partial\sigma := \{\tau \in \Delta; \tau \precneqq \sigma\}$$

form a basis of the fan topology by open sets that cannot be covered by smaller ones. Here $\preceq$ means that a cone is a face of another cone or that a set of cones is a subfan of some other fan; furthermore, $\tau \prec_1 \sigma$ means that τ is a facet of the cone σ.

Sheaf theory on that "fan space" is particularly simple, since a presheaf given on the basis already "is" a sheaf. In particular, for a sheaf $\mathcal{F}$ on Δ, the equality

$$\mathcal{F}(\langle\sigma\rangle) = \mathcal{F}_\sigma$$

of the set of sections on the affine fan $\langle\sigma\rangle$ and the stalk at the point σ holds.

Furthermore, a sheaf $\mathcal{F}$ on Δ is flabby if and only if each restriction homomorphism $\varrho^\sigma_{\partial\sigma} : \mathcal{F}(\langle\sigma\rangle) \to \mathcal{F}(\partial\sigma)$ is surjective.

In particular, we consider (sheaves of) $\mathcal{A}$-modules, where $\mathcal{A}$ is the *structure sheaf* of Δ, i.e., the graded sheaf of polynomial algebras determined by $\mathcal{A}(\langle\sigma\rangle) := A_\sigma$, the restriction homomorphism $\varrho^\sigma_\tau : A_\sigma \to A_\tau$ being the restriction of functions on σ to the face $\tau \preceq \sigma$. The set of sections $\mathcal{A}(\Lambda)$ on a subfan $\Lambda \preceq \Delta$ constitutes the algebra of conewise polynomial functions on the support $|\Lambda|$ in a natural way.

Given a homomorphism $\varphi : \mathcal{F} \to \mathcal{F}'$ of sheaves on Δ and a subfan Λ, we often write

$$F_\Lambda := \mathcal{F}(\Lambda) , \quad F_\sigma := \mathcal{F}(\langle\sigma\rangle) \quad \text{and} \quad \varphi_\Lambda : F_\Lambda \to F'_\Lambda .$$

Similarly, for a pair of subfans (Λ, Λ_0) with $\Lambda_0 \preceq \Lambda$, we define

$$F_{(\Lambda,\Lambda_0)} := \ker(\varrho^\Lambda_{\Lambda_0} : F_\Lambda \to F_{\Lambda_0}) ,$$

the submodule of sections on Λ vanishing on Λ_0. In particular, for a purely n-dimensional subfan Λ, we consider the case where Λ_0 is the *boundary fan* $\partial\Lambda$, i.e., the subfan generated by those $(n-1)$-cones which are a facet of exactly one n-cone in Λ. The sections vanishing on $\partial\Lambda$ could be regarded as an analogue of "sections with compact support".

1.3. Sheaf and Fan Constructions. Let $f : V \to W$ be a linear map inducing a morphism of fans between a fan Δ in V and a fan Λ in W, i.e., it maps each cone of Δ into a cone of Λ. Let $\mathcal{A}$ and $\mathcal{B}$ denote the corresponding sheaves of conewise polynomial functions, and let $\mathcal{F}$ on Δ and $\mathcal{G}$ on Λ be sheaves of graded $\mathcal{A}$- or $\mathcal{B}$-modules, respectively. For cones $\sigma \in \Delta$ and $\gamma \in \Lambda$ with $f(\sigma) \subset \gamma$, there is an induced homomorphism $B_\gamma \to A_\sigma$ and thus, the structure of a B_γ-module on F_σ.

We are especially interested in the following constructions:

(i) The *direct image* $f_*(\mathcal{F})$ on Λ is the $\mathcal{B}$-module sheaf defined by

$$f_*(\mathcal{F})_\gamma := F_{f^{-1}(\gamma)} \quad \text{with} \quad f^{-1}(\gamma) := \{\sigma \in \Delta\, ;\ f(\sigma) \subset \gamma\} \preceq \Delta\,.$$

The direct image of a flabby sheaf is again flabby.

(ii) The *inverse image* $f^*(\mathcal{G})$ on Δ is the $\mathcal{A}$-module sheaf determined by

$$f^*(\mathcal{G})_\sigma := A_\sigma \otimes_{B_\gamma} G_\gamma \quad \text{for } \sigma \in \Delta \text{ and the minimal } \gamma \in \Lambda \text{ with } f(\sigma) \subset \gamma\,.$$

(iii) For $W = V$, $f = \mathrm{id}_V$ and $(\hat{\Delta}, \Delta)$ with a *subdivision* $\hat{\Delta}$ of Δ instead of (Δ, Λ), we obtain thus a *refinement morphism*

$$\iota\colon \hat{\Delta} \to \Delta\,, \quad \sigma \mapsto \gamma\,.$$

In particular, we will consider the case of an affine fan $\langle\sigma\rangle$ given by an n-dimensional cone, and its stellar subdivision

$$\hat{\sigma} := \partial\sigma + \lambda := \partial\sigma \cup \{\tau + \lambda\, ;\ \tau \in \partial\sigma\}$$

with respect to a ray $\lambda := \ell \cap \sigma$, where ℓ is a one-dimensional linear subspace passing through the interior of σ.

(iv) For a cone $\sigma \in \Delta$, its closure in the fan topology is the star

$$\Delta_{\succeq\sigma} := \{\gamma \in \Delta; \gamma \succeq \sigma\}\,.$$

In general, this is not a fan. The collection

$$\Delta/\sigma := \{\pi(\gamma); \gamma \in \Delta_{\succeq\sigma}\}$$

of the image cones with respect to the projection $\pi : V \to W := V/V_\sigma$, however, is a fan, called the *transversal fan* of σ with respect to Δ. The induced map $\pi_\sigma : \Delta_{\succeq\sigma} \to \Delta/\sigma$ is a homeomorphism.

(v) Applying (iv) to the case of $\hat{\sigma}$ from (iii), the projection $\pi : V \to W := V/\ell$ maps the boundary fan $\partial\sigma$ homeomorphically onto the "*flattened boundary fan*"

$$\Lambda_\sigma := \hat{\sigma}/\lambda \cong \partial\sigma$$

in W. In that situation, choosing a linear form $T \in A^2$ with $T|_\lambda > 0$, we obtain isomorphisms $\ker(T) \cong W$ and thus $A \cong B[T]$, where we identify $B := S(W^*)$ with the subalgebra

$\pi^*(B) \subset A$. Moreover, for a sheaf $\mathcal{F}$ on $\langle\sigma\rangle$ and its image $\mathcal{G} := \pi_*(\mathcal{F}|_{\partial\sigma})$ on Λ_σ, there is a natural isomorphism of B-modules

$$G_{\Lambda_\sigma} \cong F_{\partial\sigma} . \tag{3}$$

We use notation such as $\Delta^d := \{\gamma \in \Delta; \dim \gamma = d\}$, $\Delta^{\leq d}$, etc. The fan Δ in V is said to be:

(a) *oriented* if for each cone $\sigma \in \Delta$, an orientation or_σ of V_σ is fixed in such a way that orientations for full-dimensional cones coincide,

(b) *purely n-dimensional* if each maximal cone of Δ lies in Δ^n,

(c) *irreducible* if it is not the union of two proper subfans with intersection included in $\Delta^{\leq n-2}$,

(d) *normal* if it is purely n-dimensional and for each cone $\sigma \in \Delta$, the transversal subfan Δ/σ is irreducible in V/V_σ , or, equivalently, if the support $|\Delta|$ is a normal pseudomanifold with or without boundary,

(e) *quasi-convex* if it is purely n-dimensional and the support $|\partial\Delta|$ of its boundary subfan is a real homology manifold. Note that a quasi-convex fan is normal, but not vice versa.

2. Pure sheaves on a fan. We recall the definition of the class of "pure" sheaves on a fan space that plays a key role in the sequel.

DEFINITION 2.1. A *pure sheaf* on a fan Δ is a flabby sheaf $\mathcal{F}$ of graded $\mathcal{A}$-modules such that, for each cone $\sigma \in \Delta$, the A_σ-module $F_\sigma = \mathcal{F}(\langle\sigma\rangle)$ is finitely generated and free.

We collect some useful facts about these sheaves, proved in [BBFK2] and [BreLu1] (where the class of these sheaves is denoted $\mathfrak{M}$. We note in passing that on a non-simplicial fan, pure sheaves are not locally free.)

Pure sheaves are built up from simple objects that correspond to the cones of the fan, or, equivalently, to the stalks of the structure sheaf $\mathcal{A}$. Up to a shift, such a simple sheaf is obtained from the corresponding stalk by the minimal "pure" extension process described below.

SIMPLE PURE SHEAVES: For each cone $\sigma \in \Delta$, the corresponding simple sheaf $\mathcal{L} =: {}_\sigma\mathcal{L}$ is the "minimal" pure sheaf supported on the star $\Delta_{\succeq\sigma}$ and with stalk $L_\sigma = A_\sigma$. It is constructed as follows: On the subfan $\Delta \setminus \Delta_{\succeq\sigma}$, we set $\mathcal{L} := 0$. By induction on the dimension, we extend it to the cones in $\Delta_{\succeq\sigma}$, starting with $L_\sigma := A_\sigma$. For a cone $\gamma \succeq \sigma$, we may assume that $L_{\partial\gamma}$ has been defined, and then set

$$L_\gamma := A_\gamma \otimes_{\boldsymbol{R}} \bar{L}_{\partial\gamma} .$$

The restriction homomorphism $\varrho^\gamma_{\partial\gamma}$ is defined by the following commutative diagram

$$\begin{array}{ccc} L_\gamma := A_\gamma \otimes_{\boldsymbol{R}} \bar{L}_{\partial\gamma} & \longrightarrow & \bar{L}_{\partial\gamma} \\ \big\downarrow \varrho := \mathrm{id} \otimes s & \swarrow & \| \\ L_{\partial\gamma} = A_\gamma \otimes_{A_\gamma} L_{\partial\gamma} & \longrightarrow & \bar{L}_{\partial\gamma} , \end{array}$$

where the diagonal arrow $s\colon \bar{L}_{\partial\gamma} \to L_{\partial\gamma}$ is an $\boldsymbol{R}$-linear section of the reduction map in the bottom row.

REMARK 2.2. 1. For each cone $\sigma \in \Delta$, the corresponding simple sheaf $\mathcal{L} = {}_\sigma\mathcal{L}$ is pure; it is characterized by the following properties:

(a) $\bar{L}_\sigma \cong \boldsymbol{R}$,

(b) for each cone $\tau \neq \sigma$, the reduced restriction homomorphism $\bar{L}_\tau \to \bar{L}_{\partial\tau}$ is an isomorphism.

In particular, the property (b) implies the vanishing of ${}_\sigma\mathcal{L}$ outside of the star of σ.

2. For the zero cone o, the "generic point" of Δ, the corresponding simple sheaf

$$\mathcal{E} := {}_\Delta\mathcal{E} := {}_o\mathcal{L}$$

is called the (*virtual*) *equivariant intersection cohomology sheaf* (or the minimal extension sheaf) of Δ. For a quasi-convex fan Δ, we may define its (virtual) *intersection cohomology* as

$$IH(\Delta) := \bar{E}_\Delta\,. \tag{4}$$

3. By extending scalars, each "local" sheaf ${}_\sigma\mathcal{L}$ is derived from the "global" sheaf ${}_{\Delta/\sigma}\mathcal{E}$ of the corresponding transversal fan: As in 1 (iv), we let $\pi_\sigma := \pi|_{\Delta_{\geq\sigma}} : \Delta_{\geq\sigma} \to \Delta/\sigma$ denote the homeomorphism induced from the projection $V \to V/V_\sigma$. The inverse image $\pi_\sigma^*({}_{\Delta/\sigma}\mathcal{E})$ is a flabby sheaf of graded $\mathcal{A}$-modules on the closed subset $\Delta_{\geq\sigma}$ of Δ. Its trivial extension to the whole fan space Δ then yields the sheaf ${}_\sigma\mathcal{L}$.

The following elementary decomposition theorem has been proved in [BBFK2, 2.4] and in [BreLu1, 5.3]:

THEOREM 2.3 (Decomposition theorem). *Every pure sheaf $\mathcal{F}$ on Δ admits a* (*non-canonical*) *direct sum decomposition of $\mathcal{A}$-modules*

$$\mathcal{F} \cong \bigoplus_{\sigma\in\Delta} ({}_\sigma\mathcal{L} \otimes_{\boldsymbol{R}} K_\sigma)$$

with $K_\sigma := K_\sigma(\mathcal{F}) := \ker(\bar{\varrho}^\sigma_{\partial\sigma} : \bar{F}_\sigma \to \bar{F}_{\partial\sigma})$, a finite dimensional graded vector space.

For a proof of the following special case, we refer to [BBFK2, 2.5].

EXAMPLE. Let $\iota\colon (V, \hat{\Delta}) \to (V, \Delta)$ be a refinement morphism. Then $\iota_*(\hat{\mathcal{E}})$ is a pure sheaf. Its decomposition is of the form

$$\iota_*(\hat{\mathcal{E}}) \cong \mathcal{E} \oplus \bigoplus_{\sigma\in\Delta^{\geq 2}} ({}_\sigma\mathcal{L} \otimes_{\boldsymbol{R}} K_\sigma)\,,$$

where the "correction terms" are "of higher order": they are supported on the closed subset $\Delta^{\geq 2}$, and the corresponding vector spaces K_σ are (strictly) positively graded.

REMARK 2.4. For a pure sheaf $\mathcal{F}$ on the boundary fan $\partial\sigma$ of an n-dimensional cone and the projection mapping $\pi : (V, \partial\sigma) \to (W, \Lambda_\sigma)$ corresponding to a ray λ as in 1.3, (iii), the direct image $\pi_*(\mathcal{F}|_{\partial\sigma})$ is a pure sheaf on Λ_σ.

3. The dual of a pure sheaf. In this section, the symbol $\mathcal{F}$ always denotes a pure sheaf on an oriented fan Δ. Furthermore, unless otherwise stated, the symbol Hom is understood to mean Hom_A, and $\otimes$ means $\otimes_{\boldsymbol{R}}$. Moreover, for a cone $\sigma \in \Delta$, we consider $\det V_\sigma^* := \bigwedge^{\dim\sigma} V_\sigma^*$ as a graded vector space concentrated in degree $d = 2\dim\sigma$, with the convention $\det V_o^* = \boldsymbol{R}$.

To $\mathcal{F}$, we associate its dual $\mathcal{DF}$ and show the following properties: The dual is again a pure sheaf on Δ, and for normal Δ, the module of sections $(\mathcal{DF})_\Delta$ is the dual of the module $F_{(\Delta,\partial\Delta)}$ of sections 'with compact supports' of $\mathcal{F}$.

3.1. Construction of the dual sheaf. To construct the dual $\mathcal{DF}$ of the pure sheaf $\mathcal{F}$ on Δ, we recall that it suffices to define its sections over affine fans — this will be done in such a way that duality holds by definition — and to specify the restriction morphisms between them.

SECTIONS OVER A CONE $\sigma \in \Delta$. As A_σ-module, we define $(\mathcal{DF})_\sigma = \mathcal{DF}(\langle\sigma\rangle)$ by

$$(\mathcal{DF})_\sigma := \mathrm{Hom}(F_{(\sigma,\partial\sigma)}, A_\sigma) \otimes \det V_\sigma^* . \tag{5}$$

RESTRICTION HOMOMORPHISMS. For a face $\tau \preceq \sigma$, the homomorphism ϱ_τ^σ is constructed in two steps: In the first step, we deal with the case of a facet; in the second step, we extend this recursively to the general situation of a face of arbitrary codimension.

To that end, we need *transition coefficients* $\varepsilon_\tau^\sigma = \pm 1$ for the facets τ of σ. They are defined in the following way: For $d := \dim\sigma$, the inclusion $V_\tau \subset V_\sigma$ induces a natural map $\kappa : \bigwedge^{d-1} V_\sigma^* \to \bigwedge^{d-1} V_\tau^* = \det V_\tau^*$. We choose a linear form h on V_σ^* with $V_\tau = \ker(h)$ and $h|_\sigma \geq 0$. Every element of $\det V_\sigma^*$ decomposes in the form $h \wedge \eta$ with unique image $\kappa(\eta)$. We thus obtain a homomorphism

$$\psi_h : \det V_\sigma^* \;\to\; \det V_\tau^* , \qquad h \wedge \eta \;\mapsto\; \kappa(\eta) . \tag{6}$$

With the volume forms $\omega_\sigma \in \det V_\sigma^*$ and $\omega_\tau \in \det V_\tau^*$ defining the respective orientations, we now set

$$\varepsilon_\tau^\sigma := \mathrm{sign}(\lambda) \quad \text{with} \quad \lambda := \psi_h(\omega_\sigma)/\omega_\tau \in \boldsymbol{R}_{\neq 0}$$

(i.e., λ given by $\psi_h(\omega_\sigma) = \lambda \cdot \omega_\tau$).

Step 1: *Restriction homomorphism for a facet* $\tau \prec_1 \sigma$.
Using again the linear form $h \in V_\sigma^*$, we are going to define another homomorphism

$$\varphi_h : \mathrm{Hom}(F_{(\sigma,\partial\sigma)}, A_\sigma) \to \mathrm{Hom}(F_{(\tau,\partial\tau)}, A_\tau)$$

and see that

$$\varphi_{\lambda h} = \lambda\varphi_h \quad \text{and} \quad \psi_{\lambda h} = \lambda^{-1}\psi_h$$

for every non-zero scalar $\lambda \in \boldsymbol{R}$. Thus the homomorphism

$$\varphi_h \otimes \psi_h \; : \mathrm{Hom}(F_{(\sigma,\partial\sigma)}, A_\sigma) \otimes \det V_\sigma^* \to \mathrm{Hom}(F_{(\tau,\partial\tau)}, A_\tau) \otimes \det V_\tau^*$$

does not depend on the special choice of h, so we may set

$$\varrho_\tau^\sigma := \varepsilon_\tau^\sigma \cdot \varphi_h \otimes \psi_h . \tag{7}$$

The map φ_h associates to a homomorphism $f : F_{(\sigma,\partial\sigma)} \to A_\sigma$ the homomorphism $\varphi_h(f) : F_{(\tau,\partial\tau)} \to A_\tau$, which acts in the following way: We first extend a section $s \in F_{(\tau,\partial\tau)}$ trivially to $\partial\sigma$ and then to a section $\check{s} \in F_\sigma$; we thus have $h\check{s} \in F_{(\sigma,\partial\sigma)}$ and may finally set

$$\varphi_h(f)(s) := f(h\check{s})|_\tau \in A_\tau \,. \tag{8}$$

In order to see that this definition is independent of the particular choice of $\check{s}$, we present an alternative description, following the argument of [BBFK2, p. 36]: We use three exact sequences, starting with

$$0 \to F_{(\sigma,\partial\sigma)} \to F_\sigma \to F_{\partial\sigma} \to 0\,.$$

The second one is composed of the multiplication with h and the projection onto the cokernel:

$$0 \longrightarrow A_\sigma \xrightarrow{\mu_h} A_\sigma \longrightarrow A_\tau \longrightarrow 0\,.$$

Eventually the subfan $\partial_\tau\sigma := \partial\sigma \setminus \{\tau\}$ of $\partial\sigma$ gives rise to the exact sequence

$$0 \to F_{(\tau,\partial\tau)} \to F_{\partial\sigma} \to F_{\partial_\tau\sigma} \to 0\,.$$

The associated Hom-sequences provide a diagram

$$\begin{tikzcd}
 & & \mathrm{Ext}(F_{\partial_\tau\sigma}, A_\sigma) \arrow[d] & \\
\mathrm{Hom}(F_\sigma, A_\sigma) \arrow[r] & \mathrm{Hom}(F_{(\sigma,\partial\sigma)}, A_\sigma) \arrow[r, "\alpha"] & \mathrm{Ext}(F_{\partial\sigma}, A_\sigma) \arrow[d, "\beta"] & \\
\mathrm{Hom}(F_{(\tau,\partial\tau)}, A_\sigma) \arrow[r] & \mathrm{Hom}(F_{(\tau,\partial\tau)}, A_\tau) \arrow[r, "\gamma"] & \mathrm{Ext}(F_{(\tau,\partial\tau)}, A_\sigma) \arrow[r] & \mathrm{Ext}(F_{(\tau,\partial\tau)}, A_\sigma)
\end{tikzcd}$$

with $\mathrm{Ext} = \mathrm{Ext}^1_A$. We show that γ is an isomorphism; we then have

$$\varphi_h := \gamma^{-1} \circ \beta \circ \alpha\,.$$

In fact, the rightmost arrow in the bottom row is the zero homomorphism, since it is induced by multiplication with h, which annihilates $F_{(\tau,\partial\tau)}$. On the other hand, the A_τ-module $F_{(\tau,\partial\tau)}$ is a torsion module over A_σ, so that $\mathrm{Hom}(F_{(\tau,\partial\tau)}, A_\sigma)$ vanishes.

Step 2: *Restriction homomorphism for faces of higher codimension.*

For a face $\tau \prec \sigma$ of codimension $r \geq 2$, we choose a "flag"

$$\tau_0 := \tau \prec_1 \tau_1 \prec_1 \cdots \prec_1 \tau_r := \sigma$$

of relative facets joining τ and σ. Defining the restriction homomorphism ϱ_τ^σ as the composite of the $\varrho_{\tau_i}^{\tau_{i+1}}$, we have to show that the result does not depend on the particular choice of the flag. This is easy to see in the case $r = 2$: For two flags $\gamma \prec_1 \tau \prec_1 \sigma$ and $\gamma \prec_1 \tau' \prec_1 \sigma$ and $h, h' \in V_\sigma^*$ as above, we set $g := h|_{V_{\tau'}}$ and $g' := h'|_{V_\tau}$, and then find

$$\varphi_{g'} \circ \varphi_h = \varphi_g \circ \varphi_{h'} \quad \text{and} \quad \psi_{g'} \circ \psi_h = -\psi_g \circ \psi_{h'}\,,$$

whence

$$\varrho_\gamma^\tau \circ \varrho_\tau^\sigma = \varrho_\gamma^{\tau'} \circ \varrho_{\tau'}^\sigma\,. \tag{9}$$

Thus, for general r, it suffices to verify that every two such flags can be transformed into each other in such a way that in each step, only one intermediate cone is replaced by another one. We proceed by induction on the codimension r.

To prove that claim, we may assume $\tau = o$ (otherwise, we replace Δ with Δ/τ) and $\Delta = \langle\sigma\rangle$. We want to compare the given flag (τ_j) joining $\tau = o$ and σ with a second one, say $(\tilde{\tau}_j)$. There is a chain of rays $\varrho_1 := \tau_1, \ldots, \varrho_s := \tilde{\tau}_1$ such that the two-dimensional cones $\varrho_i + \varrho_{i+1}$ belong to Δ. We now proceed by a second induction on that number s. For $s = 1$, we may pass to the fan Δ/τ_1 and use the first induction hypothesis for $r-1$. For the induction step, it evidently suffices to consider the case $s = 2$. Choosing any auxiliary flag of the form $o \prec_1 \tau_1 \prec_1 \tau_1 + \tilde{\tau}_1 \prec_1 \cdots \prec_1 \sigma$, the case $s = 1$ yields its equivalence with the start flag. On the other hand, by (9), the auxiliary flag is equivalent to the one obtained by interchanging τ_1 and $\tilde{\tau}_1$, and this in turn is equivalent to the "twiddled" flag.

3.2. Global sections. We now show that formula (5) defining duality for cones actually extends to normal subfans. To that end, we need the following preparatory results:

LEMMA 3.1. (i) *For an arbitrary fan Δ, there is a natural isomorphism*

$$\Theta : \bigoplus_{\sigma \in \Delta^n} (\mathcal{D}\mathcal{F})_\sigma \xrightarrow{\cong} \operatorname{Hom}(F_{(\Delta, \Delta^{\le n-1})}, A) \otimes \det V^* . \tag{10}$$

(ii) *If Δ is purely n-dimensional, the A-modules*

$$F_{(\Delta, \Delta^{\le n-1})} \subset F_{(\Delta, \partial\Delta)} \subset F_\Delta$$

are torsion-free and of the same rank. As a consequence, the restriction homomorphisms

$$\operatorname{Hom}(F_\Delta, A) \to \operatorname{Hom}(F_{(\Delta, \partial\Delta)}, A) \to \operatorname{Hom}(F_{(\Delta, \Delta^{\le n-1})}, A)$$

for the dual modules are injective.

(iii) *In the setup of cellular* ("Čech") *cochains and cocycles as presented in Section* 3 *of* [BBFK2], *for an arbitrary sheaf $\mathcal{G}$ on a purely n-dimensional fan Δ, there is an isomorphism*

$$Z^0(\Delta; \mathcal{G}) = G_{(\Delta, \partial\Delta)} , \tag{11}$$

and for a normal fan, we also have an isomorphism

$$Z^0(\Delta, \partial\Delta; \mathcal{G}) = G_\Delta . \tag{12}$$

PROOF. (i) For each n-dimensional cone σ, the equality $\det V_\sigma^* = \det V^*$ clearly holds. Hence, the isomorphism Θ in question is immediately obtained from the defining equality (5) for $(\mathcal{D}\mathcal{F})_\sigma$ by applying the additive functor $\operatorname{Hom}(\_, A) \otimes V^*$ to the obvious direct sum decomposition

$$F_{(\Delta, \Delta^{\le n-1})} \cong \bigoplus_{\sigma \in \Delta^n} F_{(\sigma, \partial\sigma)} .$$

(ii) For the special case $\mathcal{F} = \mathcal{E}$, this has been proved in [BBFK2, 6.1, i)]. The proof clearly carries over to arbitrary pure sheaves.

(iii) We recall that the submodule

$$Z^0(\Delta, \partial\Delta; \mathcal{G}) \subset C^0(\Delta, \partial\Delta; \mathcal{G}) = C^0(\Delta; \mathcal{G}) = \bigoplus_{\sigma\in\Delta^n} G_\sigma$$

of degree zero cocycles $g = (g_\sigma)$ relative to $\partial\Delta$ consists of those cochains that satisfy $g_\sigma|_\tau = g_{\sigma'}|_\tau$ whenever $\tau \in \Delta^{n-1}$ is an "inner" facet, i.e., a common facet of two n-cones $\sigma, \sigma' \in \Delta$, whereas for the submodule

$$Z^0(\Delta; \mathcal{G}) \subset Z^0(\Delta, \partial\Delta; \mathcal{G})$$

of absolute cocycles, we have to require in addition that the restriction of g_σ to each "outer" facet $\tau \in \partial\Delta^{n-1}$ vanishes.

For (11) and (12), we note that in both cases, the right hand side is always contained in the left hand side. In order to see the reverse inclusion, we have to show for a cochain $g = (g_\sigma) \in C^0(\Delta; \mathcal{G})$ that $g_\sigma|_\gamma = g_{\sigma'}|_\gamma$ holds whenever $\gamma \in \Delta$ is a common face of two n-cones σ and σ'. Since in the first case it suffices to consider cones $\gamma \notin \partial\Delta$ (then the fan Δ/γ is even complete!) and in the second Δ is assumed to be normal, the cones σ and σ' can be joined by a sequence of n-cones intersecting successively in common facets containing γ. It thus suffices to consider the case that γ is a facet, where the statement is obvious. □

THEOREM 3.2. *For a normal oriented fan Δ, the natural isomorphism Θ of* (10) *induces isomorphisms*

$$(\mathcal{DF})_\Delta \xrightarrow{\cong} \operatorname{Hom}(F_{(\Delta,\partial\Delta)}, A) \otimes \det V^* \tag{13}$$

and

$$(\mathcal{DF})_{(\Delta,\partial\Delta)} \xrightarrow{\cong} \operatorname{Hom}(F_\Delta, A) \otimes \det V^* . \tag{14}$$

By part (iii) of the preceding lemma it is sufficient to prove the following reformulation:

RESTATEMENT. *For any* 0*-cochain $\psi = (\psi_\sigma) \in \bigoplus_{\sigma\in\Delta^n} \mathcal{DF}_\sigma$ and its image $\Theta(\psi)$ in* $\operatorname{Hom}(F_{(\Delta,\Delta^{\le n-1})}, A) \otimes \det V^*$, *the equivalences*

$$\psi \in Z^0(\Delta, \partial\Delta; \mathcal{DF}) \Longleftrightarrow \Theta(\psi) \in \operatorname{Hom}(F_{(\Delta,\partial\Delta)}, A) \otimes \det V^* \tag{15}$$

and

$$\psi \in Z^0(\Delta; \mathcal{DF}) \Longleftrightarrow \Theta(\psi) \in \operatorname{Hom}(F_\Delta, A) \otimes \det V^* \tag{16}$$

hold.

PROOF. We choose an auxiliary function $h = \prod_{i=1}^s h_i \in A^{2s}$ as the lowest degree product of linear forms h_i that vanishes on $\bigcup_{\tau\in\Delta^{n-1}} V_\tau$; so each V_τ is the kernel V_i of some h_i. After fixing a positive volume form on V and thus, an isomorphism $\boldsymbol{R} \xrightarrow{\cong} \det V^*$, the homomorphisms ψ_{h_i} of 3, (6) provide isomorphisms $\boldsymbol{R} \cong \det V^* \cong \det V_i^*$. We may thus drop the determinant factors on the right hand side, and for each cone $\gamma \in \Delta^{\ge n-1}$, we may replace $(\mathcal{DF})_\gamma$ with $\operatorname{Hom}(F_{(\gamma,\partial\gamma)}, A_\gamma)$ and the restriction maps with $\pm\varphi_{h_i}$. Using the obvious inclusions

$$hF_{(\Delta,\partial\Delta)} \subset hF_{\Delta} \subset F_{(\Delta,\Delta^{\le n-1})}$$

of torsion-free A-modules, the right hand sides of (15) and (16) are equivalent to the inclusions

$$\chi(hF_{(\Delta,\partial\Delta)}) \subset hA \quad \text{and} \quad \chi(hF_{\Delta}) \subset hA\,, \quad \text{where} \;\; \chi := \Theta(\psi)\,.$$

Proof of the implications "$\Rightarrow$" in (15) and (16): It suffices to show that for a pertinent 0-cocycle ψ, the divisibility relation $h_i \mid \chi(hf)$ holds for each index i and for an arbitrary section f in $F_{(\Delta,\partial\Delta)}$ or F_{Δ}, respectively.

With $f_\sigma := f|_\sigma \in F_\sigma$ for an n-cone σ, we write

$$\chi(hf) = \sum_{\sigma\in\Delta^n} \psi_\sigma(hf_\sigma) \in A\,.$$

For each index $i = 1, \dots, s$, we introduce the function $g_i := h/h_i \in A^{2s-2}$. For the implication in (15), we consider a relative 0-cocycle $\psi = (\psi_\sigma) \in Z^0(\Delta, \partial\Delta; \mathcal{DF})$ and a section $f \in F_{(\Delta,\partial\Delta)}$ 'with compact support'. If $\sigma \cap V_i$ is not a facet of σ or if it belongs to $\partial\Delta$, then $g_i f_\sigma$ lies in $F_{(\sigma,\partial\sigma)}$ and thus $\psi_\sigma(hf_\sigma) = h_i\psi_\sigma(g_i f_\sigma) \in h_i A$ holds. Otherwise, there is precisely one n-cone $\sigma' \neq \sigma$ such that $\tau := \sigma \cap V_i$ is a common facet of both, σ and σ'. We now verify that h_i divides the sum $\psi_\sigma(hf_\sigma) + \psi_{\sigma'}(hf_{\sigma'})$ or, equivalently, that

$$\psi_\sigma(hf_\sigma)\,|_{V_i} = -\psi_{\sigma'}(hf_{\sigma'})\,|_{V_i}$$

holds. Using the extension $g_i f_\sigma \in F_{(\sigma,\partial_\tau\sigma)}$ of $(g_i f)|_\tau \in F_{(\tau,\partial\tau)}$ in the formula (8), we obtain

$$\psi_\sigma(hf_\sigma)|_{V_i} = (\varphi_{h_i}(\psi_\sigma))((g_i f_\sigma)|_\tau)\,.$$

By the relative cocycle condition, ψ_σ and $\psi_{\sigma'}$ restrict to the same section in $(\mathcal{DF})_\tau$. According to the choice of the transition coefficients ε^σ_τ in the definition of the restriction homomorphism ϱ^σ_τ in 3, (7), that yields

$$\varphi_{h_i}(\psi_\sigma) = -\varphi_{h_i}(\psi_{\sigma'})\,,$$

which implies our claim. If $\psi \in Z^0(\Delta; \mathcal{DF})$ is an absolute cocycle and $f \in F_\Delta$, the argument is as above, only in the case where $\tau := \sigma \cap V_i$ is an "outer" facet of σ (i.e., contained in $\partial\Delta$), one has to use the fact that $\psi_\sigma|_\tau = 0$.

Proof of the implications "$\Leftarrow$": We now assume that the homomorphism $\chi = \Theta(\psi) : F_{(\Delta,\Delta^{\le n-1})} \to A$ can be extended to the larger modules $F_{(\Delta,\partial\Delta)}$ or F_Δ, respectively (cf. parts (i, ii) in the preceding lemma). We have to verify the pertinent cocycle condition for $\psi = (\psi_\sigma)$, namely, the equality $\psi_\sigma|_\tau = \psi_{\sigma'}|_\tau$ whenever τ is a common "inner" facet of two n-cones $\sigma, \sigma' \in \Delta$, and, in the second ("absolute") case, the additional vanishing $\psi_\sigma|_\tau = 0$ if τ is an "outer" facet of σ.

For an "inner" facet $\tau = \sigma \cap \sigma'$, let i be the index with $\ker(h_i) = V_\tau$. We fix an arbitrary section $f_0 \in F_{(\tau,\partial\tau)}$ and, as for (8), extend it to sections $f \in F_\sigma$, $f' \in F_{\sigma'}$ vanishing on all the remaining facets of σ and of σ', respectively. Patching them together and extending by 0 yields a section $f_1 \in F_{(\Delta,\partial\Delta)}$. Then the equation

$$h_i\chi(f_1) = \chi(h_i f_1) = \chi(h_i f + h_i f') = \psi_\sigma(h_i f) + \psi_{\sigma'}(h_i f')\,,$$

after restriction to τ, yields

$$0 = (h_i\chi(f_1))\big|_\tau = \varphi_{h_i}(\psi_\sigma)(f_0) + \varphi_{h_i}(\psi_{\sigma'})(f_0) = (\psi_\sigma|_\tau - \psi_{\sigma'}|_\tau)(f_0)\,.$$

Finally, we leave it to the reader to consider the remaining case where χ can be extended to the largest module F_Δ, and $\tau \in \partial\Delta$ is an "outer" facet. □

3.3. Purity. We now show that duality preserves purity.

THEOREM 3.3. *The dual sheaf $\mathcal{DF}$ of a pure sheaf $\mathcal{F}$ is pure.*

PROOF. As in Corollary 4.12 in [BBFK2], the A_σ-module $F_{(\sigma,\partial\sigma)}$ is free of finite rank, thus also its dual $(\mathcal{DF})_\sigma$; hence, all we have to verify is flabbyness: For each cone $\sigma \in \Delta$, the restriction homomorphism

$$\varrho^\sigma_{\partial\sigma} : (\mathcal{DF})_\sigma \to (\mathcal{DF})_{\partial\sigma}$$

is surjective.

To that end, we first interpret $(\mathcal{DF})_{\partial\sigma}$. We may assume $\dim\sigma = n$ and use the setup of 1 (iv). For $\mathcal{G} := \pi_*(\mathcal{F}|_{\partial\sigma})$, as in 1, (3), there is a natural isomorphism

$$(\mathcal{DG})_{\Lambda_\sigma} \;\cong\; (\mathcal{DF})_{\partial\sigma}$$

of B-modules, while for the complete fan Λ_σ in W, Theorem 3.2 yields

$$(\mathcal{DG})_{\Lambda_\sigma} \;\cong\; \mathrm{Hom}_B(G_{\Lambda_\sigma}, B) \otimes \det W^*\,.$$

Using the isomorphism (3), we thus obtain a chain of isomorphisms

$$(\mathcal{DF})_{\partial\sigma} \cong (\mathcal{DG})_{\Lambda_\sigma} \cong \mathrm{Hom}_B(G_{\Lambda_\sigma}, B) \cong \mathrm{Hom}_B(F_{\partial\sigma}, B)\,.$$

Eventually, using these isomorphisms, a section $\beta \in (\mathcal{DF})_{\partial\sigma}$ may be interpreted as an element of $\mathrm{Hom}_B(F_{\partial\sigma}, B)$.

To proceed with the proof, we introduce the sheaf $\mathcal{H} := \pi^*(\mathcal{G})$ on $\hat\sigma$. There are isomorphisms

$$H_{\hat\sigma} \cong A \otimes_B G_{\Lambda_\sigma} \cong A \otimes_B F_{\partial\sigma} \quad \text{and} \quad H_{\partial\hat\sigma} \cong F_{\partial\sigma} \tag{17}$$

and a "Thom isomorphism"

$$\mu_g : H_{\hat\sigma} \xrightarrow{\cong} gH_{\hat\sigma} = H_{(\hat\sigma,\partial\hat\sigma)}\,, \quad h \mapsto gh \tag{18}$$

with a conewise linear function $g \in A^2_{(\hat\sigma,\partial\hat\sigma)}$, unique up to a non-zero scalar multiple, that is constructed conewise as follows: We fix a nontrivial linear form $f \in A^2_\lambda$. For a facet $\tau \prec_1 \sigma$, let $g_\tau \in A^2$ be the unique linear form with $\ker(g_\tau) = V_\tau$ and $g_\tau|_\lambda = f$. Then we set $g|_{\tau+\lambda} := g_\tau$.

For each facet τ of σ, the function g_τ induces an isomorphism

$$\det V^* \cong \det V_\tau^* , \quad g_\tau \wedge \eta \mapsto \eta|_{V_\tau} .$$

Hence, the composed isomorphism $\det V^* \cong \det V_\tau^* \cong \det W^*$ is independent of τ. We thus may drop the determinant factors.

We want to show that an inverse image $\alpha \in (\mathcal{DF})_\sigma = \mathrm{Hom}(F_{(\sigma,\partial\sigma)}, A)$ of $\beta \in (\mathcal{DF})_{\partial\sigma}$ with respect to $\varrho^\sigma_{\partial\sigma}$ is given by the composite

$$F_{(\sigma,\partial\sigma)} \xrightarrow{i} H_{(\hat\sigma,\partial\hat\sigma)} \xrightarrow{\mu_{1/g}} H_{\hat\sigma} \cong A \otimes_B F_{\partial\sigma} \xrightarrow{\mathrm{id}_A \otimes \beta} A \otimes_B B = A$$

where $\mu_{1/g}$ is the isomorphism "division by g" corresponding to (18), and the homomorphism i is constructed as follows: Since F_σ is a free A-module and the restriction homomorphism $H_{\hat\sigma} \to H_{\partial\hat\sigma}$ is surjective, cf. (17), the operator $\varrho^\sigma_{\partial\sigma}$ for the sheaf $\mathcal{F}$ admits a factorization of the form

$$F_\sigma \xrightarrow{j} H_{\hat\sigma} \longrightarrow H_{\partial\hat\sigma} \cong F_{\partial\sigma} .$$

Since $j(F_{(\sigma,\partial\sigma)})$ clearly is contained in $H_{(\hat\sigma,\partial\hat\sigma)}$, we may choose $i := j|_{F_{(\sigma,\partial\sigma)}}$.

To prove the equality $\alpha|_{\partial\sigma} = \beta$, it still remains to show that $\alpha|_\tau = \beta|_\tau$ for all facets $\tau \prec_1 \sigma$. Here we identify the (naturally isomorphic) algebras B and A_τ.

We fix an arbitrary section $s \in F_{(\tau,\partial\tau)} \subset F_{\partial\sigma}$, where the inclusion is given by trivial extension. Using the isomorphisms (17) and (18), any further extension $\check s$ of s to a section of $\mathcal{F}$ on the whole cone σ, regarded as section in $H_{\hat\sigma} \supset F_\sigma$, can be written in the form

$$\check s = 1 \otimes s + gd \in H_{\hat\sigma} \cong A \otimes_B F_{\partial\sigma}$$

with some correction term $d \in H_{\hat\sigma}$. Recalling the formula (8) in the definition of the homomorphism ϱ^σ_τ for $\mathcal{DF}$, we have to show that the restriction of the polynomial function $\alpha(g_\tau \cdot \check s) \in A_\sigma$ to τ coincides with $\beta(s)$. To that end, we note that $g_\tau \cdot (1 \otimes s) = g \cdot (1 \otimes s)$ holds, since the support of $1 \otimes s \in H_{\hat\sigma}$ is contained in $\tau + \lambda$. So we eventually have the equality

$$\alpha(g_\tau \cdot \check s)|_\tau = (\mathrm{id}_A \otimes \beta)(1 \otimes s + g_\tau d)|_\tau = (\mathrm{id}_A \otimes \beta)(1 \otimes s)|_\tau ,$$

and thus $\varrho^\sigma_\tau(\alpha)$ maps s to $\beta(s)$. □

3.4. Biduality. In order to see that the dual sheaf $\mathcal{DF}$ of a simple pure sheaf again is simple, we need biduality:

THEOREM 3.4 (Biduality theorem). (cf. [BreLu1, 6.23]) *Every pure sheaf on an oriented fan is reflexive*: *For such a sheaf $\mathcal{F}$, there exists a natural isomorphism*

$$\mathcal{F} \xrightarrow{\cong} \mathcal{D}(\mathcal{DF}) .$$

PROOF. Over a cone $\sigma \in \Delta$, the biduality isomorphism $F_\sigma \to \mathcal{D}\mathcal{D}\mathcal{F}_\sigma$ is obtained by these isomorphisms:

$$
\begin{aligned}
(\mathcal{D}\mathcal{D}\mathcal{F})_\sigma &= \mathrm{Hom}((\mathcal{D}\mathcal{F})_{(\sigma,\partial\sigma)}, A_\sigma) \otimes \det V_\sigma^* \\
&\cong \mathrm{Hom}(\mathrm{Hom}(F_\sigma, A_\sigma) \otimes \det V_\sigma^*, A_\sigma) \otimes \det V_\sigma^* \\
&\cong \mathrm{Hom}(\mathrm{Hom}(F_\sigma, A_\sigma) \otimes \det V_\sigma^*, A_\sigma \otimes \det V_\sigma^*) \\
&\cong \mathrm{Hom}(\mathrm{Hom}(F_\sigma, A_\sigma), A_\sigma)\,,
\end{aligned}
\tag{19}
$$

where the first isomorphism follows from Theorem 3.2 applied to the fan $\Delta := \langle\sigma\rangle$ in the vector space V_σ. The free A_σ-module F_σ is reflexive, so it can be naturally identified with its bidual, i.e., the fourth module in (19). Since this conewise construction is natural, it carries over to the sheaves. □

COROLLARY 3.5. (cf. [BreLu1, 6.26]) *For each cone $\sigma \in \Delta$, the simple pure sheaf ${}_\sigma\mathcal{L}$ satisfies*

$$\mathcal{D}({}_\sigma\mathcal{L}) \cong {}_\sigma\mathcal{L} \otimes \det V_\sigma^*\,.$$

In particular, the equivariant intersection cohomology sheaf $\mathcal{E}$ is self-dual with an isomorphism

$$\vartheta : \mathcal{E} \xrightarrow{\cong} \mathcal{D}\mathcal{E} \tag{20}$$

of degree zero.

PROOF. Clearly, by biduality, $\mathcal{D}\mathcal{F} = 0$ implies $\mathcal{F} = 0$. On the other hand, duality respects direct sum decompositions. Since the bidual $\mathcal{D}(\mathcal{D}({}_\sigma\mathcal{L})) \cong {}_\sigma\mathcal{L}$ is simple, the Decomposition Theorem 2.3 implies that the dual $\mathcal{D}({}_\sigma\mathcal{L})$ must be a simple sheaf. For a pure sheaf $\mathcal{F}$ and a cone $\sigma \in \Delta$, the A_σ-module $F_{\partial\sigma}$ is a torsion module, whence $F_\sigma = 0$ if and only if $F_{(\sigma,\partial\sigma)} = 0$. Hence a pure sheaf and its dual have the same support, so $\mathcal{D}({}_\sigma\mathcal{L})$ and ${}_\sigma\mathcal{L}$ agree up to a shift. To determine it explicitly, we use the equality ${}_\sigma L_{(\sigma,\partial\sigma)} = {}_\sigma L_\sigma = A_\sigma$, which yields $\mathcal{D}({}_\sigma\mathcal{L})_\sigma \cong \mathrm{Hom}(A_\sigma, A_\sigma) \otimes \det V_\sigma^* \cong A_\sigma \otimes \det V_\sigma^*$. □

4. The intersection product. In order to make precise the naturality of the intersection product, we introduce the following notion:

DEFINITION 4.1. *A duality correlation* on Δ is a sheaf homomorphism

$$\varphi : \mathcal{E} \to \mathcal{D}\mathcal{E}$$

of degree 0 from the equivariant intersection cohomology sheaf to its dual extending the natural identification

$$\mathcal{E}_o = \boldsymbol{R} \xrightarrow{1 \mapsto 1^*} \boldsymbol{R}^* = \mathcal{D}\mathcal{E}_o\,.$$

After multiplication with an appropriate scalar factor if necessary, any isomorphism $\mathcal{E} \to \mathcal{D}\mathcal{E}$ provides such a duality correlation. The first and main aim of this section is to prove the following result:

THEOREM 4.2. *On every fan Δ, there is a* unique duality correlation. *It defines a self duality $\mathcal{E} \cong \mathcal{DE}$ for the equivariant intersection cohomology sheaf $\mathcal{E}$.*

Existence has already been shown in Corollary 3.5; uniqueness follows from Proposition 4.5, applied to $\mathcal{F} = \mathcal{DE}$. But let us first state the global version of duality, which is needed later on in the proof of lemma 4.6:

REMARK AND DEFINITION 4.3. Let Δ be a normal n-dimensional oriented fan. If we fix a positive volume form $\omega \in \det V^*$, then every duality correlation φ gives rise to an *intersection product* on Δ, i.e., a pairing

$$E_\Delta \times E_{(\Delta,\partial\Delta)} \to A[-2n] \tag{21}$$

as follows: The isomorphism Θ of Lemma 3.1 yields an isomorphism

$$(\mathcal{DE})_\Delta \xrightarrow{\Theta} \mathrm{Hom}(E_{(\Delta,\partial\Delta)}, A) \otimes \det\ V^* \xrightarrow{\cong} \mathrm{Hom}(E_{(\Delta,\partial\Delta)}, A[-2n])\,.$$

Composed with the duality correlation φ_Δ on the level of global sections it provides a homomorphism

$$\chi_\Delta := \chi_\Delta^\omega : E_\Delta \to \mathrm{Hom}(E_{(\Delta,\partial\Delta)}, A[-2n])\,, \tag{22}$$

which is equivalent to (21).

THEOREM 4.4. (cf. [BBFK$_2$, 6.3] and [BreLu$_1$, 6.28]) *Let the oriented fan Δ be normal, and fix a positive volume form $\omega \in \det V^*$. If a duality correlation $\varphi : \mathcal{E} \to \mathcal{DE}$ is an isomorphism, then the induced pairing*

$$E_\Delta \times E_{(\Delta,\partial\Delta)} \to A[-2n]$$

is a duality pairing of reflexive A-modules. If Δ is even quasi-convex, then the A-modules E_Δ and $E_{(\Delta,\partial\Delta)}$ are free, and thus the associated reduced pairing

$$\bar{E}_\Delta \times \bar{E}_{(\Delta,\partial\Delta)} \to \bar{A}[-2n] \cong \boldsymbol{R}[-2n] \tag{23}$$

is a duality pairing of graded real vector spaces.

PROOF. Composing the isomorphisms φ_Δ and $\varphi_{(\Delta,\partial\Delta)}$ with the isomorphisms (13) and (14) of Theorem 3.2, one obtains:

$$E_\Delta \xrightarrow{\cong} \mathcal{DE}_\Delta \cong \mathrm{Hom}(E_{(\Delta,\partial\Delta)}, A)$$

and

$$E_{(\Delta,\partial\Delta)} \xrightarrow{\cong} \mathcal{DE}_{(\Delta,\partial\Delta)} \cong \mathrm{Hom}(E_\Delta, A)\,.$$

If Δ is even quasi-convex, then the modules E_Δ and $E_{(\Delta,\partial\Delta)}$ are free, see [BBFK2, 4.8, 4.12]. □

We now state and prove the proposition that yields the unicity of the duality correlation.

PROPOSITION 4.5. (see [BBFK2, 1.8 iii)] and [BreLu2, 3.14]) *For a fan Δ and two copies $\mathcal{E}$ and $\mathcal{F}$ of the equivariant intersection cohomology sheaf, every homomorphism*

$$\boldsymbol{R} = E_o \to F_o = \boldsymbol{R}$$

extends in a unique manner to a homomorphism $\mathcal{E} \to \mathcal{F}$ of degree 0.

For the proof, we assume the Vanishing Lemma 4.6. The rest of this section will then be dedicated to the proof of that lemma.

PROOF. The extension is constructed conewise by induction on the dimension: We have to show that over each non-zero cone σ, a homomorphism $\varphi_{\partial\sigma} : E_{\partial\sigma} \to F_{\partial\sigma}$ extends in a unique way to a homomorphism $\varphi_\sigma : E_\sigma \to F_\sigma$. Assuming Lemma 4.6 (i), the A-modules E_σ and F_σ are generated by homogeneous elements of degree below $\dim\sigma$. On the other hand, Lemma 4.6, (iii) yields $E^q_{(\sigma,\partial\sigma)} = 0 = F^q_{(\sigma,\partial\sigma)}$ for $q \le \dim\sigma$. Hence, the restriction maps $E^q_\sigma \to E^q_{\partial\sigma}$ and $F^q_\sigma \to F^q_{\partial\sigma}$ are isomorphisms for $q < \dim\sigma$, whence the uniqueness of φ_σ follows. The existence is a consequence of the fact that E_σ is a free A_σ-module. □

We thus have to prove the following result, the proof of which is based on the Hard Lefschetz theorem 4.7 for fans in lower dimensions:

LEMMA 4.6 (Vanishing lemma). (see [BBFK2, 1.7, 1.8 ii)] and [BreLu2, 3.13]) *For the equivariant intersection cohomology sheaf $\mathcal{E}$ on a non-zero cone σ, the following equivalent conditions hold*:

(i) $\bar{E}^q_\sigma = 0$ *for* $q \ge \dim\sigma$,
(ii) $\bar{E}^q_{(\sigma,\partial\sigma)} = 0$ *for* $q \le \dim\sigma$,
(iii) $E^q_{(\sigma,\partial\sigma)} = 0$ *for* $q \le \dim\sigma$.

PROOF. Let us first prove (i). We may assume $\dim\sigma = n$ and we use the setup of 1 (v). First of all note that

$$(B/\mathfrak{m}_B) \otimes_B E_{\partial\sigma} \cong (B/\mathfrak{m}_B)[T] \otimes_{B[T]} E_{\partial\sigma} \, .$$

Now we tensor the exact sequence

$$0 \longrightarrow (B/\mathfrak{m}_B)[T] \xrightarrow{\mu_T} (B/\mathfrak{m}_B)[T] \longrightarrow A/\mathfrak{m}_A \longrightarrow 0$$

with $E_{\partial\sigma}$ and obtain the exact sequence

$$(B/\mathfrak{m}_B) \otimes_B E_{\partial\sigma} \xrightarrow{\bar{\mu}_T} (B/\mathfrak{m}_B) \otimes_B E_{\partial\sigma} \longrightarrow (A/\mathfrak{m}_A) \otimes_A E_{\partial\sigma} \longrightarrow 0$$

with $\bar{\mu}_T := \mathrm{id}_{(B/\mathfrak{m}_B)} \otimes \mu_T$, where μ_T acts on the A-module $E_{\partial\sigma}$. Thus

$$\bar{E}_{\partial\sigma} \cong \operatorname{coker}(\bar{\mu}_T : (B/\mathfrak{m}_B) \otimes_B E_{\partial\sigma} \to (B/\mathfrak{m}_B) \otimes_B E_{\partial\sigma}) \, .$$

On the other hand, according to [BBFK2, (5.3.2)] together with 1, (3) and using the notation of 2.2, (4), we have an isomorphism

$$\bar{E}_{\partial\sigma} \cong \operatorname{coker}(\bar{\mu}_\psi : IH(\Lambda_\sigma) \to IH(\Lambda_\sigma)) \, ,$$

where $\mu_\psi : \mathcal{E}(\Lambda_\sigma) \to \mathcal{E}(\Lambda_\sigma)$ is the multiplication with the strictly convex conewise linear function

$$\psi := T \circ (\pi|_{\partial\sigma})^{-1} \in \mathcal{A}^2(\Lambda_\sigma)\,.$$

It now suffices to apply for $m := n-1$ the following theorem proved in [Ka]:

THEOREM 4.7 (Combinatorial hard Lefschetz theorem). *Let Λ be a complete fan in the m-dimensional vector space V and $\psi \in \mathcal{A}^2(\Lambda)$ a conewise linear strictly convex function. Then the "Lefschetz homomorphism|rm" $L := \bar{\mu}_\psi : \bar{E}_\Lambda \to \bar{E}_\Lambda$ induced by the multiplication $\mu_\psi : E_\Lambda \to E_\Lambda$ with ψ yields isomorphisms*

$$L^k : IH^{m-k}(\Lambda) \to IH^{m+k}(\Lambda)$$

for each $k \geq 0$. In particular, L is injective in degrees $q \leq m-1$ and surjective in degrees $q \geq m-1$.

Let us finish the proof of Lemma 4.6: The equivalence of (i) and (ii) follows from (20) and the dual pairing (23) in Theorem 4.4 in the particular case $\Delta = \langle\sigma\rangle$, while the equivalence of (ii) and (iii) is a consequence of the following fact: For a finitely generated graded A-module M, one has $M^q = 0$ for $q \leq r$ if and only if $\bar{M}^q = 0$ for $q \leq r$. □

5. Comparison with previous definitions. Let $\iota\colon (V, \hat{\Delta}) \to (V, \Delta)$ be an *oriented refinement*, i.e., if a cone in $\hat{\Delta}^d$ is contained in a cone in Δ^d, then their orientations coincide.

PROPOSITION 5.1. *For every pure sheaf $\mathcal{F}$ on $\hat{\Delta}$, there exists a canonical isomorphism*

$$\mathcal{D}(\iota_*(\mathcal{F})) \cong \iota_*(\mathcal{D}\mathcal{F})\,.$$

PROOF. For a cone $\sigma \in \Delta^d$, let $\hat{\sigma} \preceq \hat{\Delta}$ denote its refinement. Then the formula (13) of Theorem 3.2, applied to $\hat{\sigma}$, yields the isomorphism in the following chain

$$\begin{aligned}\mathcal{D}(\iota_*(\mathcal{F}))_\sigma &= \mathrm{Hom}(\iota_*(\mathcal{F})_{(\sigma,\partial\sigma)}, A) \otimes \det V_\sigma^* = \mathrm{Hom}(F_{(\hat{\sigma},\partial\hat{\sigma})}, A) \otimes \det V_\sigma^* \\ &\cong \mathcal{D}\mathcal{F}_{\hat{\sigma}} = \iota_*(\mathcal{D}\mathcal{F})_\sigma\end{aligned}$$ □

We now can prove the Compatibility Theorem stated in the introduction:

THEOREM 5.2 (Compatibility theorem). (cf. [BreLu2, 7.2]) *Let $\hat{\mathcal{E}}$ be the equivariant intersection cohomology sheaf of the oriented refinement $\hat{\Delta}$ of the normal n-dimensional fan Δ, and let $\varepsilon : \mathcal{E} \to \iota_*(\hat{\mathcal{E}})$ be a homomorphism of graded sheaves extending the identity $\mathcal{E}(o) = \boldsymbol{R} = \iota_*(\hat{\mathcal{E}})(o)$. Then the intersection products provide a commutative diagram*

$$\begin{array}{ccc}\mathcal{E}(\Delta) \times \mathcal{E}(\Delta, \partial\Delta) & \longrightarrow & \hat{\mathcal{E}}(\hat{\Delta}) \times \hat{\mathcal{E}}(\hat{\Delta}, \partial\hat{\Delta}) \\ & \searrow \quad \swarrow & \\ & A[-2n]\,. & \end{array}$$

PROOF. The homomorphism ε provides a diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\varepsilon"] \arrow[d] & \iota_*(\hat{\mathcal{E}}) \arrow[d] \\ \mathcal{D}\mathcal{E} & \mathcal{D}\iota_*(\hat{\mathcal{E}}) \cong \iota_*(\mathcal{D}\hat{\mathcal{E}}) \arrow[l, "\mathcal{D}\varepsilon"'] \end{tikzcd} ,$$

where the vertical arrows are ϑ respectively $\iota_*(\hat{\vartheta})$ with the duality correlations $\vartheta : \mathcal{E} \to \mathcal{D}\mathcal{E}$ of (20) and $\hat{\vartheta} : \hat{\mathcal{E}} \to \mathcal{D}\hat{\mathcal{E}}$. It is commutative at the zero cone and thus everywhere, see Proposition 4.5. Passing to the level of global sections yields the claim. □

Finally let us discuss the approach of [BBFK2, 6.1]. Here we use the notion of an *evaluation map*:

DEFINITION 5.3. Let Δ be an oriented purely n-dimensional fan in the vector space V, endowed with a volume form $\omega \in \det V^*$. Then, for $1 \in E^0_\Delta = E^0_o = \boldsymbol{R}$, the homomorphism

$$e^\omega_\Delta := \chi^\omega_\Delta(1) : E_{(\Delta,\partial\Delta)} \to A[-2n] ,$$

see (22), is called the *evaluation map* associated to ω.

THEOREM 5.4. *Let Δ be an oriented normal fan in a vector space V endowed with a volume form $\omega \in \det V^*$. Furthermore let*

$$\beta : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$$

be a bilinear map of $\mathcal{A}$-module sheaves extending the multiplication

$$E_o \times E_o = \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R} = E_o$$

of real numbers. Then the pairing

$$e^\omega_\Delta \circ \beta_\Delta : \mathcal{E}(\Delta) \times \mathcal{E}(\Delta, \partial\Delta) \to \mathcal{E}(\Delta, \partial\Delta) \to A[-2n] \tag{24}$$

coincides with the intersection product.

Note that for a simplicial fan Δ, the equality $\mathcal{E} = \mathcal{A}$ holds, so the bilinear map β necessarily is the multiplication of functions and thus, symmetric. In the non-simplicial case, the map β is not uniquely determined. Nevertheless, there always exists such a map β that is symmetric. For a complete fan, the intersection product is thus symmetric, which also follows from Theorem 5.2 with a simplicial subdivision $\hat{\Delta}$ of Δ.

PROOF OF 5.4. For each cone σ and a positive volume form $\omega_\sigma \in \det(V^*_\sigma)$, we define $e^{\omega_\sigma}_\sigma : E_{(\sigma,\partial\sigma)} \to A_\sigma[-2\dim\sigma]$ analogously to e^ω_Δ. Then $e^{\omega_\sigma}_\sigma \otimes \omega_\sigma$ does not depend on the choice of ω_σ, and the family of homomorphisms

$$\tilde{\varphi}_\sigma : \mathcal{E}_\sigma \to \mathcal{D}\mathcal{E}_\sigma , \quad s \mapsto (e^{\omega_\sigma}_\sigma \circ \beta)(s, \_) \otimes \omega_\sigma$$

defines a duality correlation $\tilde{\varphi} : \mathcal{E} \to \mathcal{D}\mathcal{E}$ and thus, according to Theorem 4.2, is unique. In particular, the pairing (24) is the intersection product. □

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FACHBEREICH MATHEMATIK UND STATISTIK
UNIVERSITÄT KONSTANZ, FACH D 203
D-78457 KONSTANZ
GERMANY

E-mail address: Gottfried.Barthel@uni-konstanz.de

IML-CNRS, LUMINY, CASE 907
F-13288 MARSEILLE CEDEX 9
FRANCE

E-mail address: jpb@iml.univ-mrs.fr

MATEMATISKA INSTITUTIONEN, BOX 480
UPPSALA UNIVERSITET
SE-75106 UPPSALA
SWEDEN

E-mail address: khf@math.uu.se

FACHBEREICH MATHEMATIK UND STATISTIK
UNIVERSITÄT KONSTANZ, FACH D 203
D-78457 KONSTANZ
GERMANY

E-mail address: Ludger.Kaup@uni-konstanz.de