Tohoku Math. J.
57 (2005), 201–221

GEOMETRIC FLOW ON COMPACT LOCALLY CONFORMALLY KÄHLER MANIFOLDS

YOSHINOBU KAMISHIMA AND LIVIU ORNEA

(Received July 7, 2003, revised March 12, 2004)

Abstract. We study two kinds of transformation groups of a compact locally conformally Kähler (l.c.K.) manifold. First, we study compact l.c.K. manifolds by means of the existence of holomorphic l.c.K. flow (i.e., a conformal, holomorphic flow with respect to the Hermitian metric.) We characterize the structure of the compact l.c.K. manifolds with parallel Lee form. Next, we introduce the Lee-Cauchy-Riemann (LCR) transformations as a class of diffeomorphisms preserving the specific G-structure of l.c.K. manifolds. We show that compact l.c.K. manifolds with parallel Lee form admitting a non-compact holomorphic flow of LCR transformations are rigid: such a manifold is holomorphically isometric to a Hopf manifold with parallel Lee form.

1. Introduction. Let (M, g, J) be a connected, complex Hermitian manifold of complex dimension $n \geq 2$. We denote its fundamental 2-form by ω, which is defined by $\omega(X, Y) = g(X, JY)$. If there exists a real 1-form θ satisfying the integrability condition

$$d\omega = \theta \wedge \omega \quad \text{with} \;\; d\theta = 0\,,$$

then g is said to be a *locally conformally Kähler* (l.c.K.) metric. A complex manifold M endowed with a l.c.K. metric is called a l.c.K. manifold. The conformal class of a l.c.K. metric g is said to be a l.c.K. structure on M. The closed 1-form θ is called *the Lee form* and it encodes the geometric properties of such a manifold. The vector field $\theta^\sharp$, defined by $\theta(X) = g(X, \theta^\sharp)$, is called the Lee field.

The purpose of this paper is to study two kinds of transformation groups of a compact l.c.K. manifold (M, g, J). We first consider $\mathrm{Aut}_{\mathrm{l.c.K.}}(M)$, the group of all conformal, holomorphic diffeomorphisms. We discuss its properties in §2. A holomorphic vector field Z on (M, g, J) generates a 1-dimensional complex Lie group $\mathcal{C}$. (The universal covering group of $\mathcal{C}$ is $\boldsymbol{C}$.) We call $\mathcal{C}$ a holomorphic flow on M.

DEFINITION 1.1. If a holomorphic flow $\mathcal{C}$ (resp. holomorphic vector field Z) belongs to $\mathrm{Aut}_{\mathrm{l.c.K.}}(M)$ (resp. Lie algebra of $\mathrm{Aut}_{\mathrm{l.c.K.}}(M)$), then $\mathcal{C}$ (resp. Z) is said to be a *holomorphic l.c.K. flow* (resp. *holomorphic l.c.K. vector field*).

2000 *Mathematics Subject Classification*. Primary 57S25; Secondary 53C55.

Key words and phrases. Locally conformally Kähler manifold, Lee form, contact structure, strongly pseudoconvex CR-structure, G-structure, holomorphic complex torus action, transformation groups.

The second author is a member of EDGE, Research Training Network HPRN-CT-2000-00101, supported by The European Human Potential Programme.

A nontrivial subclass of l.c.K. manifolds is formed by those (M, g, J) having parallel Lee form with respect to the Levi-Civita connection ∇^g (i.e., $\nabla^g\theta = 0$). We observe that a compact non-Kähler l.c.K. manifold (M, g, J) with parallel Lee form θ supports a holomorphic vector field $Z = \theta^\sharp - \mathrm{i}J\theta^\sharp$ which generates holomorphic isometries of g. (Compare [18], [19], [6].) We shall prove that the converse is also true:

THEOREM A. *Let (M, g, J) be a compact, connected, l.c.K. non-Kähler manifold, of complex dimension at least* 2. *If* $\mathrm{Aut}_{\mathrm{l.c.K.}}(M)$ *contains a holomorphic l.c.K. flow, then there exists a metric with parallel Lee form in the conformal class of g.*

COROLLARY A_1. *With the same hypothesis, M admits a l.c.K. metric with parallel Lee form if and only if it admits a holomorphic l.c.K. flow.*

In §3, we discuss the existence of l.c.K. metrics with parallel Lee form on the Hopf manifold. (Compare with [7].) Let $\Lambda = (\lambda_1, \ldots, \lambda_n)$ with the λ_i's complex numbers satisfying $0 < |\lambda_n| \leq \cdots \leq |\lambda_1| < 1$. By a *primary Hopf manifold M_Λ* of type Λ we mean the compact quotient manifold of $\boldsymbol{C}^n - \{0\}$ by a subgroup Γ_Λ generated by the transformation $(z_1, \ldots, z_n) \mapsto (\lambda_1 z_1, \ldots, \lambda_n z_n)$. Note that a primary Hopf manifold of type Λ of complex dimension 2 is a primary Hopf surface of Kähler rank 1. We prove the following:

THEOREM B. *The primary Hopf manifold M_Λ of type Λ supports a l.c.K. metric with parallel Lee form.*

See §3 which is devoted to the construction of such a metric. More generally, we prove the existence of a l.c.K. metric with parallel Lee form on the Hopf manifold (cf. Theorem 3.1).

In the second half of the paper we adopt the viewpoint of G-structure theory in order to study a non-compact, non-holomorphic, transformation group of a compact l.c.K. manifold (M, g, J) with parallel Lee form. Locally, the 2-form ω defines the real 1-forms $\theta, \theta \circ J$ and $n-1$ complex 1-forms θ^α and their conjugates $\bar\theta^\alpha$, where $\theta \circ J$ is called the *anti-Lee form* and is defined by $\theta \circ J(X) = \theta(JX)$. We consider the group $\mathrm{Aut}_{\mathrm{LCR}}(M)$ of transformations of M preserving the structure of unitary coframe fields $\mathcal{F} = \{\theta, \theta \circ J, \theta^1, \ldots, \theta^{n-1}, \bar\theta^1, \ldots, \bar\theta^{n-1}\}$. More precisely, an element f of $\mathrm{Aut}_{\mathrm{LCR}}(M)$ is called a *Lee-Cauchy-Riemann* (LCR) transformation if it satisfies the equations:

$$\begin{aligned} f^*\theta &= \theta\,, \\ f^*(\theta \circ J) &= \lambda \cdot (\theta \circ J)\,, \\ f^*\theta^\alpha &= \sqrt{\lambda} \cdot \theta^\beta U^\alpha_\beta + (\theta \circ J) \cdot v^\alpha\,, \\ f^*\bar\theta^\alpha &= \sqrt{\lambda} \cdot \bar\theta^\beta \overline{U}^\alpha_\beta + (\theta \circ J) \cdot \overline{v}^\alpha\,. \end{aligned}$$

Here λ, v^α, U^α_β are smooth functions with values, respectively, in $\boldsymbol{R}^+$, $\boldsymbol{C}$ and $\mathrm{U}(n-1)$. Obviously, if $\mathrm{I}(M, g, J)$ is the group of holomorphic isometries, then both $\mathrm{Aut}_{\mathrm{l.c.K.}}(M)$ and $\mathrm{Aut}_{\mathrm{LCR}}(M)$ contain $\mathrm{I}(M, g, J)$.

As the main result of this part we exhibit the rigidity of compact l.c.K. manifolds under the existence of a non-compact LCR flow:

THEOREM C. *Let (M, g, J) be a compact, connected, l.c.K. non-Kähler manifold of complex dimension at least* 2, *with parallel Lee form θ. Suppose that M admits a closed subgroup $\boldsymbol{C}^* = S^1 \times \boldsymbol{R}^+$ of Lee-Cauchy-Riemann transformations whose S^1 subgroup induces the Lee field $\theta^\sharp$. Then M is holomorphically isometric, up to scalar multiple of the metric, to the primary Hopf manifold M_Λ of type Λ endowed with the canonical l.c.K. metric as stated in Theorem B.*

ACKNOWLEDGMENT. The authors are grateful to the anonymous referee for useful criticism. The second named author thanks JSPS for financial support and the Department of Mathematics of Tokyo Metropolitan University for hospitality during the preparation of this work.

2. Locally conformally Kähler transformations.

PROPOSITION 2.1. *Let (M, g, J) be a compact l.c.K. manifold with* $\dim_{\boldsymbol{C}} M \geq 2$. *Then* $\mathrm{Aut}_{\text{l.c.K.}}(M)$ *is a compact Lie group.*

PROOF. Note that $\mathrm{Aut}_{\text{l.c.K.}}(M)$ is a closed Lie subgroup in the group of all conformal diffeomorphisms of (M, g). If $\mathrm{Aut}_{\text{l.c.K.}}(M)$ were noncompact, then by the celebrated result of Obata and Lelong-Ferrand ([15], [14]), (M, g) would be conformally equivalent with the sphere S^{2n}, $n \geq 2$. Hence M would be simply connected. It is well-known that a compact simply connected l.c.K. manifold is conformal to a Kähler manifold (cf. [6]), which is impossible because the sphere S^{2n} has no Kähler structure. □

From now on, we shall suppose that the l.c.K. manifold we work with is compact, non-Kähler and, moreover, that the Lee field is nowhere vanishing. In particular, such a manifold is not simply connected (cf. [6]). Given a l.c.K. manifold (M, g, J), let $\tilde{M}$ be the universal covering space of M, let $p : \tilde{M} \to M$ be the canonical projection and denote also by J the lifted complex structure on $\tilde{M}$. We can associate to the fundamental 2-form ω a canonical Kähler form on $\tilde{M}$ as follows. Since the Lee form θ is closed, its lift to $\tilde{M}$ is exact, hence $p^*\theta = d\tau$ for some smooth function τ on $\tilde{M}$. We put $h = e^{-\tau} \cdot p^*g$ (resp. $\Omega = e^{-\tau} \cdot p^*\omega$). It is easy to check that $d\Omega = 0$, thus h is a Kähler metric on $(\tilde{M}, J)$. In particular g *is locally conformal to the Kähler metric* h (compare with [6] and the bibliography therein). Let $f \in \mathrm{Aut}_{\text{l.c.K.}}(M)$. By definition, $f^*\omega = e^\lambda \cdot \omega$ for some function λ on M. Differentiate this equality to yield that $(f^*\theta - \theta - d\lambda) \wedge \omega = 0$. As ω is nondegenerate and $\dim_{\boldsymbol{C}} M > 1$, $f^*\theta = \theta + d\lambda$. Since $p^*\theta = d\tau$, for any lift $\tilde{f}$ of f to $\tilde{M}$ we have $d\tilde{f}^*\tau = d(\tau + p^*\lambda)$, thus $-\tilde{f}^*\tau + p^*\lambda = -\tau + c$ for some constant c. We can write $\tilde{f}^*\Omega = e^c \cdot \Omega$. If $c \neq 0$, $\tilde{f}$ is a holomorphic homothety with respect to h; when $c = 0$, $\tilde{f}$ will be an isometry.

We denote by $\mathcal{H}(\tilde{M}, \Omega, J)$ the group of all holomorphic, homothetic transformations of $\tilde{M}$ with respect to the Kähler structure (h, J). If $f_1, f_2 \in \mathcal{H}(\tilde{M}, \Omega, J)$, there exist some constants $\rho(f_i)$ $(i = 1, 2)$ satisfying $f_i^*\Omega = \rho(f_i) \cdot \Omega$ as above. It is easy to check that

$\rho(f_1 \circ f_2) = \rho(f_1) \cdot \rho(f_2)$. We obtain a continuous homomorphism:

$$\rho : \mathcal{H}(\tilde{M}, \Omega, J) \longrightarrow \boldsymbol{R}^+. \tag{2.1}$$

Let $\pi_1(M)$ be the fundamental group of M. Then we note that $\pi_1(M) \subset \mathcal{H}(\tilde{M}, \Omega, J)$. For this, if $\gamma \in \pi_1(M)$, then $\gamma^*\Omega = e^{-\gamma^*\tau} \cdot \gamma^* p^*\omega = e^{-\gamma^*\tau} \cdot p^*\omega = e^{-\gamma^*\tau+\tau} \cdot \Omega$. Since Ω is a Kähler form ($n \geq 2$), $e^{-\gamma^*\tau+\tau}$ must be constant $\rho(\gamma)$.

Let $\mathcal{C}$ be a holomorphic l.c.K. flow on M. If we denote by $\tilde{\mathcal{C}}$ a lift of $\mathcal{C}$ to $\tilde{M}$, then $\tilde{\mathcal{C}} \subset \mathcal{H}(\tilde{M}, \Omega, J)$. If V is a vector field which generates a one-parameter subgroup of $\tilde{\mathcal{C}}$, then so does JV with V and JV together generating $\tilde{\mathcal{C}}$. We define a smooth function $s : \tilde{M} \to \boldsymbol{R}$ to be $s(x) = \Omega(JV_x, V_x)$. Since $\tilde{\mathcal{C}}$ centralizes each element γ of $\pi_1(M)$, it follows that $s(\gamma x) = \Omega(JV_{\gamma x}, V_{\gamma x}) = \Omega(\gamma_* JV_x, \gamma_* V_x) = \rho(\gamma)s(x)$. If every element γ satisfies $\rho(\gamma) = 1$, i.e., $\gamma^*\Omega = \Omega$, then $\pi_1(M)$ acts as holomorphic isometries of h so that Ω would induce a Kähler metric on M. By our hypothesis, this does not occur. There exists at least one element γ such that $\rho(\gamma) \neq 1$. In particular, we note that:

(2.2) The function s is not constant on $\tilde{M}$.

On the other hand, we prove the following lemma. (The proof of the lemma is almost the same as that of [10].)

LEMMA 2.1. *$\rho(\tilde{\mathcal{C}}) = \boldsymbol{R}^+$, i.e., the group $\tilde{\mathcal{C}}$ acts by holomorphic, non-trivial homotheties with respect to the Kähler metric h on $\tilde{M}$.*

PROOF. Since $\tilde{\mathcal{C}}$ is connected, if $\rho(\tilde{\mathcal{C}}) \neq \boldsymbol{R}^+$, it must be trivial. By reduction to absurdity, suppose that $\rho(\tilde{\mathcal{C}}) = \{1\}$. Then $\tilde{\mathcal{C}}$ leaves Ω invariant. As $\{V, JV\}$ generates $\tilde{\mathcal{C}}$, it follows that $\mathcal{L}_V\Omega = \mathcal{L}_{JV}\Omega = 0$. In particular, $Vs = (JV)s = 0$. For any distribution D on $\tilde{M}$, denote by $D^\perp$ the orthogonal complement to D with respect to the metric h, where $h(\tilde{X}, \tilde{Y}) = \Omega(J\tilde{X}, \tilde{Y})$. Since $0 = (\mathcal{L}_V\Omega)(JV, \tilde{X}) = V\Omega(JV, \tilde{X}) - \Omega([V, JV], \tilde{X}) - \Omega(JV, [V, \tilde{X}])$, if $\tilde{X} \in \{V, JV\}^\perp$, then $\Omega(JV, [V, \tilde{X}]) = 0$, similarly $\Omega(V, [JV, \tilde{X}]) = 0$. The equality

$$\begin{aligned} 0 = 3d\Omega(\tilde{X}, V, JV) &= \tilde{X}\Omega(V, JV) - V\Omega(\tilde{X}, JV) + JV\Omega(\tilde{X}, V) \\ &\quad - \Omega([\tilde{X}, V], JV) - \Omega([V, JV], \tilde{X}) - \Omega([JV, \tilde{X}], V) \end{aligned}$$

implies that $\tilde{X}\Omega(V, JV) = 0$, i.e., $\tilde{X}s = 0$ for any $\tilde{X} \in \{V, JV\}^\perp$. Therefore, s becomes constant, being a contradiction to (2.2). □

2.1. The submanifold W and its pseudo-Hermitian structure. As Kerρ has one dimension, denote by $-J\xi$ the vector field whose one-parameter subgroup $\{\psi_t\}_{t\in\boldsymbol{R}}$ acts as holomorphic isometries on $\tilde{M}$.

$$\psi_t^*\Omega = \Omega\,, \quad t \in \boldsymbol{R}\,. \tag{2.3}$$

Since $-J\xi$ and ξ together generate the group $\tilde{\mathcal{C}}$, the 1-parameter subgroup $\{\varphi_t\}_{t\in\boldsymbol{R}}$ generated by ξ acts as nontrivial holomorphic homotheties with respect to Ω by Lemma 2.1. In particular, the group $\{\varphi_t\}_{t\in\boldsymbol{R}}$ is isomorphic to $\boldsymbol{R}$. Since $\varphi_t^*\Omega = \rho(\varphi_t) \cdot \Omega$ ($t \in \boldsymbol{R}$, $\rho(\varphi_t) \in \boldsymbol{R}^+$) from

(2.1) and ρ is a continuous homomorphism, $\rho(\varphi_t) = e^{at}$ for some constant $a \neq 0$. We may normalize $a = 1$ so that:

$$\varphi_t^* \Omega = e^t \cdot \Omega\,, \quad t \in \boldsymbol{R}\,. \tag{2.4}$$

LEMMA 2.2. *The group $\{\varphi_t\}_{t\in\boldsymbol{R}}$ acts properly and hence freely on $\tilde{M}$. In particular, $\xi \neq 0$ everywhere on $\tilde{M}$.*

PROOF. Recall that $\mathcal{C}$ lies in $\mathrm{Aut}_{\text{l.c.K.}}(M)$ by definition. As $\mathrm{Aut}_{\text{l.c.K.}}(M)$ is a compact Lie group, its closure $\overline{\mathcal{C}}$ in $\mathrm{Aut}_{\text{l.c.K.}}(M)$ is also compact and so isomorphic to a k-torus ($k \geq 2$). Therefore, the lift H of $\overline{\mathcal{C}}$ to $\tilde{M}$ acts properly on $\tilde{M}$. The lift H is isomorphic to $\boldsymbol{R}^l \times T^m$, where $l + m = k$. Note that $l \geq 1$ because ρ maps any compact subgroup of H to $\{1\}$, but the group $\{\varphi_t\}_{t\in\boldsymbol{R}} \subset H$ satisfies $\rho(\{\varphi_t\}) = \boldsymbol{R}^+$. Hence the group $\{\varphi_t\}_{t\in\boldsymbol{R}}$ has a nontrivial summand in $\boldsymbol{R}^l$, which implies that $\{\varphi_t\}_{t\in\boldsymbol{R}}$ is closed in H. Thus, the group $\{\varphi_t\}_{t\in\boldsymbol{R}}$ acts properly on $\tilde{M}$. If we note that $\{\varphi_t\}_{t\in\boldsymbol{R}}$ is isomorphic to $\boldsymbol{R}$, then it acts freely on $\tilde{M}$. □

PROPOSITION 2.2. *Let $s : \tilde{M} \to \boldsymbol{R}$ be the smooth map defined as $s(x) = \Omega(J\xi_x, \xi_x)$. Then 1 is a regular value of s, and hence $s^{-1}(1)$ is a codimension one, regular submanifold of $\tilde{M}$.*

PROOF. As φ_t is holomorphic, $s(\varphi_t x) = \Omega(J\xi_{\varphi_t x}, \xi_{\varphi_t x}) = \Omega(\varphi_{t*} J\xi_x, \varphi_{t*}\xi_x) = e^t \cdot s(x)$. Hence,

$$\mathcal{L}_\xi s = \lim_{t\to 0} \frac{\varphi_t^* s - s}{t} = s\,.$$

We also note that

$$\mathcal{L}_\xi \Omega = \Omega\,. \tag{2.5}$$

By Lemma 2.2, notice that $\xi \neq 0$ everywhere on $\tilde{M}$. Since $s(x) \neq 0$, $s^{-1}(1) \neq \emptyset$. For $x \in s^{-1}(1)$, $ds(\xi_x) = (\mathcal{L}_\xi s)(x) = s(x) = 1$. This proves that $ds : T_x\tilde{M} \to \boldsymbol{R}$ is onto and so $s^{-1}(1)$ is a codimension one smooth regular submanifold of $\tilde{M}$. □

Let now $W = s^{-1}(1)$. We can prove:

LEMMA 2.3. *The submanifold W is connected and the map $H : \boldsymbol{R} \times W \to \tilde{M}$, defined by $H(t, w) = \varphi_t w$, is an equivariant diffeomorphism.*

PROOF. Let W_0 be a component of $s^{-1}(1)$ and $\boldsymbol{R} \cdot W_0$ the set $\{\varphi_t w\ ;\ w \in W_0, t \in \boldsymbol{R}\}$. As $\boldsymbol{R} = \{\varphi_t\}$ acts freely and $s(\varphi_t x) = e^t s(x)$, we have $\varphi_t W_0 \cap W_0 = \emptyset$ for $t \neq 0$. Thus $\boldsymbol{R} \cdot W_0$ is an open subset of $\tilde{M}$. We prove that it is also closed. Let $\overline{\boldsymbol{R} \cdot W_0}$ be the closure of $\boldsymbol{R} \cdot W_0$ in $\tilde{M}$. We choose a limit point $p = \lim \varphi_{t_i} w_i \in \overline{\boldsymbol{R} \cdot W_0}$. Then $s(p) = \lim s(\varphi_{t_i} w_i) = \lim e^{t_i} s(w_i) = \lim e^{t_i}$. Put $t = \log s(p)$. Then $t = \lim t_i$, so $\varphi_t^{-1}(p) = \lim \varphi_{t_i}^{-1}(\lim \varphi_{t_i} w_i) = \lim w_i$. Since $s^{-1}(1)$ is regular (i.e., closed with respect to the relative topology induced from $\tilde{M}$), its component W_0 is also closed. Hence $\varphi_t^{-1} p \in W_0$. Therefore $p = \varphi_t(\varphi_t^{-1} p) \in \boldsymbol{R} \cdot W_0$, proving that $\boldsymbol{R} \cdot W_0$ is closed in $\tilde{M}$. In conclusion, $\boldsymbol{R} \cdot W_0 = \tilde{M}$. Now, if W_1 is another component of $s^{-1}(1)$, the same argument shows $\boldsymbol{R} \cdot W_1 = \tilde{M}$. As $\boldsymbol{R} \cdot W_0 = \boldsymbol{R} \cdot W_1$ and $s(W_1) = 1$, this implies $W_0 = W_1$, in other words, W is connected. □

Let $i : W \to \tilde{M}$ be the inclusion and $\pi : \tilde{M} \to W$ the canonical projection. Define a 1-form η on W to be

$$\eta = i^* \iota_\xi \Omega \,. \tag{2.6}$$

Here ι_ξ denotes the interior product with ξ. From the definition of $\{\psi_t\}_{t\in\boldsymbol{R}}$ (see the beginning of § 2.1) we have

$$\frac{d\psi_t}{dt}(x)\bigg|_{t=0} = -J\xi_x \,. \tag{2.7}$$

By (2.3), $s(\psi_t w) = s(w) = 1$ $(w \in W)$ so that the group $\{\psi_t\}_{t\in\boldsymbol{R}}$ leaves W invariant. Hence, the vector field $-J\xi$ restricts to a vector field A to W. If $\{\psi'_t\}_{t\in\boldsymbol{R}}$ is the one-parameter subgroup generated by A, then

$$\psi_t = i \circ \psi'_t \,. \tag{2.8}$$

LEMMA 2.4. *The* 1*-form* η *is a contact form on* W *for which* A *is the characteristic vector field* (*Reeb field*).

PROOF. First note that $\eta(A_w) = \iota_\xi \Omega(-J\xi_w) = \Omega(J\xi_w, \xi_w) = s(w) = 1$ $(w \in W)$. Moreover, from (2.5), $d\eta = i^* d\iota_\xi \Omega = i^*(d\iota_\xi \Omega + \iota_\xi d\Omega) = i^* \mathcal{L}_\xi \Omega = i^* \Omega$. Hence, $\eta \wedge d\eta^{n-1} \neq 0$ on W showing that η is a contact form. Noting (2.3), (2.8) and that both φ_t and ψ_θ commutes with each other, it is easy to see that

$$\begin{aligned} {\psi'}^*_t \iota_\xi \Omega = \iota_\xi \quad &\text{on } \tilde{M} \,. \\ {\psi'}^*_t \eta = \eta \quad &\text{on } W \,. \end{aligned} \tag{2.9}$$

Let Null $\eta = \{X \in TW \mid \eta(X) = 0\}$ be the contact subbundle. Since $\mathcal{L}_A\eta(X) = A\eta(X) - \eta([A, X])$ and $\mathcal{L}_A\eta = 0$ from (2.9), if $X \in$ Null η, then $\eta([A, X]) = 0$. Moreover, $d\eta(A, X) = (A\eta(X) - X\eta(A) - \eta([A, X]))/2 = 0$, which implies that $d\eta(A, X) = 0$ for all $X \in TW$, showing that A is the characteristic vector field. □

Recall that $\boldsymbol{R} \to \tilde{M} \xrightarrow{\pi} W$ is a principal fiber bundle with $T\boldsymbol{R} = \langle\xi\rangle$. By Lemma 2.3, each point $x \in \tilde{M}$ can be described uniquely as $x = \varphi_t w$. By (2.8),

$$\begin{aligned} \pi \circ \psi_\theta(x) &= \pi \circ \psi_\theta(\varphi_t w) = \pi \circ \varphi_t(\psi_\theta w) \\ &= \pi \circ i {\psi'}_\theta(w) = {\psi'}_\theta(w) = {\psi'}_\theta \circ \pi(x) \,, \end{aligned} \tag{2.10}$$

and hence, $\pi_*(-J\xi) = A$. As $i_*\pi_* X_x - X_x = a \cdot \xi_x$ for some function a, by (2.6), π maps $\{\xi, J\xi\}^\perp$ isomorphically onto Null η. Since $\{\xi, J\xi\}^\perp$ is J-invariant, there exists an almost complex structure J on Null η such that the following diagram is commutative:

$$\begin{array}{ccc} \{\xi, J\xi\}^\perp & \xrightarrow{\pi_*} & \text{Null}\,\eta \\ \big\downarrow J & & \big\downarrow J \\ \{\xi, J\xi\}^\perp & \xrightarrow{\pi_*} & \text{Null}\,\eta \,. \end{array} \tag{2.11}$$

PROPOSITION 2.3. *The pair (η, J) is a strictly pseudoconvex, pseudo-Hermitian structure on W.*

PROOF. Let $\Psi : \mathrm{Null}\,\eta \times \mathrm{Null}\,\eta \to \boldsymbol{R}$ be the bilinear form defined by $\Psi(X, Y) = d\eta(JX, Y)$. There exist $\tilde{X}, \tilde{Y} \in \{\xi, J\xi\}^{\perp}$ such that $\pi_* \tilde{X} = X$, $\pi_* \tilde{Y} = Y$. Then it is easy to see that $i_* JX \equiv J\tilde{X}$, $i_* Y \equiv \tilde{Y} \bmod \xi$. Using $d\eta = i^*\Omega$ as above, $\Psi(X, Y) = i^*\Omega(JX, Y) = \Omega(J\tilde{X}, \tilde{Y}) = h(\tilde{X}, \tilde{Y})$, and hence Ψ is positive definite. By definition, η is strictly pseudoconvex. Let $\{\xi, J\xi\}^{\perp} \otimes \boldsymbol{C} = B^{1,0} \oplus B^{0,1}$ be the canonical splitting of J. Then we prove that $[B^{1,0}, B^{1,0}] \subset B^{1,0}$. Let $\tilde{X}, \tilde{Y} \in B^{1,0}$. Since $T^{1,0}\tilde{M} = \{\xi - \mathrm{i}J\xi\} \oplus B^{1,0}$ (where $\mathrm{i} = \sqrt{-1}$) and J is integrable on $\tilde{M}$, $[\tilde{X}, \tilde{Y}] \in T^{1,0}\tilde{M}$. Put $[\tilde{X}, \tilde{Y}] = a(\xi - \mathrm{i}J\xi) + \tilde{Z}$ for some function a and $\tilde{Z} \in B^{1,0}$. As $\pi_*(-J\xi) = A$ from (2.10), $\pi_*([\tilde{X}, \tilde{Y}]) = a\mathrm{i}A + \pi_*\tilde{Z}$. By definition, $2d\eta(\pi_*\tilde{X}, \pi_*\tilde{Y}) = -\eta([\pi_*\tilde{X}, \pi_*\tilde{Y}]) = -a\mathrm{i}$. On the other hand, since Ω is J-invariant, $\Omega(\tilde{X}, \tilde{Y}) = 0$ for any $\tilde{X}, \tilde{Y} \in B^{1,0}$. As above, $i_*\pi_*\tilde{X} \equiv \tilde{X} \bmod \xi$, similarly for $\tilde{Y}$, we obtain that $d\eta(\pi_*\tilde{X}, \pi_*\tilde{Y}) = \Omega(i_*\pi_*\tilde{X}, i_*\pi_*\tilde{Y}) = \Omega(\tilde{X}, \tilde{Y}) = 0$. Hence, $a = 0$ and so $[\tilde{X}, \tilde{Y}] = \tilde{Z} \in B^{1,0}$. If we note that $\pi_* : \{\xi, J\xi\}^{\perp} \otimes \boldsymbol{C} \to \mathrm{Null}\,\eta \otimes \boldsymbol{C}$ is J-isomorphic by (2.11), then $\mathrm{Null}\,\eta \otimes \boldsymbol{C} = \pi_* B^{1,0} \oplus \pi_* B^{0,1}$ is the splitting for J, in which we have shown $[\pi_* B^{1,0}, \pi_* B^{1,0}] \subset \pi_* B^{1,0}$. Therefore J is a complex structure on $\mathrm{Null}\,\eta$. □

Consider the group of pseudo-Hermitian transformations on (W, η, J):

$$\mathrm{PSH}(W, \eta, J) = \{f \in \mathrm{Diff}(W) \mid f^*\eta = \eta,\ f_* \circ J = J \circ f_* \text{ on } \mathrm{Null}\,\eta\}. \tag{2.12}$$

COROLLARY 2.1. *The characteristic vector field A generates the subgroup $\{\psi'_t\}_{t \in \boldsymbol{R}}$ consisting of pseudo-Hermitian transformations.*

PROOF. By (2.3) and (2.9), ψ_t (resp. ψ'_t) preserves $\{\xi, J\xi\}^{\perp}$ (resp. $\mathrm{Null}\,\eta$). Then the equality $\pi \circ \psi_\theta = \psi'_\theta \circ \pi$ from (2.10) with diagram (2.11) implies that $\psi'_{t*}J = J\psi'_{t*}$ on $\mathrm{Null}\,\eta$. Therefore

$$\{\psi'_t\}_{t \in \boldsymbol{R}} \subset \mathrm{PSH}(W, \eta, J)\,. \tag{2.13}$$ □

Proof of Theorem A

2.2. Parallel Lee form. Let again φ_t be the 1-parameter subgroup generated by ξ. According to the notation in Lemma 2.3, let $Y_{\varphi_t w} \in T_{\varphi_t w}\tilde{M}$ be any vector. We have $\pi_* Y_{\varphi_t w} \in T_w W$, and hence $i_*\pi_* Y_{\varphi_t w} - \varphi_{-t\,*} Y_{\varphi_t w} = \lambda \xi_w$ for some number λ. Then,

$$\begin{aligned}\iota_\xi \Omega(i_*\pi_* Y_{\varphi_t w}) &= \Omega(\xi_w, i_*\pi_* Y_{\varphi_t w}) = \Omega(\xi_w, \varphi_{-t\,*} Y_{\varphi_t w}) + \Omega(\xi_w, \lambda\xi_w)\\ &= \varphi^*_{-t}\Omega(\varphi_{t*}\xi_w, Y_{\varphi_t w}) = e^{-t}\Omega(\xi_{\varphi_t w}, Y_{\varphi_t w}) = e^{-t}\iota_\xi\Omega(Y_{\varphi_t w})\,.\end{aligned}$$

By the definition (2.6),

$$\pi^*\eta = \pi^* i^* \iota_\xi \Omega = e^{-t}\iota_\xi\Omega, \quad \text{equivalently,} \quad e^t \pi^*\eta = \iota_\xi\Omega\,. \tag{2.14}$$

As $\Omega = \mathcal{L}_\xi \Omega = d\iota_\xi\Omega$ from (2.5), we obtain that

$$d(e^t \pi^*\eta) = \Omega \quad \text{on } \tilde{M}. \tag{2.15}$$

For the given l.c.K. metric g, the Kähler metric h is obtained as $h = e^{-\tau} \cdot p^* g$ where $d\tau = \tilde{\theta}$. As ω is the fundamental 2-form of g, note that $\Omega = e^{-\tau} \cdot p^*\omega$.

We now consider on $\tilde{M}$ the 2-form:

$$\bar{\Theta} = 2e^{-t} \cdot d(e^t \pi^* \eta) \ (= 2e^{-t} \cdot \Omega)\,. \tag{2.16}$$

Then $\bar{g}(X, Y) = \bar{\Theta}(JX, Y)$ is a l.c.K. metric. Put $\bar{\theta} = -dt$. Then, as $d\bar{\Theta} = -2e^{-t} dt \wedge d(e^t \pi^* \eta) = -dt \wedge \bar{\Theta}$, we see that $\bar{\theta}$ is the Lee form of $\bar{g}$.

LEMMA 2.5. *$\bar{\theta}$ is parallel with respect to $\bar{g}$* ($\nabla^{\bar{g}} \bar{\theta} = 0$).

PROOF. First we determine the Lee field $\bar{\theta}^\sharp$ (where $\bar{\theta}(X) = \bar{g}(X, \bar{\theta}^\sharp)$). We start from:

$$\begin{aligned}\bar{g}(\xi, Y) &= \bar{\Theta}(J\xi, Y) = 2e^{-t}(e^t dt \wedge \pi^*\eta + e^t d\pi^*\eta)(J\xi, Y)\\ &= 2(dt \wedge \pi^*\eta + d\pi^*\eta)(J\xi, Y) = 2(dt \wedge \pi^*\eta)(J\xi, Y),\end{aligned}$$

because $A = -\pi_* J\xi$ is the characteristic vector field of the contact form η. As before, a point $x \in \tilde{M}$ can be described uniquely as $\varphi_t w$ for some $w \in W$. In particular, by Lemma 2.3, the t-coordinate of x is t. Noting that $\psi_\theta(x) = \varphi_t \psi_\theta w$ and $\psi_\theta w \in W$, by the uniqueness of the t-coordinate of $\psi_\theta(x)$, $t(\psi_\theta(x)) = t$. From (2.7),

$$dt(-J\xi_x) = dt\left(\frac{d\psi_\theta}{d\theta}(x)\Big|_{\theta=0}\right) = \frac{dt}{d\theta}\Big|_{\theta=0} = 0\,. \tag{2.17}$$

The above formula becomes:

$$\begin{aligned}\bar{g}(\xi, Y) &= 2(dt \wedge \pi^*\eta)(J\xi, Y) = -dt(Y)\eta(-A)\\ &= dt(Y) = -\bar{\theta}(Y) = -\bar{g}(Y, \bar{\theta}^\sharp)\,,\end{aligned} \tag{2.18}$$

proving that $\bar{\theta}^\sharp = -\xi$. Next we observe that the flow $\{\varphi_s\}_{s \in \boldsymbol{R}}$ acts by isometries with respect to $\bar{g}$. As φ_s is holomorphic, it is enough to prove that each φ_s leaves $\bar{\Theta}$ invariant, but

$$\varphi_s^* \bar{\Theta} = 2e^{-\varphi_s^* t} d(e^{\varphi_s^* t} \varphi_s^* \pi^* \eta) = 2e^{-(s+t)} d(e^{s+t} \pi^* \eta) = 2e^{-t} d(e^t \pi^* \eta) = \bar{\Theta}\,.$$

Thus $\mathcal{L}_{\bar{\theta}^\sharp} \bar{g} = -\mathcal{L}_\xi \bar{g} = 0$. Now we put $\sigma = \bar{\theta}$ in the equality $(\mathcal{L}_{\sigma^\sharp} \bar{g})(X, Y) + 2d\sigma(X, Y) = 2\bar{g}(\nabla^{\bar{g}}_X \sigma^\sharp, Y)$, valid for any 1-form σ, take into account $d\bar{\theta} = 0$ and obtain $\nabla^{\bar{g}} \bar{\theta}^\sharp = 0$, which is equivalent with $\nabla^{\bar{g}} \bar{\theta} = 0$, so $\bar{\theta}$ is parallel with respect to $\bar{g}$ as announced. □

By the equation (2.16), $\bar{g}$ is conformal to the lifted metric $p^* g$:

$$\bar{\Theta} = \mu \cdot p^* \omega \ \ (\text{equivalently } \bar{g} = \mu \cdot p^* g)\,, \tag{2.19}$$

where $\mu = 2e^{-(t+\tau)} : \tilde{M} \to \boldsymbol{R}^+$ is a smooth map. We finally prove:

LEMMA 2.6. *$\pi_1(M)$ acts by holomorphic isometries of $\bar{g}$. In particular, $\pi_1(M)$ leaves $\bar{\theta}$ invariant.*

PROOF. We prove the following two facts:

1. $\gamma^* \pi^* \eta = \pi^* \eta$ for every $\gamma \in \pi_1(M)$.
2. $\gamma^* e^t = \rho(\gamma) \cdot e^t$, where $\rho : \pi_1(M) \to \boldsymbol{R}^+$ is the restriction of the homomorphism defined in (2.1).

First note that as $\boldsymbol{R} = \{\varphi_t\}$ centralizes $\pi_1(M)$, $\gamma_* \xi = \xi$ for $\gamma \in \pi_1(M)$. As γ is holomorphic, $\gamma_* J\xi = J\xi$. Since $\pi_1(M)$ acts on $\tilde{M}$ as holomorphic homothetic transformations, (i.e., $\gamma^*\Omega = \rho(\gamma)\cdot\Omega$), $\pi_1(M)$ preserves $\{\xi, J\xi\}^{\perp}$. If we recall that $\pi_* : \{\xi, J\xi\}^{\perp} \to \text{Null}\ \eta$ is isomorphic, then for $X \in \{\xi, J\xi\}^{\perp}$, $\gamma^*\pi^*\eta(X) = \eta(\pi_*\gamma_* X) = 0$. As $-\pi_* J\xi = A$ is the characteristic field of η, it follows that $\gamma^*\pi^*\eta(J\xi) = \eta(\pi_*\gamma_* J\xi) = \eta(\pi_* J\xi) = -1$. This shows that $\gamma^*\pi^*\eta = \pi^*\eta$ on $\tilde{M}$. On the other hand, if we note $\gamma_*\xi = \xi$, then

$$\begin{aligned}\gamma^*(\iota_\xi\Omega)(X) &= \Omega(\xi, \gamma_* X) = \Omega(\gamma_*\xi, \gamma_* X) = \gamma^*\Omega(\xi, X)\\ &= \rho(\gamma)\cdot\Omega(\xi, X) = \rho(\gamma)\cdot\iota_\xi\Omega(X)\,,\end{aligned}$$

where $\rho(\gamma)$ is a positive constant. Applying γ^* to $\pi^*\eta = e^{-t}\cdot\iota_\xi\Omega$ from (2.14), we obtain $\gamma^* e^{-t}\cdot\rho(\gamma) = e^{-t}$. Equivalently, $\gamma^* e^t = \rho(\gamma)\cdot e^t$. This shows 1 and 2. From (2.16),

$$\begin{aligned}\gamma^*\bar{\Theta} &= \gamma^*(2e^{-t}\cdot d(e^t\pi^*\eta)) = 2\rho(\gamma)^{-1}\cdot e^{-t} d(\rho(\gamma)\cdot e^t\gamma^*\pi^*\eta)\\ &= 2e^{-t}\cdot d(e^t\pi^*\eta) = \bar{\Theta}\,.\end{aligned}$$

Since $\bar{g}(X, Y) = \bar{\Theta}(JX, Y)$, $\pi_1(M)$ acts through holomorphic isometries of $\bar{g}$. We have that $\bar{\theta}(Y) = \bar{g}(Y, \bar{\theta}^\sharp) = -\bar{g}(Y, \xi)$ $(Y \in T\tilde{M})$ from (2.18). Then,

$$\gamma^*\bar{\theta}(Y) = -\bar{g}(\gamma_* Y, \xi) = -\bar{g}(\gamma_* Y, \gamma_*\xi) = -\bar{g}(Y, \xi) = \bar{\theta}(Y)\,. \qquad \square$$

From this lemma, the covering map $p : \tilde{M} \to M$ induces a l.c.K. metric $\hat{g}$ with parallel Lee form $\hat{\theta}$ on M such that $p^*\hat{g} = \bar{g}$ and $p^*\hat{\theta} = \bar{\theta}$ with $\nabla^{\hat{g}}_{p_* X}\hat{\theta}(p_* Y) = \nabla^{\bar{g}}_X\bar{\theta}(Y)$. Applying γ^* to both sides of (2.19), we derive

$$\gamma^*\bar{g} = \bar{g} = \mu\cdot p^*\,, \quad \gamma^*\mu\cdot\gamma^* p^* g = \gamma^*\mu\cdot p^* g\,.$$

Therefore $\gamma^*\mu = \mu$, which implies that μ factors through a map $\hat{\mu} : M \to \boldsymbol{R}^+$ so that $p^*\hat{g} = p^*(\hat{\mu}\cdot g)$. We have $\hat{\mu}\cdot g = \hat{g}$. The conformal class of g contains a l.c.K. metric $\hat{g}$ with parallel Lee form $\hat{\theta}$. This ends the proof of Theorem A. $\square$

As to Corollary A_1 in the Introduction, we recall the following. (Compare [18], [6, p. 37].) Let (M, g, J) be a compact, connected, non-Kähler, l.c.K. manifold with parallel Lee form θ. Then the following results hold: $g(\theta^\sharp, \theta^\sharp)$ =const,

$$\mathcal{L}_{\theta^\sharp} J = \mathcal{L}_{J\theta^\sharp} J = 0\,, \quad \mathcal{L}_{\theta^\sharp} g = \mathcal{L}_{J\theta^\sharp} g = 0\,.$$

Then $Z = \theta^\sharp - \mathrm{i}J\theta^\sharp$ is a holomorphic vector field because $[\theta^\sharp, J\theta^\sharp] = 0$ (cf. [12]). By Definition 1.1, $Z = \theta^\sharp - \mathrm{i}J\theta^\sharp$ is a holomorphic l.c.K. vector field.

PROPOSITION 2.4. *The real vector fields $\theta^\sharp$ and $J\theta^\sharp$ satisfy the following*:

1. *A flow generated by the Lee field $\theta^\sharp$ lifts to a one-parameter subgroup of nontrivial homothetic holomorphic transformations with respect to Ω.*
2. *A flow generated by the anti-Lee field $-J\theta^\sharp$ lifts to a one-parameter subgroup consisting of holomorphic isometries with respect to Ω.*

PROOF. Let $\{\hat{\varphi}_t\}_{t\in\boldsymbol{R}}$ be the flow generated by $\theta^\sharp$ on M and $\{\varphi_t\}_{t\in\boldsymbol{R}}$ its lift to $\tilde{M}$. Denote by ξ the vector field on $\tilde{M}$ induced by $\{\varphi_t\}$. Then, $p_*\xi = \theta^\sharp$. Because θ is parallel, $\{\hat{\varphi}_t\}$ (resp. $\{\varphi_t\}$) acts by holomorphic isometries with respect to g (resp. p^*g). In particular, $\{\varphi_t\}$ preserves $p^*\omega$. Then, for $\Omega = e^{-\tau}p^*\omega$, we have $\varphi_t^*\Omega = e^{-(\varphi_t^*\tau - \tau)}\Omega$. As $\rho : \{\varphi_t\}_{t\in\boldsymbol{R}} \to \boldsymbol{R}^+$

is a homomorphism and $\rho(\varphi_t) = e^{-(\varphi_t^*\tau-\tau)}$ is constant for each $t \in \boldsymbol{R}$ ($\dim_{\boldsymbol{C}} M \geq 2$), we can describe as $-(\varphi_t^*\tau - \tau) = c \cdot t$ for some constant c. Recall that h is the Kähler metric associated to Ω. If $\{\varphi_t\}$ acts as holomorphic isometries with respect to h, then the above equation implies that $c = 0$, i.e., $\varphi_t^*\tau - \tau = 0$ for every t, and so $\mathcal{L}_\xi \tau = 0$. On the other hand, as $d\tau = p^*\theta$, we have:

$$0 = \mathcal{L}_\xi \tau = d\tau(\xi) = \theta(p_*\xi) = \theta(\theta^\sharp) = \text{const.} > 0,$$

a contradiction. Thus, $\varphi_t^*\Omega = \rho(\varphi_t)\Omega = e^{c\cdot t}\Omega$ with $c \neq 0$. Hence, $\{\varphi_t\}_{t\in\boldsymbol{R}}$ is a group of nontrivial homothetic holomorphic transformations isomorphic to $\boldsymbol{R}$. On the other hand, let $\{\hat{\psi}_t\}_{t\in\boldsymbol{R}}$ (resp. $\{\psi_t\}_{t\in\boldsymbol{R}}$) be the flow generated by $-J\theta^\sharp$ on M (resp. $-J\xi$ on $\tilde{M}$). As $p_*(J\xi) = Jp_*\xi = J\theta^\sharp$,

$$\mathcal{L}_{J\xi}\tau = d\tau(J\xi) = p^*\theta(J\xi) = \theta(J\theta^\sharp) = g(J\theta^\sharp, \theta^\sharp) = 0\,,$$

and hence $\psi_t^*\tau = \tau$ for every $t \in \boldsymbol{R}$. By the fact that $\mathcal{L}_{J\theta^\sharp} g = 0$, $\mathcal{L}_{J\theta^\sharp}\omega = 0$. This implies that $\psi_t^*\Omega = \psi_t^* e^{-\tau}\psi_t^* p^*\omega = e^{-\tau} p^*\hat{\psi}_t^*\omega = e^{-\tau}p^*\omega = \Omega$. □

Let $\boldsymbol{R} \to \tilde{M} \stackrel{\pi}{\to} W$ be the principal bundle, where $\boldsymbol{R} = \{\varphi_t\}_{t\in\boldsymbol{R}}$ (cf. Lemma 2.2). Define the centralizer of $\boldsymbol{R}$ in $\mathcal{H}(\tilde{M}, \Omega, J)$ to be:

DEFINITION 2.1. $\mathcal{C}_{\mathcal{H}}(\boldsymbol{R}) = \{f \in \mathcal{H}(\tilde{M}, \Omega, J) \mid f \circ \varphi_t = \varphi_t \circ f$ for *all* $t \in \boldsymbol{R}\}$.

As $\tilde{\mathcal{C}}$ centralizes the fundamental group $\pi_1(M)$, noting the remark below (2.1), we have

$$\pi_1(M) \subset \mathcal{C}_{\mathcal{H}}(\boldsymbol{R})\,. \tag{2.20}$$

LEMMA 2.7. *There exists a homomorphism* $\nu : \mathcal{C}_{\mathcal{H}}(\boldsymbol{R}) \to \mathrm{PSH}(W, \eta, J)$ *for which* $\pi : \tilde{M} \to W$ *becomes* ν*-equivariant. Moreover, there exists a splitting homomorphism* q : $\mathrm{PSH}(W, \eta, J) \to \mathcal{C}_{\mathcal{H}}(\boldsymbol{R})$.

PROOF. By definition, any element $f \in \mathcal{C}_{\mathcal{H}}(\boldsymbol{R})$ satisfies $f_*\xi = \xi$. As $f^*\Omega = \rho(f)\Omega$, choosing $e^s = \rho(f)$, put $\gamma = \varphi_{-s} \circ f$. Then, $\gamma^*\Omega = \Omega$. In particular, γ leaves W invariant. Let γ' be the restriction of γ to W (i.e., $i \circ \gamma' = \gamma$). Using (2.6) and $\gamma_*\xi = \xi$, we have that $\gamma'^*\eta = \gamma^*\mathcal{L}_\xi\Omega = \mathcal{L}_\xi\Omega = \eta$. Hence $\gamma' \in \mathrm{PSH}(W, \eta, J)$. If we define $\nu(f) = \gamma'$, then it is easy to see that ν is a well-defined homomorphism. Let $x = \varphi_t w$ be a point in $\tilde{M}$. As $\pi(x) = w$, $\pi(fx) = \pi(\varphi_s\gamma(\varphi_t w)) = \pi(\varphi_s\varphi_t i\gamma' w) = \pi(i\gamma' w) = \gamma' w = \nu(f)\pi(x)$, so π is ν-equivariant.

For $\gamma \in \mathrm{PSH}(W, \eta, J)$, we define a diffeomorphism $\tilde{\gamma} : \tilde{M} \to \tilde{M}$ to be

$$\tilde{\gamma}(x) = \tilde{\gamma}(\varphi_t w) = \varphi_t \gamma w\,. \tag{2.21}$$

By definition, $\pi \circ \tilde{\gamma} = \gamma \circ \pi$ and the t-coordinate satisfies that $\tilde{\gamma}^* t = t$. By (2.15) and $\gamma^*\eta = \eta$, it follows that $\tilde{\gamma}^*\Omega = d(e^{\gamma^* t}\pi^*\gamma^*\eta) = d(e^t\pi^*\eta) = \Omega$. To see that $\tilde{\gamma} : \tilde{M} \to \tilde{M}$ is holomorphic, notice that $\tilde{\gamma}_*\xi = \xi$. As $\tilde{\gamma}(\psi_\theta x) = \tilde{\gamma}(\psi_\theta\varphi_t w) = \tilde{\gamma}(\varphi_t i\psi'_\theta w) = \varphi_t i\gamma\psi'_\theta w$, and $\gamma_* A = A$,

$$\begin{aligned}\tilde{\gamma}_*(-J\xi_x) &= \tilde{\gamma}_*\left(\frac{d\psi_\theta}{d\theta}(x)\bigg|_{\theta=0}\right) = \left(\frac{d\varphi_t i\gamma(\psi'_\theta w)}{d\theta}\bigg|_{\theta=0}\right)\\ &= \varphi_{t*} i_* \gamma_* \left(\frac{d\psi'_\theta}{d\theta}(w)\bigg|_{\theta=0}\right) = \varphi_{t*} i_* \gamma_* A_w = \varphi_{t*} i_* A_{\gamma w}\\ &= \varphi_{t*}(-J\xi_{\gamma w}) = -J\xi_{\tilde{\gamma} x}\,.\end{aligned} \tag{2.22}$$

Hence, $\tilde{\gamma}$ preserves $\{\xi, J\xi\}^\perp$. Since the complex structure $J : \mathrm{Null}\,\eta \to \mathrm{Null}\,\eta$ is defined by the commutative diagram (2.11), $J\gamma_*(\pi_* X) = \gamma_* J(\pi_* X)$ for $X \in \{\xi, J\xi\}^\perp$ by definition. Then $\pi_* \tilde{\gamma}_* J(X) = J\gamma_* \pi_*(X) = J\pi_* \tilde{\gamma}_*(X) = \pi_* J\tilde{\gamma}_*(X)$. As a consequence, $\tilde{\gamma}_* \circ J = J \circ \tilde{\gamma}_*$ on $\tilde{M}$. Hence, $\tilde{\gamma} \in \mathcal{C}_{\mathcal{H}}(\boldsymbol{R})$. It is easy to check that $q(\gamma) = \tilde{\gamma}$ is a homomorphism of $\mathrm{PSH}(W, \eta, J)$ into $\mathcal{C}_{\mathcal{H}}(\boldsymbol{R})$ such that $\nu \circ q = \mathrm{id}$. □

REMARK 2.1. From this lemma, there is an isomorphism $\mathcal{C}_{\mathcal{H}}(\boldsymbol{R}) \approx \boldsymbol{R} \times \mathrm{PSH}(W, \eta, J)$, where each element of $\mathcal{C}_{\mathcal{H}}(\boldsymbol{R})$ is described as $\varphi_s \cdot q(\alpha)$ for $s \in \boldsymbol{R}$, $\alpha \in \mathrm{PSH}(W, \eta, J)$. It acts on $\tilde{M}$ as

$$\varphi_s \cdot q(\alpha)(\varphi_t \cdot w) = \varphi_{s+t} \cdot \alpha w\,,$$

for which there is an equivariant principal bundle:

$$\boldsymbol{R} \longrightarrow (\mathcal{C}_{\mathcal{H}}(\boldsymbol{R}), \tilde{M}) \xrightarrow{(\nu,\pi)} (\mathrm{PSH}(W, \eta, J), W)\,.$$

2.3. Central group extension. The material in this subsection and, in particular, Proposition 2.5, will be needed in Section 4.

Consider the exact sequence:

$$1 \longrightarrow \boldsymbol{R} \longrightarrow \mathcal{C}_{\mathcal{H}}(\boldsymbol{R}) \xrightarrow{\nu} \mathrm{PSH}(W, \eta, J) \longrightarrow 1\,. \tag{2.23}$$

Suppose that $\boldsymbol{R} \cap \pi_1(M)$ is nontrivial. Then it is an infinite cyclic subgroup $\boldsymbol{Z}$ such that the quotient group $\boldsymbol{R}/\boldsymbol{Z}$ is a circle S^1. Put $Q = \nu(\pi_1(M)) \subset \mathrm{PSH}(W, \eta, J)$. We have a central group extension:

$$1 \longrightarrow \boldsymbol{Z} \longrightarrow \pi_1(M) \xrightarrow{\nu} Q \longrightarrow 1\,. \tag{2.24}$$

The above principal bundle restricts to the following one:

$$(\boldsymbol{Z}, \boldsymbol{R}) \longrightarrow (\pi_1(M), \tilde{M}) \xrightarrow{(\nu,\pi)} (Q, W). \tag{2.25}$$

As both $\boldsymbol{R}$ and $\pi_1(M)$ act properly on $\tilde{M}$, Q acts also properly discontinuously (but not necessarily freely) on W such that the quotient Hausdorff space W/Q is compact. Since $\rho(\boldsymbol{Z}) \subset \rho(\boldsymbol{R}) = \boldsymbol{R}^+$ from § 2.1, $\rho(\boldsymbol{Z})$ is an infinite cyclic subgroup of $\boldsymbol{R}^+$. We need the following lemma. (Compare [10], [5].)

LEMMA 2.8. *Let* $1 \to \boldsymbol{Z} \to \pi_1(M) \xrightarrow{\nu} Q \to 1$ *be the central extension as given in* (2.24). *Then,* $\pi_1(M)$ *has a splitting subgroup* π' *of finite index*:

$$1 \longrightarrow \boldsymbol{Z} \longrightarrow \pi' \xrightarrow{\nu} Q' \longrightarrow 1\,.$$

In particular, there exists a subgroup H' of π' which maps isomorphically onto a subgroup Q' of finite index in Q.

PROOF. Consider the homomorphism $\rho' = \rho|_{\pi_1(M)} : \pi_1(M) \to \boldsymbol{R}^+$ from (2.1). Then, $\rho'(\pi_1(M))$ is a free abelian group of rank $k \geq 1$. If we note that $\rho'(\boldsymbol{Z})$ is an infinite cyclic subgroup of $\rho'(\pi_1(M))$, then we can choose a subgroup G of finite index in $\rho'(\pi_1(M))$ such that $\rho'(\boldsymbol{Z})$ is a direct summand in G; $G = \rho'(\boldsymbol{Z}) \times \boldsymbol{Z}^{k-1}$. Put $\pi' = \rho'^{-1}(G)$ and $H' = \rho'^{-1}(\boldsymbol{Z}^{k-1})$. Then, π' has finite index in $\pi_1(M)$. Obviously, ν maps H' isomorphically onto $\nu(H') = Q'$, which is of finite index in Q. □

PROPOSITION 2.5. *The subgroup Q' acts freely on W so that the orbit space W/Q' is a closed strictly pseudoconvex pseudo-Hermitian manifold induced from the pseudo-Hermitian structure (η, J) on W.*

PROOF. Let $f = \nu'^{-1} : Q' \to H'$ be the inverse isomorphism. For each $\alpha' \in Q'$ there exists a unique element $\lambda(\alpha') \in \boldsymbol{R}$ such that $f(\alpha') = \varphi_{\lambda(\alpha')} \cdot q(\alpha')$. As we know that Q acts properly discontinuously on W from the remark below (2.25), the stabilizer at each point is finite. Suppose that $\alpha' w = w$ for some point $w \in W$. As $\alpha' \in Q_w$, $(\alpha')^l = 1$ for some l. Since φ_t is a central element and q is a homomorphism, $1 = f((\alpha')^l) = \varphi_{l\lambda(\alpha')} \cdot q((\alpha')^l) = \varphi_{l\lambda(\alpha')}$. Thus, $\lambda(\alpha') = 0$, i.e., $f(\alpha') = q(\alpha')$. By the definition of the action $(\pi', \tilde{M})$, $f(\alpha')(\varphi_t w) = q(\alpha')(\varphi_t w) = \varphi_t \alpha' w = \varphi_t w$. As π' acts freely on $\tilde{M}$, $f(\alpha') = 1$ and so $\alpha' = 1$. If we note that $Q' \subset \mathrm{PSH}(W, \eta, J)$, then (η, J) induces a pseudo-Hermitian structure $(\hat{\eta}, J)$ on W/Q'. Here we use the same notation J for the complex structure on Null $\hat{\eta}$. □

3. Examples of l.c.K. manifolds with parallel Lee form. In this section we present an explicit construction for the Hopf manifolds.

Let $S^{2n-1} = \{(z_1, \ldots, z_n) \in \boldsymbol{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 = 1\}$ be the sphere endowed with its standard contact structure

$$\eta_0 = \sum_{j=1}^{n} (x_j dy_j - y_j dx_j)\,, \tag{3.1}$$

where $z_j = x_j + \sqrt{-1}\, y_j$. Let J_0 be the restriction of the standard complex structure of $\boldsymbol{C}^n$ to $\boldsymbol{C}^n - \{0\}$. It is known that the group of pseudo-Hermitian transformations, $\mathrm{PSH}(S^{2n-1}, \eta_0, J_0)$ is isomorphic with $\mathrm{U}(n)$ (see [21], for example). We define a 1-parameter subgroup $\{\psi_t\}_{t \in \boldsymbol{R}} \subset \mathrm{PSH}(S^{2n-1}, \eta_0, J_0)$ by the formula:

$$\psi_t(z_1, \ldots, z_n) = (e^{\mathrm{i}ta_1} z_1, \ldots, e^{\mathrm{i}ta_n} z_n)\,,$$

where $\mathrm{i} = \sqrt{-1}$ and $a_1, \ldots, a_n \in \boldsymbol{R}$. The vector field induced by this action is

$$A = \sum_{j=1}^{n} a_j \left(x_j \frac{d}{dy_j} - y_j \frac{d}{dx_j} \right)$$

and satisfies $\eta_0(A) = a_1 |z_1|^2 + \cdots + a_n |z_n|^2$.

Now we require that $\eta_0(A) > 0$ everywhere on S^{2n-1}. Then the numbers a_k must satisfy (up to rearrangement):

$$0 < a_1 \leq \cdots \leq a_n \,. \tag{3.2}$$

Define a new contact form η_A on the sphere by

$$\eta_A = \frac{1}{\sum_{j=1}^{n} a_j |z_j|^2} \cdot \eta_0 \,.$$

The contact distributions of η_0 and η_A coincide, but the characteristic field of η_A is A: $\eta_A(A) = 1$, $\iota_A d\eta_A = 0$. As A generates the flow $\{\psi_t\}_{t\in \boldsymbol{R}} \subset \mathrm{PSH}(S^{2n-1}, \eta_0, J_0)$, note that $\psi_{t*} \circ J_0 = J_0 \circ \psi_{t*}$ on $\mathrm{Null}\,\eta_A$. Define a 2-form on the product $\boldsymbol{R} \times S^{2n-1}$ by:

$$\Omega_A = 2d(e^t \mathrm{pr}^* \eta_A)\,, \quad t \in \boldsymbol{R}\,.$$

Here $\mathrm{pr} : \boldsymbol{R} \times S^{2n-1} \to S^{2n-1}$ is the projection. If $\boldsymbol{R} = \{\varphi_s\}_{s\in\boldsymbol{R}}$ acts on $\boldsymbol{R} \times S^{2n-1}$ by left translations: $\varphi_s(t, z) = (s + t, z)$, then the group $\boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ acts by homothetic transformations with respect to Ω_A:

$$(\varphi_s \times \alpha)^* \Omega_A = e^s \cdot \Omega_A\,, \quad \alpha \in \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)\,. \tag{3.3}$$

In general, $\mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ is the centralizer of $\{\psi_t\}_{t\in\boldsymbol{R}}$ in $\mathrm{U}(n)$. In view of the formula of ψ_t, $\mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ contains at least the maximal torus of $\mathrm{U}(n)$:

$$T^n \subset \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)\,. \tag{3.4}$$

(For example, if all a_j are distinct, $\mathrm{PSH}(S^{2n-1}, \eta_A, J_0) = T^n$.)

Let $N = d/dt$ be the vector field induced on $\boldsymbol{R} \times S^{2n-1}$ by the $\boldsymbol{R}$-action. Taking into account that $T(\boldsymbol{R} \times S^{2n-1}) = N \oplus A \oplus \mathrm{Null}\,\eta_A$, we define an almost complex structure J_A on $\boldsymbol{R} \times S^{2n-1}$ by

$$J_A N = -A\,, \quad J_A A = N\,,$$
$$J_A|\mathrm{Null}\,\eta_A = J_0\,,$$

and show its integrability. Indeed, let

$$T(\boldsymbol{R} \times S^{2n-1}) \otimes \boldsymbol{C} = \{T^{1,0} + (A - \mathrm{i}N)\} \oplus \{T^{0,1} + (A + \mathrm{i}N)\}$$

be the splitting corresponding to J_A (here $T^{1,0} + T^{0,1} = \mathrm{Null}\,\eta_A \otimes \boldsymbol{C}$). As $J_A|\mathrm{Null}\,\eta_A = J_0$, $[T^{1,0}, T^{0,1}] \subset T^{1,0}$. Recalling that A is the characteristic field of η_A, we see that $[X, A] \in \mathrm{Null}\,\eta_A$ for any $X \in \mathrm{Null}\,\eta_A$. If $X \in T^{1,0}$, then $[X, A - \mathrm{i}N] = [X, A] = \lim_{t\to 0}(X - \psi_{-t*}X)/t$. Noting that $\psi_t \in \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ (i.e., $\psi_{t*}J_0 = J_0\psi_{t*}$),

$$J_A[X, A - \mathrm{i}N] = J_0[X, A] = \lim_{t\to 0} \frac{J_0X - \psi_{-t*}J_0X}{t} = [J_0X, A]$$
$$= [\mathrm{i}X, A] = \mathrm{i}[X, A] = \mathrm{i}[X, A - \mathrm{i}N]\,.$$

Thus $[X, A - \mathrm{i}N] \in \{T^{1,0} + (A - \mathrm{i}N)\}$. Hence J_A is integrable. By the definition of J_A, it is easy to check that the elements of $\boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ are holomorphic with respect to J_A. Moreover, Ω_A is J_A-invariant. Hence, Ω_A is a Kähler form on the complex manifold

$(\boldsymbol{R} \times S^{2n-1}, J_A)$ on which $\boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ acts as the group of holomorphic homothetic transformations. Define a Hermitian metric $\tilde{g}_A$ and its fundamental 2-form $\tilde{\omega}_A$ by setting

$$\tilde{\omega}_A = 2e^{-t} \cdot \Omega_A\,. \tag{3.5}$$
$$\tilde{g}_A(X, Y) = \tilde{\omega}_A(J_A X, Y)\,, \quad X, Y \in T(\boldsymbol{R} \times S^{2n-1})\,.$$

(Compare (2.16).) By (3.3), $\boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ acts as holomorphic isometries of $(\tilde{g}_A, J_A)$. When we choose a properly discontinuous group $\Gamma \subset \boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ acting freely on $\boldsymbol{R} \times S^{2n-1}$, $\tilde{g}_A$ (resp. $\tilde{\omega}_A$) induces a Hermitian metric g_A (resp. the fundamental 2-form ω_A) on the quotient complex manifold $(\boldsymbol{R} \times S^{2n-1}/\Gamma, \hat{J}_A)$, where the complex structure $\hat{J}_A$ is induced from J_A. We have to check that g_A is a l.c.K. metric with parallel Lee form. Let $p : \boldsymbol{R} \times S^{2n-1} \to \boldsymbol{R} \times S^{2n-1}/\Gamma$ be the projection so that $p^*\omega_A = \tilde{\omega}_A$. Since $\tilde{\omega}_A = e^{-t} \cdot \Omega_A$, we have $d\tilde{\omega}_A = -dt \wedge \tilde{\omega}_A$. Thus $\tilde{g}_A$ is a l.c.K. metric with Lee form $d(-t)$ on $\boldsymbol{R} \times S^{2n-1}$. If we note that the group $\boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ leaves $d(-t)$ invariant, i.e., $(\varphi_s \times \alpha)^* d(-t) = d(-(s+t)) = d(-t)$, then $d(-t)$ induces a 1-form θ on $\boldsymbol{R} \times S^{2n-1}/\Gamma$ such that $p^*\theta = d(-t)$. The equation $d\tilde{\omega}_A = -dt \wedge \tilde{\omega}_A$ implies that $d\omega_A = \theta \wedge \omega_A$ on $\boldsymbol{R} \times S^{2n-1}/\Gamma$. As $d\theta = 0$, g_A is a l.c.K. metric with Lee form θ. For the rest, the same argument as in the proof of Lemma 2.5 can be applied to show that θ is the parallel Lee form of g_A. Finally, we examine the complex structure $\hat{J}_A$ on $\boldsymbol{R} \times S^{2n-1}/\Gamma$.

Let $H : \boldsymbol{R} \times S^{2n-1} \to \boldsymbol{C}^n - \{0\}$ be the diffeomorphism defined by

$$H(t, (z_1, \ldots, z_n)) = (e^{-a_1 t} z_1, \ldots, e^{-a_n t} z_n),$$

where $\{a_1, \ldots, a_n\}$ satisfies the condition (3.2). We shall show that H is (J_A, J_0)-biholomorphic. We have:

$$H_*(N_{(s,z)}) = \left.\frac{dH(t+s, z)}{dt}\right|_{t=0} = (-a_1 \cdot e^{-a_1 s} \cdot z_1, \ldots, -a_n \cdot e^{-a_n s} \cdot z_n)\,;$$
$$H_*(J_A N_{(s,z)}) = H_*(-A_{(s,z)}) = -H_*\left(\left(s, \left.\frac{d}{dt}(e^{\mathrm{i}ta_1} z_1, \ldots, e^{\mathrm{i}ta_n} z_n)\right|_{t=0}\right)\right)$$
$$= -(\mathrm{i}a_1 e^{-a_1 s} z_1, \ldots, \mathrm{i}a_n e^{-a_n s} z_n) = J_0 H_*(N_{(s,z)})\,.$$

From $H_*(A_{(s,z)}) = -J_0 H_*(N_{(s,z)})$, we derive $J_0 H_*(A_{(s,z)}) = H_*(N_{(s,z)}) = H_*(J_A A)$. Now let $X \in \mathrm{Null}\,\eta_A \subset TS^{2n-1}$ and let $\sigma(t)$ be an integral curve of X on S^{2n-1}: $\dot{\sigma}(t) = X$, $\dot{\sigma}(0) = X_z$. We can view X as a pair: $X_{(s,z)} = (s, \dot{\sigma}(0))$. Then

$$H_*(X_{(s,z)}) = \frac{d}{dt} H(s, \sigma(t))|_{t=0} = (e^{-a_1 s}\dot{\sigma}_1(0), \ldots, e^{-a_n s}\dot{\sigma}_n(0))\,.$$

From this we obtain

$$H_*(J_A X_{(s,z)}) = H_*((s, J_0\dot{\sigma}(0))) = H_*((s, (\mathrm{i}\dot{\sigma}_1(0), \ldots, \mathrm{i}\dot{\sigma}_n(0))))$$
$$= (\mathrm{i}e^{-a_1 s}\dot{\sigma}_1(0), \ldots, \mathrm{i}e^{-a_n s}\dot{\sigma}_n(0))$$
$$= J_0(e^{-a_1 s}\dot{\sigma}_1(0), \ldots, e^{-a_n s}\dot{\sigma}_n(0)) = J_0 H_*(X_{(s,z)})\,.$$

Therefore $H : (\boldsymbol{R} \times S^{2n-1}, J_A) \to (\boldsymbol{C}^n - \{0\}, J_0)$ is biholomorphic.

Let $\mathrm{Hol}(\boldsymbol{C}^n - \{0\}, J_0)$ be the group of all biholomorphic transformations. We can obtain a faithful homomorphism $\boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_A, J_0) \to \mathrm{Hol}(\boldsymbol{C}^n - \{0\}, J_0)$ by associating to each $\gamma \in \boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ the biholomorphic map $H \circ \gamma \circ H^{-1}$. Let Γ_H be the image of Γ in $\mathrm{Hol}(\boldsymbol{C}^n - \{0\}, J_0)$.

DEFINITION 3.1. The quotient complex manifold $(\boldsymbol{C}^n - \{0\})/\Gamma_H$ is called a Hopf manifold.

Since our map H induces a holomorphic diffeomorphism $\hat{H} : (\boldsymbol{R} \times S^{2n-1})/\Gamma \to (\boldsymbol{C}^n - \{0\})/\Gamma_H$, letting $\hat{H}^* g = g_A$ for the *l.c.K.* metric g_A on $(\boldsymbol{R} \times S^{2n-1})/\Gamma$, we have shown:

THEOREM 3.1. *The Hopf manifold $(\boldsymbol{C}^n - \{0\})/\Gamma_H$ admits a l.c.K. metric g with parallel Lee form θ.*

By (3.4), $T^n \subset \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$. Choose $s \in \boldsymbol{R} - \{0\}$ and n complex numbers $c_1, \dots, c_n \in S^1$. Let $(s, (c_1, \dots, c_n)) \in \boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ and consider an infinite cyclic subgroup $\boldsymbol{Z}$ generated by this element. Then the corresponding group $\boldsymbol{Z}_H$ is generated by the element $(e^{-a_1 s} \cdot c_1, \dots, e^{-a_n s} \cdot c_n)$ acting on $\boldsymbol{C}^n - \{0\}$. Let $\Lambda = (\lambda_1, \dots, \lambda_n)$, with $\lambda_j = e^{-a_j s} \cdot c_j$ and so $\boldsymbol{Z}_H = \langle (\lambda_1, \dots, \lambda_n) \rangle$. The condition (3.2) ensures that the complex numbers λ_j satisfy

$$0 < |\lambda_n| \leq \cdots \leq |\lambda_1| < 1 .$$

Put $M_\Lambda = (\boldsymbol{C}^n - \{0\})/\boldsymbol{Z}_H$. We call M_Λ a *primary Hopf manifold of type* Λ. Indeed, for $n = 2$, one recovers the primary Hopf surfaces of Kähler rank 1. In particular, we derive Theorem B in the Introduction.

REMARK 3.1. Note that the manifolds M_Λ are all diffeomorphic with $S^1 \times S^{2n-1}$ and that for $c_1 = \cdots = c_n = 1$ and $a_1 = \cdots = a_n$, we obtain the standard Hopf manifold, the first known example of a l.c.K. manifold with parallel Lee form, cf. [18].

In [7] a l.c.K. metric with parallel Lee form is constructed on the primary Hopf surface $M_{\lambda_1, \lambda_2} = (\boldsymbol{C}^2 - \{0\})/\Gamma$, $\Gamma \cong \boldsymbol{Z}$ being generated by $(z_1, z_2) \mapsto (\lambda_1 z_1, \lambda_2 z_2)$, $|\lambda_1| \geq |\lambda_2| > 1$. There the diffeomorphism between M_{λ_1, λ_2} and $S^1 \times S^3$ is used to construct a potential for the Kähler metric h (in the notation of the present paper) on the universal cover. The same diffeomorphism is then used to transport the l.c.K. structure on $S^1 \times S^3$ and to show that the induced Sasakian structure on S^3 is a deformation of the standard Sasakian structure of the 3-sphere. See also [1] where a complete list of compact, complex surfaces admitting l.c.K. metrics with parallel Lee form is provided.

4. Lee-Cauchy-Riemann transformations. In this section, we study the group $\mathrm{Aut}_{\mathrm{LCR}}(M)$ described in the Introduction.

Let $\{\theta, \theta \circ J, \theta^\alpha, \bar{\theta}^\alpha\}_{\alpha=1,\dots,n-1}$ be a unitary, local coframe field adapted to a l.c.K. manifold (M, g, J) with parallel Lee form. Consider the subgroup G of $GL(2n, \boldsymbol{R})$ consisting of the following elements:

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u & v^\alpha & \bar{v}^\alpha \\ 0 & 0 & \sqrt{u}\,U^\alpha_\beta & 0 \\ 0 & 0 & 0 & \sqrt{u}\,\bar{U}^\alpha_\beta \end{pmatrix} \;\middle|\; u \in \boldsymbol{R}^+, v^\alpha \in \boldsymbol{C}, U^\alpha_\beta \in \mathrm{U}(n-1) \right\}.$$

Let $G \to P \to M$ be the principal bundle of the G-structure consisting of the above coframes $\{\theta, \theta \circ J, \theta^\alpha, \bar{\theta}^\alpha\}$. If we note that G is isomorphic to the semidirect product $\boldsymbol{C}^{n-1} \rtimes (\mathrm{U}(n-1) \times \boldsymbol{R}^+)$, then the Lie algebra $\mathfrak{g}$ is isomorphic to $\boldsymbol{C}^{n-1} \rtimes \mathfrak{u}(n-1) + \boldsymbol{R}$. Note that the subgroup $\boldsymbol{C}^{n-1}$ is of even rank, while $\mathfrak{u}(n-1) + \boldsymbol{R}$ is of order 2. In particular, the matrix group $\mathfrak{g} \subset \mathfrak{g}l(2n, \boldsymbol{R})$ has no element of rank 1, i.e., it is *elliptic* (cf. [11]). As M is assumed to be compact, it is known that the group of automorphisms $\mathcal{U}$ of P is a finite dimensional Lie group.

DEFINITION 4.1. The group of all diffeomorphisms of M onto itself which preserve the above G-structure is denoted by $\mathrm{Aut}_{\mathrm{LCR}}(M, g, J, \theta)$ (or simply by $\mathrm{Aut}_{\mathrm{LCR}}(M)$). We call $\mathrm{Aut}_{\mathrm{LCR}}(M)$ the group of Lee-Cauchy-Riemann transformations on a l.c.K. manifold (M, g, J) adapted to the parallel Lee form θ.

By definition, if $f \in \mathrm{Aut}_{\mathrm{LCR}}(M)$, then $f^* : P \to P$ is a bundle automorphism satisfying

$$\begin{aligned} &f^*\theta = \theta\,, \\ &f^*(\theta \circ J) = \lambda \cdot (\theta \circ J) \text{ for some positive, smooth function } \lambda\,, \\ &f^*\theta^\alpha = \sqrt{\lambda} \cdot \theta^\beta V^\alpha_\beta + (\theta \circ J) \cdot w^\alpha\,, \\ &f^*\bar{\theta}^\alpha = \sqrt{\lambda} \cdot \bar{\theta}^\beta \bar{V}^\alpha_\beta + (\theta \circ J) \cdot \bar{w}^\alpha \end{aligned} \tag{4.1}$$

for functions V^α_β, w^α with values in $\mathrm{U}(n-1)$, respectively in $\boldsymbol{C}$. Note that the group of holomorphic isometries $\mathrm{I}(M, g, J)$ is contained in $\mathrm{Aut}_{\mathrm{LCR}}(M)$. In fact, an element $f \in \mathrm{I}(M, g, J)$ satisfies $f^*\theta = \theta$, $f^*(\theta \circ J) = (\theta \circ J)$ and $f^*\omega = \omega$. Let $\{\theta^\sharp, J\theta^\sharp\}^\perp$ be the orthogonal complement of the complex plane field $\{\theta^\sharp, J\theta^\sharp\}$ with respect to g. It is obviously J-invariant. If we observe that $\omega|\{\theta^\sharp, J\theta^\sharp\}^\perp = -\mathrm{i}\sum_{\alpha,\beta} \delta_{\alpha\beta}\theta^\alpha \wedge \bar{\theta}^\beta$, then $f^*\theta^\alpha = \theta^\beta U^\alpha_\beta$, $f^*\bar{\theta}^\alpha = \bar{\theta}^\beta \bar{U}^\alpha_\beta$ for some $\mathrm{U}(n-1)$-valued function U^α_β.

LEMMA 4.1. *Any element $f \in \mathrm{Aut}_{\mathrm{LCR}}(M)$ preserves $\{\theta^\sharp, J\theta^\sharp\}^\perp$ and is holomorphic on it.*

PROOF. Let $X \in \{\theta^\sharp, J\theta^\sharp\}^\perp$. The equations $f^*\theta = \theta$, $f^*(\theta \circ J) = \lambda \cdot (\theta \circ J)$ show that

$$\begin{aligned} g(f_*X, \theta^\sharp) &= \theta(f_*X) = \theta(X) = g(X, \theta^\sharp) = 0\,, \\ g(f_*X, J\theta^\sharp) &= -g(Jf_*X, \theta^\sharp) = -\theta(Jf_*X) = -\theta \circ J(f_*X) \\ &= -\lambda \cdot \theta \circ J(X) = -g(X, (\theta \circ J)^\sharp) = g(X, J\theta^\sharp) = 0\,. \end{aligned} \tag{4.2}$$

Thus f_* applies $\{\theta^\sharp, J\theta^\sharp\}^\perp$ onto itself. Moreover, if $\theta^\sharp_\alpha$ is a dual frame field to θ^α (similarly for $\bar{\theta}^\alpha$), then the frame $\{\theta^\sharp_\alpha, \bar{\theta}^\sharp_\alpha\}_{\alpha=1,\ldots,n-1}$ spans $\{\theta^\sharp, J\theta^\sharp\}^\perp \otimes \boldsymbol{C}$. The equation $f^*\theta^\alpha =$

$\sqrt{\lambda}\cdot\theta^{\beta}V^{\alpha}_{\beta}+(\theta\circ J)\cdot w^{\alpha}$ implies that $f_*\theta^{\sharp}_{\alpha}=\sqrt{\lambda}\cdot\theta^{\sharp}_{\beta}V^{\beta}_{\alpha}$ (similary for $f_*\bar{\theta}^{\sharp}_{\alpha}$). Therefore $f_*\circ J=J\circ f_*$ on $\{\theta^{\sharp}, J\theta^{\sharp}\}^{\perp}$. □

When a noncompact LCR flow exists on a compact l.c.K. manifold M with parallel Lee form, we shall prove a rigidity similar to the one implied by a noncompact CR-flow on a compact CR-manifold (cf. [15], [9]).

Proof of Theorem C

4.1. Existence of spherical CR-structure on W/Q'. Let $1\to\boldsymbol{Z}\to\pi'\overset{\nu}{\to}Q'\to 1$ be the split central group extension from Lemma 2.8. Put $M'=\tilde{M}/\pi'$. Then it is easy to see that the Lee form θ, the LCR-action $\boldsymbol{C}^*$ lift to those of M', so we retain the same notation for M'. We put $\boldsymbol{C}^*=S^1\times\boldsymbol{R}^+$, where $\boldsymbol{R}^+=\{\hat{\phi}_t\}_{t\in\boldsymbol{R}}$ is a LCR flow on M'. By hypothesis, $S^1=\{\hat{\varphi}_t\}_{t\in\boldsymbol{R}}$ induces the Lee field $\theta^{\sharp}$. From 1 of Proposition 2.4, S^1 lifts to a nontrivial holomorphic homothetic flow $\boldsymbol{R}=\{\varphi_t\}_{t\in\boldsymbol{R}}$ on $\tilde{M}$ with respect to Ω. We obtain a LCR-action of $\boldsymbol{R}\times\boldsymbol{R}^+$ on $\tilde{M}$ for which $\boldsymbol{R}$ acts properly as before. Consider the commutative diagram of principal bundles:

$$
\begin{array}{ccccc}
\boldsymbol{Z} & \longrightarrow & \pi' & \overset{\nu}{\longrightarrow} & Q' \\
\downarrow & & \downarrow & & \downarrow \\
\boldsymbol{R} & \longrightarrow & (\boldsymbol{R}\times\boldsymbol{R}^+,\tilde{M}) & \overset{(\tilde{\nu},\pi)}{\longrightarrow} & (\boldsymbol{R}^+, W) \\
\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
S^1 & \longrightarrow & (S^1\times\boldsymbol{R}^+, M') & \overset{(\hat{\nu},\hat{\pi})}{\longrightarrow} & (\boldsymbol{R}^+, W/Q')\,.
\end{array}
\tag{4.3}
$$

From the bottom line, the projection $\hat{\nu}$ maps the group $\boldsymbol{R}^+=\{\hat{\phi}_t\}_{t\in\boldsymbol{R}}$ onto a group $\boldsymbol{R}^+=\{\bar{\phi}_t\}_{t\in\boldsymbol{R}}$ acting on W/Q'.

LEMMA 4.2. *The group $\boldsymbol{R}^+=\{\bar{\phi}_t\}_{t\in\boldsymbol{R}}$ acts by CR-transformations on W/Q' with respect to the CR-structure induced from the strictly pseudoconvex, pseudo-Hermitian structure $(\hat{\eta}, J)$.*

PROOF. As ξ generates the flow $\boldsymbol{R}=\{\varphi_t\}_{t\in\boldsymbol{R}}$, $p_*\xi=\theta^{\sharp}$ on M' by hypothesis and so $p:\tilde{M}\to M'$ maps the complex plane field $\{\xi, J\xi\}$ onto $\{\theta^{\sharp}, J\theta^{\sharp}\}$. By Lemma 4.1, each $\hat{\phi}_t\in \mathrm{Aut}_{\mathrm{LCR}}(M')$ preserves $\{\theta^{\sharp}, (\theta\circ J)^{\sharp}\}^{\perp}$. So its lift ϕ_t preserves the J-invariant distribution $\{\xi, J\xi\}^{\perp}$. Since $\pi_*:(\{\xi, J\xi\}^{\perp}, J)\to(\mathrm{Null}\ \eta, J)$ is J-isomorphic and each ϕ_t is holomorphic on $\{\xi, J\xi\}^{\perp}$, $\hat{\pi}_*:(\{\theta^{\sharp}, (\theta\circ J)^{\sharp}\}^{\perp}, J)\to(\mathrm{Null}\ \hat{\eta}, J)$ is also J-isomorphic through the commutative diagram and thus each $\bar{\phi}_t$ is holomorphic on Null $\hat{\eta}$; $(\bar{\phi}_{t*}\circ J=J\circ\bar{\phi}_{t*})$. Therefore, $\boldsymbol{R}^+=\{\bar{\phi}_t\}_{t\in\boldsymbol{R}}$ is a closed, noncompact subgroup of CR-transformations of W/Q' with respect to $(\mathrm{Null}\ \hat{\eta}, J)$. □

By this lemma, we obtain a compact strictly pseudoconvex CR-manifold W/Q' admitting a closed, noncompact CR-transformations $\boldsymbol{R}^+$. Then we apply the result of [9] to show that W/Q' is CR-equivalent to the sphere S^{2n-1} with the standard CR-structure. In particular

$Q' = \{1\}$ and thus Q is a finite subgroup of $\mathrm{PSH}(W, \eta, J)$ from Lemma 2.8. By the definition of spherical CR-structure (cf. [13], [8]), there exists a developing pair:

$$(\mu, \text{ dev}) : (\mathrm{Aut}_{\mathrm{CR}}(W), W) \to (\mathrm{PU}(n, 1), S^{2n-1})$$

for which dev is a CR-diffeomorphism and $\mu : \mathrm{Aut}_{\mathrm{CR}}(W) \to \mathrm{PU}(n, 1)$ is the holonomy isomorphism. Here $\mathrm{PU}(n, 1) = \mathrm{Aut}_{\mathrm{CR}}(S^{2n-1})$ and $\mathrm{Aut}_{\mathrm{CR}}(W)$ is the group of all CR-automorphisms of W containing the groups $\boldsymbol{R}^+$ and $\mathrm{PSH}(W, \eta, J) \supset Q$.

As S^1 $(\subset \boldsymbol{C}^*)$ acts on M without fixed points (but not necessarily freely, i.e., with possible subset of exceptional orbits $S^1 \cdot x$ for which the stabilizer S^1_x is a non-trivial cyclic subgroup of S^1; cf. [3]), the quotient space $M/S^1 = W/Q (\approx S^{2n-1}/\mu(Q))$ is an orbifold, so such a finite subgroup Q may exist.

On the other hand, we recall some facts from the theory of hyperbolic groups (cf. [4]). The noncompact closed $\mu(\boldsymbol{R}^+)$-action on S^{2n-1} is characterized as whether it is either loxodromic $(= \boldsymbol{R}^+)$ or parabolic $(= \mathcal{R})$ for which $\boldsymbol{R}^+$ has exactly two fixed points $\{0, \infty\}$ or $\mathcal{R}$ has the unique fixed point $\{\infty\}$ on S^{2n-1}. Moreover, the centralizer $\mathcal{C}_{\mathrm{PU}(n,1)}(\mu(\boldsymbol{R}^+))$ of $\mu(\boldsymbol{R}^+)$ in $\mathrm{PU}(n, 1)$ is one of the following groups up to conjugacy:

$$\mathcal{R} \times \mathrm{U}(n-1) \quad \text{or} \quad \boldsymbol{R}^+ \times \mathrm{U}(n-1)\,. \tag{4.4}$$

Since $\pi_1(M)$ centralizes $\boldsymbol{R} \times \boldsymbol{R}^+$, note that Q centralizes $\boldsymbol{R}^+$ (cf. (2.24)). The holonomy group $\mu(Q)$ belongs to $\mathcal{C}_{\mathrm{PU}(n,1)}(\mu(\boldsymbol{R}^+))$. As $\mu(Q)$ is a finite subgroup, (4.4) implies that

$$\mu(Q) \subset \mathrm{U}(n-1)\,. \tag{4.5}$$

4.2. Rigidity of (M, g, J) under the LCR action of $\boldsymbol{R}^+$. Let (η_0, J_0) be the standard strictly pseudoconvex pseudo-Hermitian structure on S^{2n-1} (cf. (3.1)). By definition, there exists a positive function u on W such that

$$\mathrm{dev}^* \eta_0 = u \cdot \eta\,. \tag{4.6}$$

By Lemma 2.4, we know that A is the characteristic CR-vector field on W for (η, J). If $\{\psi'_t\}$ is the flow generated by A, then note from (2.1.3) that $\{\psi'_t\} \subset \mathrm{PSH}(W, \eta, J)$. Because W is compact, $\mathrm{PSH}(W, \eta, J)$ is compact. As $\mathrm{PSH}(W, \eta, J) \subset \mathrm{Aut}_{\mathrm{CR}}(W)$, the closure of the holonomy image $\mu(\{\psi'_t\})$ (which is a connected abelian group) lies in the maximal torus T^n of the maximal compact subgroup $\mathrm{U}(n)$ in $\mathrm{PU}(n, 1)$ up to conjugacy. We can describe it as

$$\mu(\psi'_t) = (e^{ia_1 \cdot t}, \ldots, e^{ia_n \cdot t})\,, \quad t \in \boldsymbol{R}$$

for some $a_i \in \boldsymbol{R}$ $(i = 1, \ldots, n)$. On the other hand, let $\mathcal{A} = \mathrm{dev}_*(A)$. Since dev is equivariant, $\mathrm{dev}(\psi'_t w) = \mu(\psi'_t)\mathrm{dev}(w)$ on $S^{2n-1} = \{z = (z_1, z_2, \ldots, z_n) \in \boldsymbol{C}^n \mid |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 = 1\}$, we have

$$\mathcal{A}_z = \frac{d\mu(\psi'_t)}{dt} = \sum_{j=1}^{n} a_j \left(x_j \frac{d}{dy_j} - y_j \frac{d}{dx_j} \right), \quad z = \mathrm{dev}(w), \quad z_j = x_j + \mathrm{i} y_j\,. \tag{4.7}$$

As $\eta(A) = 1$, we have

$$u(w) = \mathrm{dev}^* \eta_0(A) = \eta_0(\mathcal{A}_z) = \sum_{j=1}^{n} a_j \cdot |z_j|^2 \,. \tag{4.8}$$

Since $u > 0$ from (4.6), we can assume that, up to rearranging the order of indices

$$0 < a_1 \leq \cdots \leq a_n \,. \tag{4.9}$$

As dev^{-1} maps the pseudo-Hermitain structure (η, J) on W to $(\mathrm{dev}^{-1*}\,\eta, J_0)$ on S^{2n-1}, we put

$$\eta_{\mathcal{A}} = \mathrm{dev}^{-1*}\eta \,. \tag{4.10}$$

Using (4.8), we obtain

$$\eta_{\mathcal{A}} = \frac{1}{\sum_{j=1}^{n} a_j \cdot |z_j|^2} \cdot \eta_0 \quad \text{on} \quad S^{2n-1} \,. \tag{4.11}$$

When we note that $\eta_0 = u' \cdot \eta_{\mathcal{A}}$ where $u' = u \circ \mathrm{dev}^{-1}$, and $T(\boldsymbol{R} \times S^{2n-1}) = \{d/dt, \mathcal{A}\} \oplus$ Null η_0, denote the complex structure $J_{\mathcal{A}}$ on $\boldsymbol{R} \times S^{2n-1}$ by

$$\begin{aligned} &J_{\mathcal{A}} \frac{d}{dt} = -\mathcal{A}, \quad J_{\mathcal{A}}\mathcal{A} = \frac{d}{dt} \,, \\ &J_{\mathcal{A}} | \mathrm{Null}\ \eta_0 = J_0 \,. \end{aligned} \tag{4.12}$$

(Compare §3.) Let $\mathrm{Pr} : \boldsymbol{R} \times S^{2n-1} \to S^{2n-1}$ be the canonical projection. In view of (3.5), setting

$$\begin{aligned} &\Omega_{\mathcal{A}} = d(e^t \cdot \mathrm{Pr}^* \eta_{\mathcal{A}}) \,, \quad \tilde{\omega}_{\mathcal{A}} = 2e^{-t} \cdot \Omega_{\mathcal{A}} \,, \\ &\tilde{g}_{\mathcal{A}}(X, Y) = \tilde{\omega}_{\mathcal{A}}(J_{\mathcal{A}}X, Y) \,, \end{aligned} \tag{4.13}$$

we obtain a l.c.K. structure $(\Omega_{\mathcal{A}}, J_{\mathcal{A}})$ on the product $\boldsymbol{R} \times S^{2n-1}$ endowed with the group $\boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_{\mathcal{A}}, J_0)$ of holomorphic homothetic transformations.

PROPOSITION 4.1. *There exists an equivariant holomorphic isometry between the l.c.K. manifolds* $(\mathcal{C}_{\mathcal{H}}(\boldsymbol{R}), \tilde{M}, \Omega, J)$ *and* $(\boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_{\mathcal{A}}, J_0), \boldsymbol{R} \times S^{2n-1}, \Omega_{\mathcal{A}}, J_{\mathcal{A}})$.

PROOF. Let $G : \tilde{M} \to \boldsymbol{R} \times S^{2n-1}$ be a diffeomorphism defined by $G(\varphi_t w) = (t, \mathrm{dev}(w))$. Note that $\mathrm{Pr} \circ G = \mathrm{dev} \circ \pi$ on $\tilde{M}$. As every element of $\mathcal{C}_{\mathcal{H}}(\boldsymbol{R})$ is described as $\varphi_s \cdot q(\alpha)$ from Remark 2.1, define a homomorphism $\Psi : \mathcal{C}_{\mathcal{H}}(\boldsymbol{R}) \to \boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_{\mathcal{A}}, J_0)$ by setting

$$\Psi(\varphi_s \cdot q(\alpha)) = (s, \mu(\alpha)) \,.$$

Recall that the action $q(\alpha)(\varphi_t w) = \varphi_t \alpha w$ from (2.21). Then,

$$\begin{aligned} G(\varphi_s \cdot q(\alpha)(\varphi_t w)) &= G(\varphi_{s+t} \cdot \alpha w) = (s + t, \mathrm{dev}(\alpha w)) = (s + t, \mu(\alpha)\,\mathrm{dev}(w)) \\ &= (s, \mu(\alpha))(t, \mathrm{dev}(w)) = \Psi(\varphi_s \cdot q(\alpha)) G(\varphi_t w) \,. \end{aligned}$$

Hence, $(\Psi, G) : (\mathcal{C}_{\mathcal{H}}(\boldsymbol{R}), \tilde{M}) \to (\boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_{\mathcal{A}}, J_0), \boldsymbol{R} \times S^{2n-1})$ is equivariantly diffeomorphic. Next, since $G^* t = t$ for the t-coordinate of $\boldsymbol{R} \times S^{2n-1}$ and $\mathrm{dev}^*\,\eta_{\mathcal{A}} = \eta$ from

(4.10), it follows that

$$G^*\Omega_{\mathcal{A}} = G^*d(e^t\cdot \mathrm{Pr}^*\eta_{\mathcal{A}}) = d(e^{G^*t}\cdot G^*\mathrm{Pr}^*\eta_{\mathcal{A}}) = d(e^t\cdot\pi^*\eta) = \Omega\,. \tag{4.14}$$

By definition, $G_*\xi = d/dt$. Moreover, when $x=\varphi_s w$,

$$G(\psi_t(x)) = G(\varphi_s\psi_t w) = G(\varphi_s i{\psi'}_t w) = (s, \mathrm{dev}({\psi'}_t w)) = (s, \mu({\psi'}_t)\,\mathrm{dev}(w))\,.$$

By (2.7) and (4.7),

$$G_*(-J\xi_x) = \frac{dG\psi_t}{dt}(x)\Big|_{t=0} = \mathcal{A}_{Gx} = -J_{\mathcal{A}}\left(\frac{d}{dt}\right)_{Gx}\,.$$

Thus $G_*(J\xi) = J_{\mathcal{A}}G_*\xi$. As $G^*\Omega_{\mathcal{A}} = \Omega$ from (4.14), G maps $\{\xi, J\xi\}^\perp$ onto $\{d/dt, \mathcal{A}\}^\perp$. Consider the commutative diagram:

$$\begin{array}{ccc} (\{\xi, J\xi\}^\perp, J) & \xrightarrow{\pi_*} & (\mathrm{Null}\,\eta, J) \\ \big\downarrow G_* & & \big\downarrow \mathrm{dev}_* \\ (\{d/dt, \mathcal{A}\}^\perp, J_{\mathcal{A}}) & \xrightarrow{\mathrm{Pr}_*} & (\mathrm{Null}\,\eta_0, J_0)\,. \end{array} \tag{4.15}$$

Here note that $J_{\mathcal{A}} = J_0$ on $\mathrm{Null}\,\eta_{\mathcal{A}} = \mathrm{Null}\,\eta_0$. For $X\in\{\xi, J\xi\}^\perp$,

$$\mathrm{Pr}_*G_*J(X) = \mathrm{dev}_*(J\pi_*X) = J_0\mathrm{dev}_*\pi_*(X) = J_{\mathcal{A}}\mathrm{Pr}_*G_*(X) = \mathrm{Pr}_*J_{\mathcal{A}}G_*(X)\,,$$

thus, $G_*J(X) = J_{\mathcal{A}}G_*(X)$. Hence, G is $(J, J_{\mathcal{A}})$-biholomorphic. Moreover, as $G^*\tilde{\omega}_{\mathcal{A}} = G^*(2e^{-t}\Omega_{\mathcal{A}}) = 2e^{-t}\Omega = \bar{\Theta}$ and $\bar{g}(X,Y) = \bar{\Theta}(JX, Y)$, we obtain that $G^*\tilde{g}_{\mathcal{A}} = \bar{g}$. Therefore, (Ψ, G) induces a holomorphic isometry from $(M, \hat{g}, J)$ onto $(\boldsymbol{R}\times S^{2n-1}/\Psi(\pi_1(M)), \hat{g}_{\mathcal{A}}, \hat{J}_{\mathcal{A}})$. □

4.3. The Hopf manifold $\boldsymbol{R}\times S^{2n-1}/\Psi(\pi_1(M))$. We prove that $\boldsymbol{R}\times S^{2n-1}/\Psi(\pi_1(M))$ is a primary Hopf manifold M_Λ for some Λ obtained in §3. Each element of $\pi_1(M)$ is of the form $\gamma = \varphi_s\cdot q(\alpha)$ for some $s\in\boldsymbol{R}$, where $\nu(\gamma) = \alpha\in Q = \nu(\pi_1(M))$. By the definition of Ψ, $\Psi(\gamma) = (s, \mu(\alpha))$. We show that $\Psi(\pi_1(M))$ has no torsion element. For this, if $\Psi(\gamma)$ is of finite order (say, l), then $1 = (0,1) = \Psi(\gamma^l) = (ls, \mu(\alpha^l))$. Then, $s=0$ so that $\Psi(\gamma) = (0, \mu(\alpha))$. On the other hand, recall from (4.5) that $\mu(Q)\subset \mathrm{U}(n-1)$ up to conjugacy, and so $\mu(Q)$ has a fixed point $w_0\in S^{2n-1}$. Since $\Psi(\pi_1(M))$ acts freely on $\boldsymbol{R}\times S^{2n-1}$, while $\Psi(\gamma)(t, w_0) = (t, \mu(\alpha)w_0) = (t, w_0)$, it follows that $\Psi(\gamma) = 1$. Moreover, if $\gamma_1 = \varphi_{s_1}\cdot q(\alpha_1)$, $\gamma_2 = \varphi_{s_2}\cdot q(\alpha_2)$, then $\Psi([\gamma_1,\gamma_2]) = (0, \mu([\alpha_1,\alpha_2])$. For the same reason, $\Psi([\pi_1(M),\pi_1(M)]) = \{1\}$. Hence, $\pi_1(M)$ is a finitely generated torsionfree abelian group. If we recall from (2.24) that $1\to\boldsymbol{Z}\to\pi_1(M)\overset{\nu}{\to} Q\to 1$ is the central group extension where Q is finite, then $\pi_1(M)$ itself is an infinite cyclic group. Since $\Psi(\pi_1(M))\subset\boldsymbol{R}\times\mathrm{PSH}(S^{2n-1}, \eta_{\mathcal{A}}, J_0)$ and the projection maps $\Psi(\pi_1(M))$ onto $\mu(Q)$ in $\mathrm{PSH}(S^{2n-1}, \eta_{\mathcal{A}}, J_0)$, $\mu(Q)$ is a finite cyclic group. As $\mathrm{PSH}(S^{2n-1}, \eta_{\mathcal{A}}, J_0)$ has the maximal torus T^n (cf. (3.4)), we obtain that $\Psi(\pi_1(M))\subset\boldsymbol{R}\times T^n$ up to conjugacy. A generator of $\Psi(\pi_1(M))$ is described as $(s, (c_1,\ldots,c_n))\in\boldsymbol{R}\times T^n$. Noting (4.9), let $\lambda_j = e^{-a_js}c_j$ and

$\Lambda = (\lambda_1, \dots, \lambda_n)$. By Theorem 3.1 and the remark below it, $\boldsymbol{R} \times S^{2n-1}/\Psi(\pi_1(M))$ is a primary Hopf manifold M_Λ of type Λ. This finishes the proof of Theorem C in the Introduction.

REFERENCES

[1] F. A. BELGUN, On the metric structure of non-Kähler complex surfaces, Math. Ann. 317 (2000), 1–40.

[2] D. E. BLAIR, Contact manifolds in Riemannian geometry, Lecture Notes in Math. 509, Springer-Verlag, Berlin-New York, 1976.

[3] G. BREDON, Introduction to compact transformation groups, Pure Appl. Math. 46, Academic Press, New York-London, 1972.

[4] S. S. CHEN AND L. GREENBERG, Hyperbolic spaces, Contribution to analysis (a collection of papers dedicated to Lipman Bers), 49–87, Academic Press, New York, 1974.

[5] P. CONNER AND F. RAYMOND, Injective operation of the toral groups, Topology 10 (1971), 283–296.

[6] S. DRAGOMIR AND L. ORNEA, Locally conformal Kähler geometry, Progr. Math. 155, Birkhäuser Boston, Boston, Mass., 1998.

[7] P. GAUDUCHON AND L. ORNEA, Locally conformally Kähler metrics on Hopf surfaces, Ann. Inst. Fourier (Grenoble) 48 (1998), 1107–1127.

[8] W. GOLDMAN, Complex hyperbolic geometry, Oxford Math. Monogr., Oxford Sci. Publ., The Clarendon Press, Oxford Univ. Press, New York, 1999.

[9] Y. KAMISHIMA, Geometric flows on compact manifolds and global rigidity, Topology 35 (1996), 439–450.

[10] Y. KAMISHIMA, Holomorphic torus actions on compact locally conformal Kähler manifolds, Compos. Math. 124 (2000), 341–349.

[11] S. KOBAYASHI, Transformation groups in differential geometry, Ergeb. Math. Grenzgeb. 70, Springer-Verlag, New York-Heidelberg, 1972.

[12] S. KOBAYASHI AND K. NOMIZU, Foundations of differential geometry II, Interscience Tracts in Pure and Appl. Math. 15 II, Interscience Publisheres John Wiley & Sons, Inc., New York-London-Sydney, 1969.

[13] R. KULKARNI, On the principle of uniformization, J. Differential Geom. 13 (1978), 109–138.

[14] J. LELONG-FERRAND, Transformations conformes et quasi-conformes des variétés riemanniennes compactes, Acad. Roy. Belg. Cl. Sci. Mém. Collect. 8 (2) 39 (1971), 1–44.

[15] M. OBATA, The conjectures on conformal transformations of Riemannian manifolds, J. Differential Geom. 6 (1971/72), 247–258.

[16] M. S. RAGHUNATHAN, Discrete subgroups of Lie groups, Ergeb. Math. Grenzgeb. 68, Springer-Verlag, New York-Heidelberg, 1972.

[17] F. TRICERRI, Some examples of locally conformal Kähler manifolds, Rend. Sem. Mat. Univ. Politec. Torino 40 (1982), 81–92.

[18] I. VAISMAN, Locally conformal Kähler manifolds with parallel Lee form, Rend. Mat. (6) 12 (1979), 263–284.

[19] I. VAISMAN, Generalized Hopf manifolds, Geom. Dedicata 13 (1982), 231–255.

[20] S. M. WEBSTER, On the transformation group of a real hypersurface, Trans. Amer. Math. Soc. 231 (1977), 179–190.

[21] S. M. WEBSTER, Pseudo-Hermitian steuctures on a real hypersurface, J. Differential Geom. 13 (1978), 25–41.

DEPARTMENT OF MATHEMATICS
TOKYO METROPOLITAN UNIVERSITY
MINAMI-OHSAWA 1–1, HACHIOJI
TOKYO 192–0397
JAPAN

E-mail address: kami@comp.metro-u.ac.jp

UNIVERSITY OF BUCHAREST
FACULTY OF MATHEMATICS
14 ACADEMIEI STR.
70109 BUCHAREST
ROMANIA

E-mail address: Liviu.Ornea@imar.ro