

HARDY SPACES ASSOCIATED TO THE SECTIONS

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Abstract. In this paper we define the Hardy space $H_{\mathcal{F}}^1(\mathbf{R}^n)$ associated with a family \mathcal{F} of sections and a doubling measure μ , where \mathcal{F} is closely related to the Monge-Ampère equation. Furthermore, we show that the dual space of $H_{\mathcal{F}}^1(\mathbf{R}^n)$ is just the space $BMO_{\mathcal{F}}(\mathbf{R}^n)$, which was first defined by Caffarelli and Gutiérrez. We also prove that the Monge-Ampère singular integral operator is bounded from $H_{\mathcal{F}}^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n, d\mu)$.

1. Introduction. In 1996, Caffarelli and Gutiérrez [CG1] studied real variable theory related to the Monge-Ampère equation. They gave a Besicovitch type covering lemma for a family \mathcal{F} of convex sets in Euclidean n -space \mathbf{R}^n , where $\mathcal{F} = \{S(x, t); x \in \mathbf{R}^n \text{ and } t > 0\}$ and $S(x, t)$ is called a *section* (see the definition below) satisfying certain axioms of affine invariance. In terms of the sections, Caffarelli and Gutiérrez set up a variant of the Calderón-Zygmund decomposition by applying this covering lemma and the doubling condition of a Borel measure μ . The decomposition plays an important role in the study of the linearized Monge-Ampère equation [CG2]. As an application of the above decomposition, Caffarelli and Gutiérrez defined the Hardy-Littlewood maximal operator M and $BMO_{\mathcal{F}}(\mathbf{R}^n)$ space associated to a family \mathcal{F} of sections and the doubling measure μ , and obtained the weak type $(1,1)$ boundedness of M and the John-Nirenberg inequality for $BMO_{\mathcal{F}}(\mathbf{R}^n)$ in [CG1].

Let us recall the definition of sections and the doubling measure listed below. For $x \in \mathbf{R}^n$ and $t > 0$, let $S(x, t)$ denote an open and bounded convex set containing x . We call $S(x, t)$ a *section* if the family $\{S(x, t); x \in \mathbf{R}^n, t > 0\}$ is monotone increasing in t , i.e., $S(x, t) \subset S(x, t')$ for $t \leq t'$, and satisfies the following three conditions:

(A) There exist positive constants K_1, K_2, K_3 and ϵ_1, ϵ_2 such that given two sections $S(x_0, t_0), S(x, t)$ with $t \leq t_0$ satisfying

$$S(x_0, t_0) \cap S(x, t) \neq \emptyset,$$

and an affine transformation T that “normalizes” $S(x_0, t_0)$, that is,

$$B(0, 1/n) \subset T(S(x_0, t_0)) \subset B(0, 1),$$

there exists $z \in B(0, K_3)$ depending on $S(x_0, t_0)$ and $S(x, t)$, which satisfies

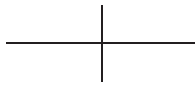
$$B(z, K_2(t/t_0)^{\epsilon_2}) \subset T(S(x, t)) \subset B(z, K_1(t/t_0)^{\epsilon_1}),$$

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and

$$T(z) \in B(z, (1/2)K_2(t/t_0)^{\epsilon_2}).$$

Here and below $B(x, t)$ denotes the Euclidean ball centered at x with radius t .

(B) There exists a constant $\delta > 0$ such that given a section $S(x, t)$ and $y \notin S(x, t)$, if T is an affine transformation that “normalizes” $S(x, t)$, then for any $0 < \epsilon < 1$

$$B(T(y), \epsilon^\delta) \cap T(S(x, (1 - \epsilon)t)) = \emptyset.$$

(C) $\bigcap_{t>0} S(x, t) = \{x\}$ and $\bigcup_{t>0} S(x, t) = \mathbf{R}^n$.

In addition, we also assume that a Borel measure μ which is finite on compact sets is given, $\mu(\mathbf{R}^n) = \infty$, and satisfies the following *doubling property* with respect to \mathcal{F} , that is, there exists a constant A such that

$$(1.1) \quad \mu(S(x, 2t)) \leq A\mu(S(x, t)) \quad \text{for any section } S(x, t) \in \mathcal{F}.$$

An important example of the family \mathcal{F} of sections is given as follows. Let $\phi : \mathbf{R}^n \rightarrow \mathbf{R}$ be a convex smooth function. For any given point $x \in \mathbf{R}^n$, let $\mathcal{L}(x)$ be a supporting hyperplane of ϕ at the point $(x, \phi(x))$. For $t > 0$, define the set

$$S_\phi(x, t) = \{y \in \mathbf{R}^n ; \phi(y) < \mathcal{L}(x) + t\}.$$

Then

$$\mathcal{F} = \{S_\phi(x, t) ; x \in \mathbf{R}^n \text{ and } t > 0\}$$

is a family of sections that satisfies the properties (A), (B) and (C). Moreover, the Monge-Ampère measure generated by the convex function ϕ

$$\det D^2 \phi = \mu$$

satisfies the doubling condition (1.1) under certain condition of ϕ . For instance, if the graph of ϕ contains no lines, then μ satisfies the doubling condition (1.1) (see [C, CG1]). The terminology *section* comes from the fact that $S_\phi(x, t)$ is obtained by projecting on \mathbf{R}^n the bounded part of the graph of ϕ cut by a hyperplane parallel to the supporting hyperplane at $(x, \phi(x))$.

In [CG1], Caffarelli and Gutiérrez defined the space $BMO_{\mathcal{F}}(\mathbf{R}^n)$ associated with the family \mathcal{F} and the Borel measure μ satisfying the doubling condition (1.1). Let f be a real-valued function defined on \mathbf{R}^n . We say that $f \in BMO_{\mathcal{F}}(\mathbf{R}^n)$ if

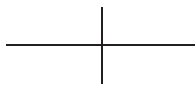
$$\|f\|_* := \sup_{S \in \mathcal{F}} \frac{1}{\mu(S)} \int_S |f(x) - m_S(f)| d\mu(x) < \infty,$$

where $m_S(f)$ denotes the mean of f over the section S defined by

$$m_S(f) = \frac{1}{\mu(S)} \int_S f(x) d\mu(x).$$

Similar to the classic case, Caffarelli and Gutiérrez [CG1] also proved the following John-Nirenberg inequality for $BMO_{\mathcal{F}}$:





There exist positive constants C_1 and C_2 dependent only on the measure μ such that, for every continuous $f \in BMO_{\mathcal{F}}(\mathbf{R}^n)$ and every section S ,

$$\frac{1}{\mu(S)} \int_S \exp\left(C_1 \frac{|f(x) - m_S(f)|}{\|f\|_*}\right) d\mu(x) \leq C_2.$$

Hence, it is an important and interesting problem to ask whether it is possible to set up a Hardy space with respect to the family of sections \mathcal{F} and a doubling measure. In this paper we are going to construct such a Hardy space. We first introduce $(1, q)$ -atoms and the atomic Hardy space $H_{\mathcal{F}}^{1,q}(\mathbf{R}^n)$ for $q > 1$ with respect to the family \mathcal{F} . Then we show that the atomic Hardy spaces $H_{\mathcal{F}}^{1,q}(\mathbf{R}^n)$ are all equivalent for any $q > 1$. Thus we may define the Hardy space $H_{\mathcal{F}}^1(\mathbf{R}^n)$. We will further prove that the dual space of $H_{\mathcal{F}}^1(\mathbf{R}^n)$ is just the space $BMO_{\mathcal{F}}(\mathbf{R}^n)$, which was defined by Caffarelli and Gutiérrez in [CG1]. Moreover, as an application of the atomic decomposition, we will also prove that the Monge-Ampère singular integral operator (defined later) is bounded from $H_{\mathcal{F}}^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n, d\mu)$.

We now define a $(1, q)$ -atom and the atomic Hardy space with respect to a family \mathcal{F} of sections and a doubling measure μ .

DEFINITION 1.1. Let $1 < q \leq \infty$. A function $a(x) \in L^q(\mathbf{R}^n, d\mu)$ is called a $(1, q)$ -atom if there exists a section $S(x_0, t_0) \in \mathcal{F}$ such that

- (i) $\text{supp}(a) \subset S(x_0, t_0)$;
- (ii) $\int_{\mathbf{R}^n} a(x) d\mu(x) = 0$;
- (iii) $\|a\|_{L_{\mu}^q} \leq [\mu(S(x_0, t_0))]^{-1/q'}$, where $\|a\|_{L_{\mu}^q} = (\int_{\mathbf{R}^n} |a(x)|^q d\mu(x))^{1/q}$ and $1/q + 1/q' = 1$.

The atomic Hardy space $H_{\mathcal{F}}^{1,q}(\mathbf{R}^n)$ is defined by

$$(1.2) \quad H_{\mathcal{F}}^{1,q}(\mathbf{R}^n) = \left\{ f \in \mathcal{S}' ; f(x) \stackrel{\mathcal{S}'}{=} \sum_j \lambda_j a_j(x), \text{ each } a_j \text{ is a } (1, q)\text{-atom and } \sum_j |\lambda_j| < \infty \right\},$$

where $\mathcal{S}(\mathbf{R}^n)$ denotes the space of Schwartz functions and $\mathcal{S}'(\mathbf{R}^n)$ is the dual space of $\mathcal{S}(\mathbf{R}^n)$. Define the $H_{\mathcal{F}}^{1,q}$ norm of f by

$$\|f\|_{H_{\mathcal{F}}^{1,q}} = \inf \left\{ \sum_j |\lambda_j| \right\},$$

where the infimum is taken over all decompositions of $f = \sum_j \lambda_j a_j$ above.

The first result of this paper is

THEOREM 1.1. For $q > 1$, $H_{\mathcal{F}}^{1,q}(\mathbf{R}^n) = H_{\mathcal{F}}^{1,\infty}(\mathbf{R}^n)$.

By Theorem 1.1, we may take the atomic Hardy space $H_{\mathcal{F}}^{1,q}$ for any $q > 1$ as the definition of the Hardy space $H_{\mathcal{F}}^1(\mathbf{R}^n)$. Our second task is to show the following duality.

THEOREM 1.2. The dual space of $H_{\mathcal{F}}^1(\mathbf{R}^n)$ is the space $BMO_{\mathcal{F}}(\mathbf{R}^n)$.



In 1997, Caffarelli and Gutiérrez [CG3] defined a class of the Monge-Ampère singular integral operators as follows. Suppose that $0 < \alpha \leq 1$ and $c_1, c_2 > 0$. Let $\{k_i(x, y)\}_{i=1}^\infty$ be a sequence of kernels satisfying the following conditions:

$$(1.3) \quad \text{supp}k_i(\cdot, y) \subset S(y, 2^i) \text{ for all } y \in \mathbf{R}^n;$$

$$(1.4) \quad \text{supp}k_i(x, \cdot) \subset S(x, 2^i) \text{ for all } x \in \mathbf{R}^n;$$

$$(1.5) \quad \int_{\mathbf{R}^n} k_i(x, y) d\mu(y) = \int_{\mathbf{R}^n} k_i(x, y) d\mu(x) = 0 \text{ for all } x, y \in \mathbf{R}^n;$$

$$(1.6) \quad \sup_i \int_{\mathbf{R}^n} |k_i(x, y)| d\mu(y) \leq c_1 \text{ for all } x \in \mathbf{R}^n;$$

$$(1.7) \quad \sup_i \int_{\mathbf{R}^n} |k_i(x, y)| d\mu(x) \leq c_2 \text{ for all } y \in \mathbf{R}^n;$$

$$(1.8) \quad \text{If } T \text{ is an affine transformation that normalizes the section } S(y, 2^i), \text{ then}$$

$$|k_i(u, y) - k_i(v, y)| \leq \frac{c_2}{\mu(S(y, 2^i))} |T(u) - T(v)|^\alpha;$$

$$(1.9) \quad \text{If } T \text{ is an affine transformation that normalizes the section } S(x, 2^i), \text{ then}$$

$$|k_i(x, u) - k_i(x, v)| \leq \frac{c_2}{\mu(S(x, 2^i))} |T(u) - T(v)|^\alpha.$$

Denote $K(x, y) = \sum_i k_i(x, y)$. The Monge-Ampère singular integral operator H is defined by

$$H(f)(x) = \int_{\mathbf{R}^n} K(x, y) f(y) d\mu(y).$$

Caffarelli and Gutiérrez [CG3] proved that H is bounded from $L^2(\mathbf{R}^n, d\mu)$ to $L^2(\mathbf{R}^n, d\mu)$. Subsequently, Incognito [In] gave the weak type (1,1) estimate of H . Using the atomic decomposition of $H_{\mathcal{F}}^1(\mathbf{R}^n)$, we have the following result for the operator H .

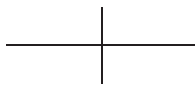
THEOREM 1.3. *The operator H is a bounded operator from $H_{\mathcal{F}}^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n, d\mu)$.*

As an application of Theorem 1.3, we have a different method from [In] to obtain the following corollary.

COROLLARY 1.1. *The operator H is bounded on $L^p(\mathbf{R}^n, d\mu)$, $1 < p < \infty$.*

Indeed, it follows from Theorem 1.3 and the $L^2(\mathbf{R}^n, d\mu)$ boundedness of H (see [CG3]) that we can easily get the $L^p(\mathbf{R}^n, d\mu)$ boundedness of H for $1 < p < 2$ by applying the interpolation theorem. We then use the duality to get the $L^p(\mathbf{R}^n, d\mu)$ boundedness of H for $2 < p < \infty$.

The organization of this paper is as follows. In Section 2 we recall some elementary properties of the Hardy-Littlewood maximal operator with respect to sections, and two covering lemmas. The equivalence of all atomic Hardy spaces $H_{\mathcal{F}}^{1,q}(\mathbf{R}^n)$ will be proved in Section 3. In Section 4, we will show that the dual space of $H_{\mathcal{F}}^1(\mathbf{R}^n)$ is $BMO_{\mathcal{F}}(\mathbf{R}^n)$. Finally, the $(H_{\mathcal{F}}^1, L_{\mu}^1)$ boundedness of the Monge-Ampère singular integral operator H will be proved in Section 5. Finally, we would like to point out that the basic idea of proving our main results in this paper is based on a noted paper [CW2] by Coifman and Weiss.



2. Elementary properties of sections and covering lemmas. From the properties (A) and (B) of sections, Aimar, Forzani, and Toledano [AFT] obtained the following *engulfing property*: There exists a constant $\theta \geq 1$, depending only on δ , K_1 , and ϵ_1 , such that for each $y \in S(x, t)$,

$$(D) \quad S(x, t) \subset S(y, \theta t) \text{ and } S(y, t) \subset S(x, \theta t).$$

Define a function ρ on $\mathbf{R}^n \times \mathbf{R}^n$ by

$$\rho(x, y) = \inf\{t > 0 ; y \in S(x, t)\}.$$

Using the engulfing property (D), Incognito [In] obtained the following conclusions:

$$(E) \quad \rho(x, y) \leq \theta \rho(y, x) \text{ for all } x, y \in \mathbf{R}^n.$$

$$(F) \quad \rho(x, y) \leq \theta^2(\rho(x, z) + \rho(z, y)) \text{ for all } x, y, z \in \mathbf{R}^n.$$

Obviously, from the definition of ρ , it is easy to see that

$$(G) \quad \text{for a given section } S(x, t), y \in S(x, t) \text{ if and only if } \rho(x, y) < t.$$

In [CG1], Caffarelli and Gutiérrez defined the Hardy-Littlewood maximal operator M with respect to a family \mathcal{F} of sections and the doubling measure μ by

$$(2.1) \quad Mf(x) = \sup_{t>0} \frac{1}{\mu(S(x, t))} \int_{S(x, t)} |f(y)| d\mu(y).$$

We now give some elementary properties of the operator M .

LEMMA 2.1. *Let M be the Hardy-Littlewood maximal operator defined by (2.1).*

(i) *M is of weak type $(1, 1)$, that is, there exists a constant C_0 such that for all $\lambda > 0$ and any $f \in L^1(\mathbf{R}^n, d\mu)$*

$$\mu(\{x \in \mathbf{R}^n ; Mf(x) > \lambda\}) \leq \frac{C_0}{\lambda} \|f\|_{L^1_\mu}.$$

(ii) *M is of type (p, p) for $1 < p \leq \infty$, that is, there exists a constant C_1 such that for any $f \in L^p(\mathbf{R}^n, d\mu)$*

$$\|Mf\|_{L^p_\mu} \leq C_1 \|f\|_{L^p_\mu}.$$

(iii) *For all $\lambda > 0$, the set $P^\lambda = \{x \in \mathbf{R}^n ; Mf(x) > \lambda\}$ is a open set in \mathbf{R}^n .*

(iv) *Let $f \in L^1(\mathbf{R}^n, d\mu)$ and $\text{supp}(f) \subset S_0 := S(x_0, t_0) \in \mathcal{F}$. Then there exists a constant $C_2 = C_2(A, \theta)$ such that, when $\lambda > C_2 \cdot m_{S_0}(|f|)$,*

$$P^\lambda = \{x \in \mathbf{R}^n ; Mf(x) > \lambda\} \subset S(x_0, 2\theta^2(1 + \theta)t_0),$$

where $m_{S_0}(|f|)$ is the mean of $|f|$ over the section S_0 .

PROOF. See [CG1] for the proof of conclusion (i). From (i) and the obvious boundedness of M on $L^\infty(\mathbf{R}^n, d\mu)$, by applying the Marcinkiewicz interpolation theorem, we get (ii).

Now let us turn to the proof of (iii). Denote by E^c the complement of $E \subset \mathbf{R}^n$. It suffices to show that $(P^\lambda)^c = \{x \in \mathbf{R}^n ; Mf(x) \leq \lambda\}$ is a closed set for all $\lambda > 0$. Let $\{x_k\}_{k=1}^\infty \subset (P^\lambda)^c$ be a sequence of points such that $x_k \rightarrow x$ as $k \rightarrow \infty$. We have to show that,



for any $t > 0$ and $S(x, t) \in \mathcal{F}$,

$$(2.2) \quad \frac{1}{\mu(S(x, t))} \int_{S(x, t)} |f(y)| d\mu(y) \leq \lambda.$$

Denote $S_k = S(x_k, t)$ and $f_k(y) = f(y)\chi_{S(x, t) \Delta S_k}(y)$ for all $k = 1, 2, \dots$, where

$$S(x, t) \Delta S_k = (S(x, t) \setminus S_k) \cup (S_k \setminus S(x, t)).$$

Thus, $|f_k(y)| \leq |f(y)|$ for all k and $\lim_{k \rightarrow \infty} f_k(y) = 0$ (μ -a.e.). Applying the Lebesgue dominated convergence theorem, we have

$$\lim_{k \rightarrow \infty} \frac{1}{\mu(S(x, t))} \int_{S(x, t)} |f_k(y)| d\mu(y) = 0.$$

On the other hand,

$$\frac{1}{\mu(S(x, t))} \int_{S_k} |f(y)| d\mu(y) = \frac{\mu(S_k)}{\mu(S(x, t))} \frac{1}{\mu(S_k)} \int_{S_k} |f(y)| d\mu(y) \leq \frac{\mu(S_k)}{\mu(S(x, t))} \cdot \lambda.$$

Hence

$$\begin{aligned} \frac{1}{\mu(S(x, t))} \int_{S(x, t)} |f(y)| d\mu(y) &\leq \frac{1}{\mu(S(x, t))} \int_{S(x, t) \Delta S_k} |f(y)| d\mu(y) \\ &\quad + \frac{1}{\mu(S(x, t))} \int_{S_k} |f(y)| d\mu(y) \\ &\leq \frac{1}{\mu(S(x, t))} \int_{S(x, t)} |f_k(y)| d\mu(y) + \frac{\mu(S_k)}{\mu(S(x, t))} \cdot \lambda. \end{aligned}$$

Taking $k \rightarrow \infty$, we obtain (2.2).

Finally, we prove the conclusion (iv). Let $x \in \mathbf{R}^n$ and suppose $\rho(x_0, x) \geq 2\theta^2(1 + \theta)t_0$ (equivalently, $x \notin S(x_0, 2\theta^2(1 + \theta)t_0)$ by the property (G) of sections). Then for any $t \leq t_0$, $S(x, t) \cap S(x_0, t_0) = \emptyset$. Indeed, if $y \in S(x, t) \cap S(x_0, t_0)$, then by the properties (E), (F) and (G) of sections

$$\begin{aligned} 2\theta^2(1 + \theta)t_0 \leq \rho(x_0, x) &\leq \theta^2(\rho(x_0, y) + \rho(y, x)) \leq \theta^2(\rho(x_0, y) + \theta\rho(x, y)) \\ &< \theta^2(t_0 + \theta t) \leq \theta^2(1 + \theta)t_0. \end{aligned}$$

The contradiction shows that such y cannot exist. Thus $\int_{S(x, t)} |f(y)| d\mu(y) = 0$ for any section $S(x, t)$ with $t \leq t_0$. Hence, whenever $x \notin S(x_0, 2\theta^2(1 + \theta)t_0)$,

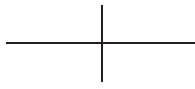
$$Mf(x) = \sup_{t > t_0} \frac{1}{\mu(S(x, t))} \int_{S(x, t)} |f(y)| d\mu(y).$$

On the other hand, for a section $S(x, t)$ with $t > t_0$, we only consider the case that $S(x, t) \cap S(x_0, t_0) \neq \emptyset$. In this case, we take $z \in S(x, t) \cap S(x_0, t_0)$. Using the properties (E) and (F) of sections again, we have

$$S(x_0, t_0) \subset S(z, \theta t_0) \subset S(z, \theta t).$$

On the other hand, by $z \in S(x, t) \subset S(x, \theta t)$ we get $S(z, \theta t) \subset S(x, \theta^2 t)$. Hence

$$(2.3) \quad S(x_0, t_0) \subset S(x, \theta^2 t).$$



By (2.3) and the doubling condition (1.1) of the measure μ ,

$$(2.4) \quad \frac{\mu(S(x_0, t_0))}{\mu(S(x, t))} \leq \frac{\mu(S(x, \theta^2 t))}{\mu(S(x, t))} \leq A^{1+2\log_2 \theta}.$$

Denoting $C_2 = A^{1+2\log_2 \theta}$, we obtain by (2.4) that for $x \notin S(x_0, 2\theta^2(1+\theta)t_0)$ and $t > t_0$

$$\begin{aligned} \frac{1}{\mu(S(x, t))} \int_{S(x, t)} |f(y)| d\mu(y) &\leq \frac{\mu(S(x_0, t_0))}{\mu(S(x, t))} \frac{1}{\mu(S(x_0, t_0))} \int_{S(x_0, t_0)} |f(y)| d\mu(y) \\ &\leq C_2 \cdot m_{S_0}(|f|). \end{aligned}$$

This shows that whenever $x \notin S(x_0, 2\theta^2(1+\theta)t_0)$, we have $Mf(x) \leq C_2 \cdot m_{S_0}(|f|)$. Therefore, if $\lambda > C_2 \cdot m_{S_0}(|f|)$, then $P^\lambda \subset S(x_0, 2\theta^2(1+\theta)t_0)$. This completes the proof of Lemma 2.1.

LEMMA 2.2 (Vitali-Wiener type covering lemma for sections). *Let $E \subset \mathbf{R}^n$ be a bounded set. If for each $x \in E$ there exists a section $S(x, t(x)) \subset E$ with $t(x) > 0$, then there exists a sequence $\{x_j\}_{j=1}^\infty \subset E$ such that*

- (i) $\{S(x_j, t(x_j))\}_{j=1}^\infty$ is a disjoint sequence of sections;
- (ii) $\bigcup_{j=1}^\infty S(x_j, 4\theta^3 t(x_j)) \supset E$.

PROOF. Denote $\mathcal{F}_E = \{S(x, t(x)); x \in E\}$. Since E is a bounded set, we may assume that

$$L = \sup\{t(x); S(x, t(x)) \in \mathcal{F}_E\} < \infty.$$

Take $x_1 \in E$ such that $t(x_1) > L/2$. If $E \setminus S(x_1, 4\theta^3 t(x_1)) = \emptyset$, then we stop. Otherwise, we take $x_2 \in E \setminus S(x_1, 4\theta^3 t(x_1))$ such that

$$t(x_2) > \frac{1}{2} \sup\{t(x); S(x, t(x)) \in \mathcal{F}_E \text{ and } x \in E \setminus S(x_1, 4\theta^3 t(x_1))\}.$$

If $E \setminus \{S(x_1, 4\theta^3 t(x_1)) \cup S(x_2, 4\theta^3 t(x_2))\} = \emptyset$, then we stop. Otherwise, we take $x_3 \in E \setminus \{S(x_1, 4\theta^3 t(x_1)) \cup S(x_2, 4\theta^3 t(x_2))\}$ such that

$$t(x_3) > \frac{1}{2} \sup\{t(x); S(x, t(x)) \in \mathcal{F}_E \text{ and } x \in E \setminus \{S(x_1, 4\theta^3 t(x_1)) \cup S(x_2, 4\theta^3 t(x_2))\}\}.$$

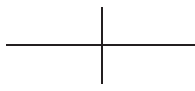
If $E \subset \bigcup_{j=1}^3 S(x_j, 4\theta^3 t(x_j))$, then we stop. Otherwise, we will continue the same process. In general, for the j th-stage we pick $x_j \in E \setminus \bigcup_{i=1}^{j-1} S(x_i, 4\theta^3 t(x_i))$ such that

$$(2.5) \quad t(x_j) > \frac{1}{2} \sup \left\{ t(x); S(x, t(x)) \in \mathcal{F}_E \text{ and } x \in E \setminus \bigcup_{i=1}^{j-1} S(x_i, 4\theta^3 t(x_i)) \right\}.$$

Continuing in this way, we construct a sequence of sections in \mathcal{F}_E , possibly infinite and denoted by $\{S(x_j, t(x_j))\}_{j=1}^\infty$, satisfying the following conditions:

- (a) For $j > 1$, $x_j \notin \bigcup_{i=1}^{j-1} S(x_i, 4\theta^3 t(x_i))$.
- (b) For $i < j$, $t(x_i) > (1/2)t(x_j)$.





We first show that $\{S(x_j, t(x_j))\}$ is disjoint. Suppose that $y \in S(x_i, t(x_i)) \cap S(x_j, t(x_j))$. Without loss of generality, we may assume that $i < j$. Hence $t(x_i) > (1/2)t(x_j)$. By the properties (E), (F) and (G), we have

$$\begin{aligned} \rho(x_i, x_j) &\leq \theta^2(\rho(x_i, y) + \rho(y, x_j)) \leq \theta^2(\rho(x_i, y) + \theta\rho(x_j, y)) \\ &< \theta^2(t(x_i) + \theta t(x_j)) < \theta^2(1 + 2\theta)t(x_i) \\ &< 4\theta^3 t(x_i). \end{aligned}$$

Using the property (G) again, we get $x_j \in S(x_i, 4\theta^3 t(x_i))$. However, this contradicts the condition (a).

Now we prove that $E \subset \bigcup_{j=1}^{\infty} S(x_j, 4\theta^3 t(x_j))$. If it is not the case, then there exists $x_0 \in E$ such that $x_0 \notin \bigcup_{j=1}^{\infty} S(x_j, 4\theta^3 t(x_j))$. So, there exists a section $S(x_0, t(x_0)) \in \mathcal{F}_E$ with $t(x_0) > 0$. Since $\{S(x_j, t(x_j))\}_{j=1}^{\infty}$ is disjoint and $\bigcup_{j=1}^{\infty} S(x_j, t(x_j)) \subset E$ is bounded, we have

$$\infty > |E| \geq \left| \bigcup_{j=1}^{\infty} S(x_j, t(x_j)) \right| = \sum_{j=1}^{\infty} |S(x_j, t(x_j))|,$$

where $|E|$ denotes the Lebesgue measure of the set E . From this we get

$$\lim_{j \rightarrow \infty} |S(x_j, t(x_j))| = 0,$$

and hence

$$(2.6) \quad \lim_{j \rightarrow \infty} t(x_j) = 0,$$

because, for each j , $S(x_j, t(x_j))$ is a bounded, convex, open set in \mathbf{R}^n . By (2.6) we may choose j large enough such that $2t(x_j) < t(x_0)$. However, this contradicts $t(x_j) > (1/2)t(x_0)$ by (2.5), because

$$x_0 \in E \setminus \bigcup_{k=1}^{\infty} S(x_k, 4\theta^3 t(x_k)) \subset E \setminus \bigcup_{i=1}^{j-1} S(x_i, 4\theta^3 t(x_i)).$$

Thus we finish the proof of Lemma 2.2.

Before proving the following covering lemma, let us recall another property of sections. In [AFT], the authors proved that if a family \mathcal{F} of sections satisfies the properties (A), (B) and (C), then there exists a quasi-metric $d(x, y)$ on \mathbf{R}^n with respect to \mathcal{F} defined by

$$d(x, y) = \inf\{r; x \in S(y, r) \text{ and } y \in S(x, r)\}.$$

The triangular constant of the quasi-metric d is just the θ appeared in the property (D), that is,

$$d(x, y) \leq \theta(d(x, z) + d(z, y)) \quad \text{for any } x, y, z \in \mathbf{R}^n.$$

Moreover, denoting by $B_d(x, r) = \{y \in \mathbf{R}^n; d(x, y) < r\}$ the d -ball of center x with radius r , we have the following facts.



LEMMA 2.3. *Let E be an open set in \mathbf{R}^n and E^c denote the complement of E . For any $x \in E$, write $r = d(x, E^c) = \inf\{d(x, y) ; y \in E^c\}$. Then*

- (i) $d(x, E^c) > 0$;
- (ii) $B_d(x, r) \subset E$;
- (iii) $B_d(x, 2r) \cap E^c \neq \emptyset$.

PROOF. (i) If $d(x, E^c) = 0$, then there exists a sequence $\{y_n\} \in E^c$ such that $d(x, y_n) < 1/n$ for each n . Hence, $y_n \in S(x, 1/n)$ for every n . On the other hand, since E is open, there is an $\varepsilon > 0$ such that $B(x, \varepsilon) = \{y \in \mathbf{R}^n ; |x - y| < \varepsilon\} \subset E$. By the property (C) of sections,

$$y_n \in S(x, 1/n) \subset B(x, \varepsilon) \subset E \quad \text{when } n \text{ is large enough.}$$

But this is impossible because $\{y_n\} \in E^c$ for all n .

- (ii) If $B_d(x, r) \cap E^c \neq \emptyset$, then there exists $y_0 \in B_d(x, r) \cap E^c$. Thus

$$r = d(x, E^c) = \inf\{d(x, y) ; y \in E^c\} \leq d(x, y_0) < r.$$

This contradiction shows that $B_d(x, r) \subset E$.

(iii) If $B_d(x, 2r) \subset E$, then we have $y \in B_d(x, 2r) \subset E$ whenever $d(x, y) < 2r$. On the other hand, there exists a sequence $\{y_n\} \subset E^c$ such that $d(x, y_n) < d(x, E^c) + 1/n = r + 1/n$ for all $n \in \mathbf{N}$. Since $r > 0$, we have $r + 1/n < 2r$, when n is large enough. Thus $y_n \in B_d(x, 2r) \subset E$ for n large enough. However, this contradicts $\{y_n\} \subset E^c$ for all n .

The following relationship between a section and a d -ball can be found in [AFT].

- (H) For any $x \in \mathbf{R}^n$ and any $r > 0$, $S(x, r/2\theta) \subset B_d(x, r) \subset S(x, r)$.

Now let us state and prove the Whitney type covering lemma for sections.

LEMMA 2.4 (Whitney type covering lemma for sections). *Suppose that $E \subset \mathbf{R}^n$ is a bounded open set in \mathbf{R}^n and $C \geq 1$. Then there exists a sequence of sections $\{S(x_k, t_k)\}_{k=1}^\infty$ satisfying the following:*

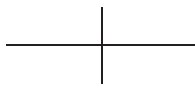
- (i) Let $S_k = S(x_k, t_k)$. Then $E = \bigcup_{k=1}^\infty S_k$.
- (ii) Let $\tilde{S}_k = S(x_k, 16C\theta^3 t_k)$. Then for each k , $\tilde{S}_k \cap E^c \neq \emptyset$.
- (iii) Let $\bar{S}_k = S(x_k, 2C\theta t_k)$. Then $\{\bar{S}_k\}_{k=1}^\infty$ is a Θ -disjoint collection, that is, there exists a constant $\Theta = \Theta(A, \theta, C)$ such that $\sum_{k=1}^\infty \chi_{\bar{S}_k}(x) \leq \Theta$.

PROOF. Let $r(x) = d(x, E^c)$ for $x \in E$. By property (H), we have

$$(2.7) \quad \begin{aligned} S\left(x, \frac{r(x)}{8\theta^3 C}\right) &\subset B_d\left(x, \frac{r(x)}{4\theta^2}\right) \subset S\left(x, \frac{r(x)}{4\theta^2}\right) \subset B_d\left(x, \frac{r(x)}{2\theta}\right) \\ &\subset S\left(x, \frac{r(x)}{2\theta}\right) \subset B_d(x, r(x)) \subset E. \end{aligned}$$

Therefore, the family of sections $\{S(x, r(x)/4\theta^3 8\theta^3 C) ; x \in E\}$ satisfies the condition of Lemma 2.2. By the conclusions of Lemma 2.2, there exists a sequence $\{x_k\}_{k=1}^\infty \subset E$ such that

- (a) $\{S(x_k, r_k/4\theta^3 8\theta^3 C)\}_{k=1}^\infty$ is a disjoint sequence of sections,
- (b) $\bigcup_{k=1}^\infty S(x_k, r_k/8\theta^3 C) \supset E$,



where and below we denote $r(x_k)$ by r_k for simplicity. By (2.7) and (b) we obtain

$$(2.8) \quad E \subset \bigcup_{k=1}^{\infty} S\left(x_k, \frac{r_k}{8\theta^3 C}\right) \subset \bigcup_{k=1}^{\infty} S\left(x_k, \frac{r_k}{4\theta^2}\right) \subset \bigcup_{k=1}^{\infty} B_d\left(x_k, \frac{r_k}{2\theta}\right) \subset E.$$

We first prove that $\{B_d(x_k, r_k/2\theta)\}_{k=1}^{\infty}$ is a Θ -disjoint collection. Let $z_0 \in B_d(x_k, r_k/2\theta)$ and denote $R_0 = d(z_0, E^c)$. Then

$$r_k = d(z_k, E^c) \leq \theta[d(x_k, z_0) + d(z_0, E^c)] \leq \theta\left(\frac{r_k}{2\theta} + R_0\right) = \frac{r_k}{2} + \theta R_0.$$

Thus $r_k \leq 2\theta R_0$. From this, we have

$$(2.9) \quad B_d(x_k, r_k/2\theta) \subset B_d(z_0, 2\theta R_0) \quad \text{for each } k.$$

Indeed, for any $y \in B_d(x_k, r_k/2\theta)$,

$$d(z_0, y) \leq \theta[d(z_0, x_k) + d(x_k, y)] \leq \theta(r_k/2\theta + r_k/2\theta) \leq 2\theta R_0.$$

On the other hand, we see that

$$\begin{aligned} R_0 = d(z_0, E^c) &\leq \theta[d(z_0, x_k) + d(x_k, E^c)] \\ &\leq \theta\left(\frac{r_k}{2\theta} + r_k\right) = \left(\frac{1}{2} + \theta\right)r_k = \left(\frac{1}{2} + \theta\right)4\theta^3 8\theta^3 C \cdot \frac{r_k}{4\theta^3 8\theta^3 C}. \end{aligned}$$

Equivalently,

$$(2.10) \quad \frac{r_k}{4\theta^3 8\theta^3 C} \geq \frac{R_0}{(1/2 + \theta)4\theta^3 8\theta^3 C}.$$

Now we assume that

$$(2.11) \quad z_0 \in \bigcap_j B_d(x_{k_j}, r_{k_j}/2\theta).$$

To simplify the notation we denote $x_j = x_{k_j}$ and $r_j = r_{k_j}$. Then by (2.9), for each j ,

$$B_d\left(x_j, \frac{r_j}{4\theta^3 8\theta^3 C}\right) \subset B_d\left(x_j, \frac{r_j}{2\theta}\right) \subset B_d(z_0, 2\theta R_0).$$

Note that for each j , $B_d(x_j, r_j/4\theta^3 8\theta^3 C) \subset S(x_j, r_j/4\theta^3 8\theta^3 C)$ by (H). Hence, the sequence $\{B_d(x_j, r_j/4\theta^3 8\theta^3 C)\}_{j=1}^{\infty}$ is also disjoint by (a). Thus by (2.10)

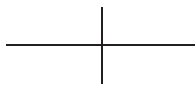
$$d(x_i, x_j) \geq \min\left\{\frac{r_i}{4\theta^3 8\theta^3 C}, \frac{r_j}{4\theta^3 8\theta^3 C}\right\} \geq \frac{R_0}{(1/2 + \theta)4\theta^3 8\theta^3 C}.$$

By Lemma 1.1 in [CW1], there exists a constant $\Theta = \Theta(A, \theta, C)$ such that the numbers of j in (2.11) cannot be greater than K . By the Θ -disjointness of $\{B_d(x_k, r_k/2\theta)\}_{k=1}^{\infty}$ and (2.7), we obtain the Θ -disjointness of $\{S(x_k, r_k/4\theta^2)\}_{k=1}^{\infty}$.

Finally, we take $t_k = r_k/8\theta^3 C$. Then by (2.8) we get the conclusions (i) and (iii) of Lemma 2.4. As for the conclusion (ii), it is a direct result of Lemma 2.3 (ii), because

$$\tilde{S}_k = S(x_k, 16C\theta^3 t_k) = S(x_k, 2r_k) \supset B_d(x_k, 2r_k).$$





Therefore we complete the proof of Lemma 2.4.

The following fact is obvious.

LEMMA 2.5. *Suppose that $F_k \subset E_k$ for each k , and $\{E_k\}_{k=1}^\infty$ is a Θ -disjoint collection. Then $\{F_k\}_{k=1}^\infty$ is also a Θ -disjoint collection.*

REMARK 2.1. By the conclusion (iii) of Lemma 2.4 and Lemma 2.5, $\{S_k\}_{k=1}^\infty$ is also a Θ -disjoint collection, since $S_k \subset \bar{S}_k$ for each k .

3. Proof of theorem 1.1. First it is easy to see that for all $q > 1$, $H_{\mathcal{F}}^{1,\infty}(\mathbf{R}^n) \subset H_{\mathcal{F}}^{1,q}(\mathbf{R}^n)$. We now show that the opposite inclusion also holds. It suffices to prove that every $(1, q)$ -atom $a(x)$ has the representation

$$(3.1) \quad a(x) = \sum_j \alpha_j a_j(x),$$

where each $a_j(x)$ is a $(1, \infty)$ -atom and $\sum_j |\alpha_j| < \infty$.

Since $a(x)$ is a $(1, q)$ -atom, there exists a section $S_0 = S(x_0, t_0) \in \mathcal{F}$ such that $\text{supp}(a) \subset S(x_0, t_0)$. We denote $b(x) = \mu(S_0)a(x)$. Then

$$(3.2) \quad \text{(i) } \text{supp}(b) \subset S_0, \quad \text{(ii) } \int b(x)d\mu(x) = 0, \quad \text{and} \quad \text{(iii) } \|b\|_{L^q(\mu)} \leq (\mu(S_0))^{1/q}.$$

On the other hand, we take the constant $C = \theta(1 + \theta)$ in Lemma 2.4. Then by (1.1) we have

$$(3.3) \quad \frac{\mu(\bar{S}_k)}{\mu(S_k)} \leq A^{2+\log_2 \theta^2(1+\theta)} := K_0 \quad \text{for every } k.$$

For a positive integer m , let $N^m = N \times N \times \dots \times N$ and $N^0 = \{0\}$. We denote the general element in N^m by j_m . We prove the following proposition by an inductive argument on m .

PROPOSITION 3.1. *There exists a sequence of sections $\{S_{j_\ell}\} \subset \mathcal{F}$, $j_\ell \in N^\ell$, $\ell = 0, 1, \dots$, such that for each natural number m*

$$(3.4) \quad b(x) = D_0 \Theta \alpha \sum_{\ell=0}^{m-1} \alpha^\ell \sum_{j_\ell \in N^\ell} \mu(\bar{S}_{j_\ell}) a_{j_\ell}(x) + \sum_{j_m \in N^m} h_{j_m}(x),$$

where $\alpha = \alpha(q, A, \theta)$, $D_0 = D_0(A, \theta)$, and

- (I) $a_{j_\ell}(x)$ is a $(1, \infty)$ -atom supported in \bar{S}_{j_ℓ} , $j_\ell \in N^\ell$, $\ell = 0, 1, \dots, m - 1$;
- (II) $\bigcup_{j_m \in N^m} S_{j_m} \subset \{x \in \mathbf{R}^n; (M_q b)(x) > \alpha^m / 2\}$, and $(M_q b)(x) = [M(|b|^q)(x)]^{1/q}$;
- (III) $\{\bar{S}_{j_\ell}\}$ is a Θ^ℓ -disjoint collection;
- (IV) the functions $h_{j_m}(x)$ are supported in S_{j_m} ;
- (V) $\int h_{j_m}(x)d\mu(x) = 0$;
- (VI) $|h_{j_m}(x)| \leq |b(x)| + D_0 \alpha^m \chi_{S_{j_m}}(x)$;
- (VII) $[m_{S_{j_m}}(|h_{j_m}|^q)]^{1/q} \leq 2D_0 \alpha^m$.



We first show that if the properties from (I) to (VII) hold for each $m \in N$, then (3.1) holds. By (3.3), (II), (III), Lemma 2.5 and Lemma 2.1 (i), we have

$$(3.5) \quad \begin{aligned} \sum_{j_m \in \mathcal{N}^m} \mu(\bar{S}_{j_m}) &\leq K_0 \sum_{j_m \in \mathcal{N}^m} \mu(S_{j_m}) \leq K_0 \Theta^m \mu\left(\bigcup_{j_m \in \mathcal{N}^m} S_{j_m}\right) \\ &\leq K_0 \Theta^m \mu(\{x \in \mathbf{R}^n; (M_q b)(x) > \alpha^m/2\}) \\ &\leq K_0 \Theta^m C_0 (2/\alpha^m)^q \|b\|_{L_\mu^q}^q. \end{aligned}$$

In the last inequality, we use the conclusion (i) of Lemma 2.1. By (iii) in (3.2)

$$\sum_{m=1}^{\infty} \alpha^m \sum_{j_m \in \mathcal{N}^m} \mu(\bar{S}_{j_m}) \leq C_0 K_0 2^q \sum_{m=1}^{\infty} (\Theta \alpha^{1-q})^m \mu(S_0).$$

Hence, if we choose α such that $\alpha > \Theta^{1/(q-1)}$, then

$$(3.6) \quad \sum_{m=1}^{\infty} \alpha^m \sum_{j_m \in \mathcal{N}^m} \mu(\bar{S}_{j_m}) \leq B \mu(S_0),$$

where $B = B(q, A, \theta, \alpha)$ is independent of $a(x)$.

By (IV) and (VII) we have

$$(3.7) \quad \int |h_{j_m}(x)| d\mu(x) \leq \mu(S_{j_m}) \left(\frac{1}{\mu(S_{j_m})} \int_{S_{j_m}} |h_{j_m}(x)|^q d\mu(x) \right)^{1/q} \leq \mu(S_{j_m}) \cdot 2D_0 \alpha^m.$$

Denote $H_m(x) = \sum_{j_m \in \mathcal{N}^m} h_{j_m}(x)$. Then (3.5) and (3.7) imply

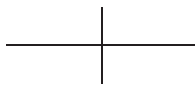
$$(3.8) \quad \begin{aligned} \int |H_m(x)| d\mu(x) &\leq \sum_{j_m \in \mathcal{N}^m} \int |h_{j_m}(x)| d\mu(x) \\ &\leq 2D_0 \alpha^m \sum_{j_m \in \mathcal{N}^m} \mu(S_{j_m}) \leq 2^{q+1} C_0 K_0 D_0 (\Theta \alpha^{1-q})^m \|b\|_{L_\mu^q}^q. \end{aligned}$$

Thus, if $\alpha > \Theta^{1/(q-1)}$, then by (3.8)

$$(3.9) \quad \lim_{m \rightarrow \infty} \int |H_m(x)| d\mu(x) \leq C \mu(S_0) \cdot \lim_{m \rightarrow \infty} (\Theta \alpha^{1-q})^m = 0.$$

On the other hand, by (I) and (3.6),

$$(3.10) \quad \begin{aligned} &\int D_0 \Theta \alpha \sum_{i=m}^{\infty} \alpha^i \sum_{j_i \in \mathcal{N}^i} \mu(\bar{S}_{j_i}) |a_{j_i}(x)| d\mu(x) \\ &= \int_{\bar{S}_{j_i}} D_0 \Theta \alpha \sum_{i=m}^{\infty} \alpha^i \sum_{j_i \in \mathcal{N}^i} \mu(\bar{S}_{j_i}) |a_{j_i}(x)| d\mu(x) \end{aligned}$$



$$\begin{aligned} &\leq D_0 \Theta \alpha \sum_{i=m}^{\infty} \alpha^i \sum_{j_i \in \mathcal{N}^i} \mu(\bar{S}_{j_i}) \|a_{j_i}\|_{L_{\mu}^{\infty}} \cdot \mu(\bar{S}_{j_i}) \\ &\leq D_0 \Theta \alpha \sum_{i=m}^{\infty} \alpha^i \sum_{j_i \in \mathcal{N}^i} \mu(\bar{S}_{j_i}) \rightarrow 0 \quad (m \rightarrow \infty). \end{aligned}$$

It follows from (3.9) and (3.10) that, when $m \rightarrow \infty$,

$$\frac{D_0 \Theta \alpha}{\mu(S_0)} \sum_{\ell=0}^{m-1} \alpha^{\ell} \sum_{j_{\ell} \in \mathcal{N}^{\ell}} \mu(\bar{S}_{j_{\ell}}) a_{j_{\ell}}(x) + \frac{1}{\mu(S_0)} \sum_{j_m \in \mathcal{N}^m} h_{j_m}(x)$$

converges to $b(x)/\mu(S_0) = a(x)$ in the L_{μ}^1 norm. Thus, in the sense of distribution we have

$$a(x) = \frac{D_0 \Theta \alpha}{\mu(S_0)} \sum_{m=1}^{\infty} \alpha^m \sum_{j_m \in \mathcal{N}^m} \mu(\bar{S}_{j_m}) a_{j_m}(x),$$

where each $a_{j_m}(x)$ is a $(1, \infty)$ -atom and

$$\frac{D_0 \Theta \alpha}{\mu(S_0)} \sum_{m=0}^{\infty} \alpha^m \sum_{j_m \in \mathcal{N}^m} \mu(\bar{S}_{j_m}) \leq B < \infty.$$

From this, we obtain $a(x) \in H_{\mathcal{F}}^{1, \infty}(\mathbf{R}^n)$. Hence, to prove Theorem 1.1, it remains only to show that the properties from (I) to (VII) hold for each $m \in \mathbf{N}$.

PROOF OF PROPOSITION 3.1. We first show that these properties are valid for $m = 1$. Let $E^{\alpha} = \{x \in \mathbf{R}^n; (M_q b)(x) > \alpha\}$. By (iii) in (3.2) and Lemma 2.1 (iv), if $\alpha^q > C_2 \geq C_2 \cdot m_{S_0}(|b|^q)$, then

$$E^{\alpha} \subset S(x_0, 2\theta^2(1+\theta)t_0) := \bar{S}_0.$$

From this and Lemma 2.1 (iii), E^{α} is a bounded open set if $\alpha^q > C_2$. By Lemma 2.1 (i), we have

$$(3.11) \quad \mu(E^{\alpha}) \leq C_0 (\|b\|_{L_{\mu}^q} / \alpha)^q \leq C_0 \alpha^{-q} \mu(S_0).$$

Applying Lemma 2.4 to E^{α} with the constant $C = \theta(1+\theta)$, we obtain a sequence of sections $\{S_j = S(x_j, t_j)\}_{j=1}^{\infty}$ satisfying

$$(II) \quad \bigcup_j S_j = E^{\alpha} \subset \{x \in \mathbf{R}^n; (M_q b)(x) > \alpha^m/2\},$$

$$(III) \quad \{\bar{S}_j = S(x_j, 2\theta^2(1+\theta)t_j)\} \text{ is a } \Theta\text{-disjoint collection,}$$

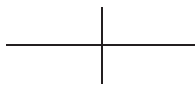
and for each j

$$(3.12) \quad \bar{S}_j \cap (E^{\alpha})^c \neq \emptyset, \quad \text{where } \bar{S}_j = S(x_j, 16\theta^4(1+\theta)t_j).$$

If we denote by $\chi_j(x)$ the characteristic function of S_j , then $\sum_{j=1}^{\infty} \chi_j(x) \leq \Theta$ by Remark 2.1. Let

$$\eta_j(x) = \begin{cases} \chi_j(x) / \sum_j \chi_j(x) & \text{if } x \in E^{\alpha}, \\ 0 & \text{if } x \notin E^{\alpha}, \end{cases}$$





and

$$g_0(x) = \begin{cases} b(x) & \text{if } x \notin E^\alpha, \\ \sum_j m_{S_j}(\eta_j b)\chi_j(x) & \text{if } x \in E^\alpha. \end{cases}$$

In addition, $h_j(x) = b(x)\eta_j(x) - m_{S_j}(\eta_j b)\chi_j(x)$ for any $x \in \mathbf{R}^n$. Then $b(x) = g_0(x) + \sum_{j=1}^{\infty} h_j(x)$ for any $x \in \mathbf{R}^n$.

By the property (C) of sections and the fact that the Hardy-Littlewood maximal operator M related to sections is of weak type (1,1) (see Lemma 2.1 (i)), it is easy to check that the Lebesgue differential theorem holds for the family \mathcal{F} of sections. So, if $x \notin E^\alpha$, we have

$$|g_0(x)| \leq |b(x)| \leq (M_q b)(x) \leq \alpha.$$

On the other hand, by (3.12) there exists $z_j \in \tilde{S}_j \cap (E^\alpha)^c$. By the property (D) of sections, we have

$$(3.13) \quad \tilde{S}_j = S(x_j, 16\theta^4(1+\theta)t_j) \subset S(z_j, 16\theta^5(1+\theta)t_j)$$

and

$$(3.14) \quad S(z_j, 16\theta^4(1+\theta)t_j) \subset S(x_j, 16\theta^5(1+\theta)t_j).$$

The above (3.13) yields

$$(3.15) \quad S(x_j, t_j) \subset \tilde{S}_j \subset S(z_j, 16\theta^5(1+\theta)t_j),$$

which implies

$$\begin{aligned} \left(\frac{1}{\mu(S_j)} \int_{S_j} |b(x)|^q d\mu(x) \right)^{1/q} &\leq \left(\frac{\mu(S(z_j, 16\theta^5(1+\theta)t_j))}{\mu(S_j)} \right)^{1/q} \\ &\times \left(\frac{1}{\mu(S(z_j, 16\theta^5(1+\theta)t_j))} \int_{S(z_j, 16\theta^5(1+\theta)t_j)} |b(x)|^q d\mu(x) \right)^{1/q} \\ &\leq \left(\frac{\mu(S(z_j, 16\theta^5(1+\theta)t_j))}{\mu(S_j)} \right)^{1/q} \cdot (M_q b)(z_j). \end{aligned}$$

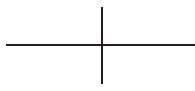
Using the inclusion relations (3.14) and (3.15) again, we have

$$\begin{aligned} \frac{\mu(S(z_j, 16\theta^5(1+\theta)t_j))}{\mu(S_j)} &= \frac{\mu(S(z_j, 16\theta^5(1+\theta)t_j))}{\mu(S(z_j, 16\theta^4(1+\theta)t_j))} \\ &\quad \times \frac{\mu(S(z_j, 16\theta^4(1+\theta)t_j))}{\mu(S(x_j, 16\theta^5(1+\theta)t_j))} \cdot \frac{\mu(S(x_j, 16\theta^5(1+\theta)t_j))}{\mu(S_j)} \\ &\leq A^{1+\log_2 \theta} \cdot A^{5+\log_2 \theta^5(1+\theta)}, \end{aligned}$$

and hence

$$(3.16) \quad \left(\frac{1}{\mu(S_j)} \int_{S_j} |b(x)|^q d\mu(x) \right)^{1/q} \leq (A^{6+\log_2 \theta^6(1+\theta)})^{1/q} (M_q b)(z_j).$$





Thus, if $x \in E^\alpha$, by Remark 2.1 together with (3.16) and noting that $z_j \in (E^\alpha)^c$, we obtain

$$\begin{aligned} |g_0(x)| &\leq \sum_{\substack{\text{at most} \\ \Theta \text{ terms}}} \frac{1}{\mu(S_j)} \int_{S_j} |b(x)\eta_j(x)| d\mu(x) \\ &\leq \sum_{\text{ibid}} \left(\frac{1}{\mu(S_j)} \int_{S_j} |b(x)|^q d\mu(x) \right)^{1/q} \leq \Theta D_0 \alpha, \end{aligned}$$

where $D_0 = (A^{6+\log_2 \theta^6(1+\theta)})^{1/q}$. This shows that

(1) $|g_0(x)| \leq \Theta D_0 \alpha$ for any $x \in \mathbf{R}^n$.

Since $E^\alpha \subset \bar{S}_0$ and $g_0(x) = b(x)$ for $x \notin E^\alpha$, by (i) in (3.2), we have

(2) $\text{supp}(g_0) \subset \bar{S}_0$.

By the definition of $h_j(x)$, we have

(IV) $\text{supp}(h_j) \subset S_j$ for each j ,

(V) $\int h_j(x) d\mu(x) = 0$ for each j .

Noting that $\|h_j\|_{L_\mu^1} \leq 2\|\chi_j\|_{L_\mu^1} = 2 \int_{S_j} |b(x)| d\mu(x)$ and by Remark 2.1, we have

$$\begin{aligned} \sum_j \|h_j\|_{L_\mu^1} &\leq 2 \sum_j \int_{S_j} |b(x)| d\mu(x) \leq 2\Theta \int_{\cup_j S_j} |b(x)| d\mu(x) \\ &\leq 2\Theta \|b\|_{L_\mu^1} \leq 2\Theta \|b\|_{L_\mu^q(\mu(S_0))}^{1/q'} \leq 2\Theta \mu(S_0). \end{aligned}$$

Hence $g_0(x) + \sum_{j=1}^\infty h_j(x)$ converges to $b(x)$ in the L_μ^1 norm. In fact, it is also convergent almost everywhere, since the sum has at most Θ terms. Thus, by (V) and (ii) in (3.2), we obtain

(3) $\int g_0(x) d\mu(x) = 0$.

Set $a_0(x) = g_0(x)(D_0\Theta\alpha\mu(\bar{S}_0))^{-1}$. From the facts (1), (2), and (3), we see that $a_0(x)$ is a $(1, \infty)$ -atom supported in the section \bar{S}_0 , which is just (I). So, we have

$$b(x) = D_0\Theta\alpha\mu(\bar{S}_0)a_0(x) + \sum_{j=1}^\infty h_j(x),$$

which is (3.4) for $m = 1$. It follows from (3.16) that

$$(3.17) \quad m_{S_j}(|b\eta_j|) \leq \left(\frac{1}{\mu(S_j)} \int_{S_j} |b(x)|^q d\mu(x) \right)^{1/q} \leq D_0 \cdot (M_q b)(z_j) \leq D_0 \alpha,$$

since $z_j \notin E^\alpha$. Hence $|h_j(x)| \leq |b(x)| + m_{S_j}(|b\eta_j|)\chi_j(x) \leq |b(x)| + D_0\alpha\chi_j(x)$ by (3.17), and (VI) holds. Finally, using (3.17) again, it is easy to check that (VII) is also valid. Thus we prove Proposition 3.1 for $m = 1$.

We now assume that Proposition 3.1 holds for m , and show that it is also true for $m + 1$. Let $E_{j_m} = \{x \in \mathbf{R}^n ; (M_q h_{j_m})(x) > \alpha^{m+1}\}$. By the hypothesis (IV), $\text{supp}(h_{j_m}) \subset S_{j_m} = S(x_{j_m}, t_{j_m})$. If $\alpha^q > C_2(2D_0)^q$, then by (VII) we have

$$C_2 m_{S_{j_m}}(|h_{j_m}|^q) \leq C_2((2D_0\alpha^m)^q < \alpha^{q(m+1)}).$$



Apply Lemma 2.1 (iv) to get $E_{j_m} \subset \bar{S}_{j_m} := S(x_{j_m}, 2\theta^2(1+\theta)t_{j_m})$. Thus E_{j_m} is a bounded open set if $\alpha^q > C_2(2D_0)^q$ by Lemma 2.1 (iii). Applying Lemma 2.4 for E_{j_m} with the constant $C = \theta(1+\theta)$, we obtain a sequence of sections $\{S_{j_m}^i = S(x_{j_m}^i, t_{j_m}^i)\}_{i=1}^\infty$ such that

- (4) $\bigcup_i S_{j_m}^i = E_{j_m} \subset \{x \in \mathbf{R}^n ; (M_q h_{j_m})(x) > \alpha^{m+1}/2\}$,
- (5) $\{\bar{S}_{j_m}^i := S(x_{j_m}^i, 2\theta^2(1+\theta)t_{j_m}^i)\}_{i=1}^\infty$ is a Θ -disjoint collection,
- (6) for each i , $\bar{S}_{j_m}^i \cap (E_{j_m})^c \neq \emptyset$, where $\bar{S}_{j_m}^i := S(x_{j_m}^i, 16\theta^4(1+\theta)t_{j_m}^i)$.

By the hypothesis (III) for m , we know that $\{\bar{S}_{j_m}\}$ is a Θ^m -disjoint collection, since the totality of sections in the family $\{\bar{S}_{j_m}^i\}$ is Θ^{m+1} -disjoint for all $j_m \in N^m$ and $i \in N$. This shows that (III) holds for $m+1$.

Now denote the characteristic function of section $S_{j_m}^i$ by $\chi_{j_m}^i(x)$. Then it follows from (5) and Lemma 2.5 that $\sum_{i=1}^\infty \chi_{j_m}^i(x) \leq \Theta$. Let

$$\eta_{j_m}^i(x) = \begin{cases} \chi_{j_m}^i(x) / \sum_\ell \chi_{j_m}^\ell(x) & \text{if } x \in E_{j_m}, \\ 0 & \text{if } x \notin E_{j_m}, \end{cases}$$

and

$$g_{j_m}(x) = \begin{cases} h_{j_m}(x) & \text{if } x \notin E_{j_m}, \\ \sum_i m_{S_{j_m}^i} (h_{j_m} \eta_{j_m}^i) \chi_{j_m}^i(x) & \text{if } x \in E_{j_m}. \end{cases}$$

In addition, we have $h_{j_m}^i(x) = h_{j_m}(x) \eta_{j_m}^i(x) - m_{S_{j_m}^i} (h_{j_m} \eta_{j_m}^i) \chi_{j_m}^i(x)$ for any $x \in \mathbf{R}^n$.

If $x \notin E_{j_m}$, then

$$|g_{j_m}(x)| \leq |h_{j_m}(x)| \leq (M_q h_{j_m})(x) \leq \alpha^{m+1}.$$

On the other hand, by (6) and by making use of the properties of sections and the same idea as in proving (3.16), we may get

$$(3.18) \quad \left(\frac{1}{\mu(S_{j_m}^i)} \int_{S_{j_m}^i} |h_{j_m}(x)|^q d\mu(x) \right)^{1/q} \leq (A^{6+\log_2 \theta^6(1+\theta)})^{1/q} (M_q h_{j_m})(z_j) \leq D_0 \alpha^{m+1},$$

where $z_j \in \bar{S}_{j_m}^i \cap (E_{j_m})^c$ and $D_0 = (A^{6+\log_2 \theta^6(1+\theta)})^{1/q}$. Hence, if $x \in E_{j_m}$, then by (5), Lemma 2.5 and (3.18) we have

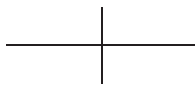
$$\begin{aligned} |g_{j_m}(x)| &\leq \sum_{\substack{\text{at most} \\ \Theta \text{ terms}}} \frac{1}{\mu(S_{j_m}^i)} \int_{S_{j_m}^i} |h_{j_m}(x) \eta_{j_m}^i(x)| d\mu(x) \\ &\leq \sum_{\text{ibid}} \left(\frac{1}{\mu(S_{j_m}^i)} \int_{S_{j_m}^i} |h_{j_m}(x)|^q d\mu(x) \right)^{1/q} \\ &\leq \Theta D_0 \alpha^{m+1}. \end{aligned}$$

Thus we obtain

$$(7) \quad |g_{j_m}(x)| \leq \Theta D_0 \alpha^{m+1} \text{ for any } x \in \mathbf{R}^n.$$

Since $E_{j_m} \subset \bar{S}_{j_m}$, by the definition of $g_{j_m}(x)$ we have

$$(8) \quad \text{supp}(g_{j_m}) \subset \bar{S}_{j_m}.$$



In addition, it is obvious that $\text{supp}(h_{j_m}^i) \subset S_{j_m}^i$ and $\int h_{j_m}^i(x) d\mu(x) = 0$ for each j . Thus (IV) and (V) hold for $m + 1$. Since $\|h_{j_m}^i\|_{L_\mu^1} \leq 2\|h_{j_m} \chi_{j_m}^i\|_{L_\mu^1} = 2 \int_{S_{j_m}^i} |h_{j_m}(x)| d\mu(x)$, by (5) together with Lemma 2.5 we have

$$\begin{aligned} \sum_i \|h_{j_m}^i\|_{L_\mu^1} &\leq 2 \sum_i \int_{S_{j_m}^i} |h_{j_m}(x)| d\mu(x) \leq 2\Theta \int_{\bigcup_i S_{j_m}^i} |h_{j_m}(x)| d\mu(x) \\ &\leq 2\Theta \|h_{j_m}\|_{L_\mu^1} \leq 2\Theta \|h_{j_m}\|_{L_\mu^q} (\mu(S_{j_m}))^{1/q'} \leq 2\Theta \mu(S_{j_m}). \end{aligned}$$

Hence $g_{j_m}(x) + \sum_{i=1}^\infty h_{j_m}^i(x)$ converges to $h_{j_m}(x)$ in the L_μ^1 norm (it is also convergent almost everywhere). Thus, by the cancellation properties of $h_{j_m}(x)$ and $h_{j_m}^i(x)$ for each i , we have

$$(9) \quad \int g_{j_m}(x) d\mu(x) = 0.$$

If we set $a_{j_m}(x) = g_{j_m}(x)(D_0\Theta\alpha^{m+1}\mu(\bar{S}_{j_m}))^{-1}$, then from (7), (8) and (9) we see that $a_{j_m}(x)$ is a $(1, \infty)$ -atom supported in the section \bar{S}_{j_m} . This shows that (I) is valid for $m + 1$. By the definition of $h_{j_m}^i(x)$, the hypothesis on $h_{j_m}(x)$ for m , and (3.18), we have

$$\begin{aligned} |h_{j_m}^i(x)| &\leq \left\{ |h_{j_m}(x)| + \frac{1}{\mu(S_{j_m}^i)} \int_{S_{j_m}^i} |h_{j_m}(x)| d\mu(x) \right\} \chi_{j_m}^i(x) \\ &\leq \left\{ |b(x)| + 2D_0\alpha^m + \left(\frac{1}{\mu(S_{j_m}^i)} \int_{S_{j_m}^i} |h_{j_m}(x)|^q d\mu(x) \right)^{1/q} \right\} \chi_{j_m}^i(x) \\ &\leq \{|b(x)| + 2D_0\alpha^m + D_0\alpha^{m+1}\} \chi_{j_m}^i(x) \\ &\leq |b(x)| + 2D_0\alpha^{m+1} \chi_{j_m}^i(x) \end{aligned}$$

provided $\alpha > 2$, which means that (VI) holds for $m + 1$. By (3.18), we see that (VII) is also valid for $m + 1$, since by the definition of $h_{j_m}^i$ we know that $(m_{S_{j_m}^i} (|h_{j_m}^i|^q))^{1/q} \leq 2(m_{S_{j_m}^i} (|h_{j_m} \eta_{j_m}^i|^q))^{1/q}$.

Finally, by (VI) we see that

$$(M_q h_{j_m})(x) \leq (M_q b)(x) + 2D_0\alpha^m \quad \text{for all } x \in \mathbf{R}^n.$$

Thus, for any $x \in E_{j_m}$, we have

$$(3.19) \quad \alpha^{m+1} < (M_q h_{j_m})(x) \leq (M_q b)(x) + 2D_0\alpha^m < (M_q b)(x) + \alpha^{m+1}/2$$

as long as $\alpha > 4D_0$. Then, by (4) and (3.19), we obtain

$$\bigcup_{\substack{j_m \in \mathbf{N}^m \\ i \in \mathbf{N}}} S_{j_m}^i = \bigcup_{j_m \in \mathbf{N}^m} \left(\bigcup_{i \in \mathbf{N}} S_{j_m}^i \right) \subset \bigcup_{j_m \in \mathbf{N}^m} E_{j_m} \subset \{x \in \mathbf{R}^n; (M_q b)(x) > \alpha^{m+1}/2\}.$$

So, (II) holds for $m + 1$.

In consequence, to complete the proof of Proposition 3.1 we only need to take α to be

$$\alpha > \max\{\Theta^{1/(q-1)}, C_2^{1/q}, 2D_0C_2^{1/q}, 2, 4D_0\},$$

since each of these numbers depends only on q, A and θ and is independent of m .



4. Proof of theorem 1.2. We need to give an equivalent definition of $BMO_{\mathcal{F}}(\mathbf{R}^n)$ with respect to the family \mathcal{F} and the doubling Borel measure μ . Let f be a real-valued function defined on \mathbf{R}^n . We say that $f \in BMO_{\mathcal{F}}^q(\mathbf{R}^n)$, $1 < q < \infty$, if

$$\|f\|_{q,*} = \sup_{S \in \mathcal{F}} \left(\frac{1}{\mu(S)} \int_S |f(x) - m_S(f)|^q d\mu(x) \right)^{1/q} < \infty.$$

PROPOSITION 4.1. For any $1 < q < \infty$, $BMO_{\mathcal{F}}^q(\mathbf{R}^n) = BMO_{\mathcal{F}}(\mathbf{R}^n)$.

PROOF. By Hölder's inequality, it is easy to get $BMO_{\mathcal{F}}^q(\mathbf{R}^n) \subset BMO_{\mathcal{F}}(\mathbf{R}^n)$. On the other hand, we assume that $f \in BMO_{\mathcal{F}}(\mathbf{R}^n)$ with $\|f\|_* = 1$. Then there exist positive numbers $\varepsilon_0 < 1$ and Γ depending only on A in (1.1) and the constants in the properties (A) and (B) of sections, such that, for any section $S \in \mathcal{F}$ and each $k = 0, 1, 2, \dots$,

$$(4.1) \quad \mu(\{x \in S; |f(x) - m_S(f)| > \Gamma + k\Gamma\}) \leq \varepsilon_0^k \mu(\{x \in S; |f(x) - m_S(f)| > \Gamma\}).$$

(See (6-6) in [CG1, p. 1091] for the proof.) Thus

$$\begin{aligned} \frac{1}{\mu(S)} \int_S |f(x) - m_S(f)|^q d\mu(x) &= \frac{q}{\mu(S)} \int_0^\infty \alpha^{q-1} \mu(\{x \in S; |f(x) - m_S(f)| > \alpha\}) d\alpha \\ &= \frac{q}{\mu(S)} \int_0^\Gamma \alpha^{q-1} \mu(\{x \in S; |f(x) - m_S(f)| > \alpha\}) d\alpha \\ &\quad + \frac{q}{\mu(S)} \int_\Gamma^\infty \alpha^{q-1} \mu(\{x \in S; |f(x) - m_S(f)| > \alpha\}) d\alpha \\ &:= I_1 + I_2. \end{aligned}$$

Here we have

$$(4.2) \quad I_1 \leq \frac{q}{\mu(S)} \int_0^\Gamma \alpha^{q-1} \cdot \mu(S) d\alpha \leq \Gamma^q < \infty.$$

On the other hand, by (4.1) and noting that $\varepsilon_0 < 1$, we get

$$\begin{aligned} (4.3) \quad I_2 &= \frac{q}{\mu(S)} \int_0^\infty (\alpha + \Gamma)^{q-1} \mu(\{x \in S; |f(x) - m_S(f)| > \alpha + \Gamma\}) d\alpha \\ &= \frac{q}{\mu(S)} \sum_{k=0}^\infty \int_{k\Gamma}^{(k+1)\Gamma} (\alpha + \Gamma)^{q-1} \mu(\{x \in S; |f(x) - m_S(f)| > \alpha + \Gamma\}) d\alpha \\ &\leq \frac{q}{\mu(S)} \sum_{k=0}^\infty [(k+1)\Gamma + \Gamma]^{q-1} \mu(\{x \in S; |f(x) - m_S(f)| > k\Gamma + \Gamma\}) \cdot \Gamma \\ &\leq \frac{q}{\mu(S)} \sum_{k=0}^\infty (k+2)^{q-1} \Gamma^q \varepsilon_0^k \mu(\{x \in S; |f(x) - m_S(f)| > \Gamma\}) \\ &\leq q\Gamma^q \sum_{k=0}^\infty (k+2)^{q-1} \varepsilon_0^k \leq Cq\Gamma^q. \end{aligned}$$

From (4.2) and (4.3), we conclude that $BMO_{\mathcal{F}}^q(\mathbf{R}^n) \supset BMO_{\mathcal{F}}(\mathbf{R}^n)$.

PROOF OF THEOREM 1.2. To prove Theorem 1.2, we need to show that if $g \in BMO_{\mathcal{F}}(\mathbf{R}^n)$, then

$$(4.4) \quad l_g(f) = \int_{\mathbf{R}^n} f(x)g(x)d\mu(x)$$

is a bounded linear functional on $H_{\mathcal{F}}^{1,q}(\mathbf{R}^n)$, and conversely that for any bounded linear functional l on $H_{\mathcal{F}}^1(\mathbf{R}^n)$, there exists $b \in BMO_{\mathcal{F}}(\mathbf{R}^n)$ such that

$$l(f) = \int_{\mathbf{R}^n} f(x)b(x)d\mu(x) \quad \text{for all } f \in H_{\mathcal{F}}^1(\mathbf{R}^n).$$

By the conclusions of Theorem 1.1 and Proposition 4.1, it suffices to show that the dual space of the atomic Hardy space $H_{\mathcal{F}}^{1,q}(\mathbf{R}^n)$ is $BMO_{\mathcal{F}}^{q'}(\mathbf{R}^n)$ for some q with $1 < q < \infty$, that is, $(H_{\mathcal{F}}^{1,q}(\mathbf{R}^n))' = BMO_{\mathcal{F}}^{q'}(\mathbf{R}^n)$, where $1/q + 1/q' = 1$.

We first prove that $BMO_{\mathcal{F}}^{q'}(\mathbf{R}^n) \subset (H_{\mathcal{F}}^{1,q}(\mathbf{R}^n))'$. Write $D = H_{\mathcal{F}}^{1,q}(\mathbf{R}^n) \cap L_c^q(\mathbf{R}^n, d\mu)$, where $L_c^q(\mathbf{R}^n, d\mu)$ consists of all functions in $L^q(\mathbf{R}^n, d\mu)$ with compact supports. Since the set of all functions with the form $\sum_{k=1}^N \lambda_k a_k(x)$ is dense in $H_{\mathcal{F}}^{1,q}(\mathbf{R}^n)$, D is a dense subset of $H_{\mathcal{F}}^{1,q}(\mathbf{R}^n)$. Then we will see that, for any $g \in BMO_{\mathcal{F}}^{q'}(\mathbf{R}^n)$, the linear functional l_g defined in (4.4) is bounded on the dense subset D of $H_{\mathcal{F}}^{1,q}(\mathbf{R}^n)$.

For $N \in \mathbf{N}$, we set

$$g_N(x) = \begin{cases} N & \text{if } g(x) \geq N, \\ g(x) & \text{if } |g(x)| < N, \\ -N & \text{if } g(x) \leq -N. \end{cases}$$

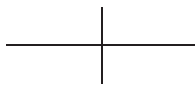
Then it is easy to verify that $g_N(x) \in BMO_{\mathcal{F}}^{q'}(\mathbf{R}^n)$ and $\|g_N\|_{q',*} \leq 4\|g\|_{q',*}$.

Set $f(x) = \sum_{k=1}^{\infty} \lambda_k a_k(x) \in D$, where $a_k(x)$ is a $(1, q)$ -atom supported in a section $S_k \in \mathcal{F}$. Thus, by the definition of the $(1, q)$ -atom, we have

$$(4.5) \quad \begin{aligned} \left| \int_{\mathbf{R}^n} f(x)g_N(x)d\mu(x) \right| &\leq \sum_{k=1}^{\infty} |\lambda_k| \left| \int_{\mathbf{R}^n} a_k(x)g_N(x)d\mu(x) \right| \\ &\leq \sum_{k=1}^{\infty} |\lambda_k| \left| \int_{S_k} a_k(x)[g_N(x) - m_{S_k}(g_N)]d\mu(x) \right| \\ &\leq \sum_{k=1}^{\infty} |\lambda_k| \|a_k\|_{L_{\mu}^q} \left(\int_{S_k} |g_N(x) - m_{S_k}(g_N)|^{q'} d\mu(x) \right)^{1/q'} \\ &\leq \sum_{k=1}^{\infty} |\lambda_k| \left(\frac{1}{\mu(S_k)} \int_{S_k} |g_N(x) - m_{S_k}(g_N)|^{q'} d\mu(x) \right)^{1/q'} \\ &\leq \|f\|_{H_{\mathcal{F}}^{1,q}} \cdot 4\|g\|_{q',*}. \end{aligned}$$

Since $g(x) \in BMO_{\mathcal{F}}^{q'}(\mathbf{R}^n)$ is a locally q' -th integrable function on \mathbf{R}^n ,

$$|f(x)g_N(x)| \leq |f(x)g(x)| \in L^1(\mathbf{R}^n, d\mu).$$



By the Lebesgue dominated convergence theorem and (4.5),

$$\left| \int_{\mathbf{R}^n} f(x)g(x)d\mu(x) \right| = \left| \lim_{N \rightarrow \infty} \int_{\mathbf{R}^n} f(x)g_N(x)d\mu(x) \right| \leq \|f\|_{H_{\mathcal{F}}^{1,q}} \cdot 4\|g\|_{q',*}.$$

This shows that the linear functional l_g is bounded on D , and $\|l_g\| \leq 4\|g\|_{q',*}$. Consequently, l_g has a unique bounded extension on $H_{\mathcal{F}}^{1,q}(\mathbf{R}^n)$, since D is a dense subset of $H_{\mathcal{F}}^{1,q}(\mathbf{R}^n)$. In this sense we then have $BMO_{\mathcal{F}}^{q'}(\mathbf{R}^n) \subset (H_{\mathcal{F}}^{1,q}(\mathbf{R}^n))'$.

In order to prove the inverse inclusion $(H_{\mathcal{F}}^{1,q}(\mathbf{R}^n))' \subset BMO_{\mathcal{F}}^{q'}(\mathbf{R}^n)$, we need to show that if l is a bounded linear functional on $H_{\mathcal{F}}^{1,q}(\mathbf{R}^n)$, then there exists $b(x) \in BMO_{\mathcal{F}}^{q'}(\mathbf{R}^n)$ such that for any $f \in H_{\mathcal{F}}^{1,q}(\mathbf{R}^n)$

$$l(f) = \int_{\mathbf{R}^n} f(x)b(x)d\mu(x).$$

The proof will be divided into the following three steps.

Step 1. Let us first prove $(H_{\mathcal{F}}^{1,q}(\mathbf{R}^n))' \subset (L_0^q(S, d\mu))'$, where $S = S(x, t) \in \mathcal{F}$ is any section in \mathbf{R}^n and

$$L_0^q(S, d\mu) = \left\{ f \in L^q(\mathbf{R}^n, d\mu); f = 0 \text{ } \mu\text{-a.e. on } S^c \text{ and } \int_S f(x)d\mu(x) = 0 \right\}.$$

Indeed, when $f(x) \in L_0^q(S, d\mu)$, it is easy to check that $a(x) = f(x)(\mu(S))^{-1/q'}\|f\|_{L_{\mu}^q(S)}^{-1}$ is a $(1, q)$ -atom. Thus $f(x) = a(x)(\mu(S))^{1/q'}\|f\|_{L_{\mu}^q(S)} \in H_{\mathcal{F}}^{1,q}(\mathbf{R}^n)$ and $\|f\|_{H_{\mathcal{F}}^{1,q}} \leq (\mu(S))^{1/q'}\|f\|_{L_{\mu}^q(S)}$. Therefore, we have

$$(4.6) \quad |l(f)| \leq \|l\| \cdot (\mu(S))^{1/q'}\|f\|_{L_{\mu}^q(S)},$$

which shows that l is also a bounded linear functional on $L_0^q(S, d\mu)$. Since $L_0^q(S, d\mu) \subset L^q(S, d\mu)$, using the Hahn-Banach extension theorem, we know that l has a unique bounded extension on $L^q(S, d\mu)$. Since $1 < q < \infty$, by the Riesz representation theorem, there exists $b(x) \in L^{q'}(S, d\mu)$ such that

$$(4.7) \quad l(f) = \int_S f(x)b(x)d\mu(x) \quad \text{for all } f \in L_0^q(S, d\mu).$$

Furthermore, we have the following fact:

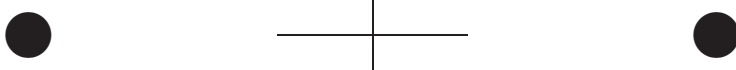
If $\int_S f(x)b(x)d\mu(x) = 0$ for all $f \in L_0^q(S, d\mu)$, then $b(x)$ is constant for almost every $x \in S$.

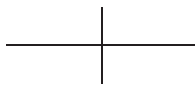
Indeed, since S is a bounded convex set, for any $h(x) \in L^q(S, d\mu)$ we have $h(x) - m_S(h) \in L_0^q(S, d\mu)$. Thus

$$0 = \int_S b(x)[h(x) - m_S(h)]d\mu(x) = \int_S h(x)[b(x) - m_S(b)]d\mu(x) \quad \text{for all } h \in L^q(S, d\mu).$$

Hence $b(x) = m_S(b)$ almost every $x \in S$.

Step 2. Fix $x_0 \in \mathbf{R}^n$ and choose a sequence of positive increasing numbers $\{t_j\}_{j=1}^{\infty}$ such that $\lim_{j \rightarrow \infty} t_j = \infty$. Then, by the property (C) of sections, $\{S(x_0, t_j)\}_{j=1}^{\infty}$ is a sequence





of sections with $\bigcup_{j=1}^{\infty} S_j = \mathbf{R}^n$, where $S_j = S(x_0, t_j)$. By (4.7), for each S_j , there exists $b_j(x) \in L^{q'}(S_j, d\mu)$ satisfying (4.7).

Consider an arbitrary $f \in L_0^q(S_1, d\mu)$. There exists $b_1(x) \in L^{q'}(S_1, d\mu)$ such that

$$(4.8) \quad l(f) = \int_{S_1} f(x)b_1(x)d\mu(x).$$

By $S_2 \supset S_1$, we have $L_0^q(S_2, d\mu) \supset L_0^q(S_1, d\mu)$ and $f \in L_0^q(S_2, d\mu)$. Therefore, there exists $b_2(x) \in L^{q'}(S_2, d\mu)$ such that

$$(4.9) \quad l(f) = \int_{S_2} f(x)b_2(x)d\mu(x) = \int_{S_1} f(x)b_2(x)d\mu(x),$$

since $\text{supp}(f) \subset S_1$. From (4.8) and (4.9), we get

$$(4.10) \quad \int_{S_1} f(x)[b_1(x) - b_2(x)]d\mu(x) = 0 \quad \text{for all } f \in L_0^q(S_1, d\mu).$$

Applying the fact shown in Step 1, we have $b_1(x) - b_2(x) = C_1$ for almost every $x \in S_1$. Now we write

$$b(x) = \begin{cases} b_1(x) & \text{if } x \in S_1, \\ b_2(x) + C_1 & \text{if } x \in S_2 \setminus S_1. \end{cases}$$

Then we obtain

$$l(f) = \int_{S_j} f(x)b(x)d\mu(x) \quad \text{for any } f \in L_0^q(S_j, d\mu), \quad j = 1, 2.$$

By a method quite similar to the above, we may obtain a function $b(x)$ satisfying

$$(4.11) \quad l(f) = \int_{S_j} f(x)b(x)d\mu(x) \quad \text{for any } f \in L_0^q(S_j, d\mu), \quad j = 1, 2, \dots$$

Step 3. Now we prove that the above $b(x) \in BMO_{\mathcal{F}}^{q'}(\mathbf{R}^n)$ and satisfies

$$(4.12) \quad l(f) = \int_{\mathbf{R}^n} f(x)b(x)d\mu(x) \quad \text{for any } f \in H_{\mathcal{F}}^{1,q}(\mathbf{R}^n).$$

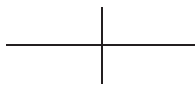
We need the following fact about sections in \mathbf{R}^n .

Assume that $S_0 = S(y_0, r) \in \mathcal{F}$ is an arbitrary section in \mathbf{R}^n . Then there exists j_0 such that $S_{j_0} \supset S_0$, where $S_{j_0} = S(x_0, t_{j_0})$ is the j_0 -th section of the sequence in Step 2.

Indeed, by $\bigcup_{j=1}^{\infty} S_j = \mathbf{R}^n$, there exists a section $S_i = S(x_0, t_i)$ such that $S(x_0, t_i) \cap S(y_0, r) \neq \emptyset$ with $t_i \geq r$. Then there exists $z \in S(x_0, t_i) \cap S(y_0, r)$. From the property (D) of sections, we have $S(y_0, r) \subset S(z, \theta r) \subset S(z, \theta t_i)$. Since $z \in S(x_0, t_i) \subset S(x_0, \theta t_i)$, using the property (D) again, we know $S(z, \theta t_i) \subset S(x_0, \theta^2 t_i)$ and therefore $S(y_0, r) \subset S(x_0, \theta^2 t_i)$. Now if we take j_0 such that $t_{j_0} \geq \theta^2 t_i$, then $S(y_0, r) \subset S(x_0, t_{j_0})$.

Now, let us return to the proof of (4.12). For any $f \in H_{\mathcal{F}}^{1,q}(\mathbf{R}^n)$, we may write $f(x) = \sum_{k=1}^{\infty} \lambda_k a_k(x)$, where $a_k(x)$ is a $(1, q)$ -atom supported in the section $S_k \in \mathcal{F}$. By the fact





above, for each k there exists j_k such that $S_k \subset S_{j_k} = S(x_0, t_{j_k})$. By the definition of $(1, q)$ -atom, we have $a_k(x) \in L^q_0(S_{j_k}, d\mu)$. Thus by (4.11),

$$(4.13) \quad l(a_k) = \int_{S_{j_k}} a_k(x)b(x)d\mu(x) = \int_{\mathbf{R}^n} a_k(x)b(x)d\mu(x).$$

Since the functional l is linear, by (4.13) we obtain

$$l(f) = \sum_{k=1}^{\infty} \lambda_k l(a_k) = \sum_{k=1}^{\infty} \lambda_k \int_{\mathbf{R}^n} a_k(x)b(x)d\mu(x) = \int_{\mathbf{R}^n} f(x)b(x)d\mu(x).$$

Finally, to finish the proof of Step 3, it remains to show that $b(x) \in BMO_{\mathcal{F}}^{q'}(\mathbf{R}^n)$. For any section $S \in \mathcal{F}$, let $h(x) \in L^q(S, d\mu)$ with $\text{supp}(h) \subset S$ and $\|h\|_{L^q_\mu} \leq 1$. Then $a(x) = (1/2)(\mu(S))^{-1/q'}[h(x) - m_S(h)]\chi_S(x)$ is a $(1, q)$ -atom supported in S and $\|a\|_{L^q_\mu} \leq 1$. Thus, (4.13) implies that

$$\left| \int_S a(x)b(x)d\mu(x) \right| = |l(a)| \leq \|l\|.$$

Hence

$$(\mu(S))^{-1/q'} \left| \int_S [h(x) - m_S(h)]b(x)d\mu(x) \right| \leq 2\|l\|.$$

That is,

$$(4.14) \quad (\mu(S))^{-1/q'} \left| \int_S h(x)[b(x) - m_S(b)]d\mu(x) \right| \leq 2\|l\|.$$

From (4.14), we have

$$(\mu(S))^{-1/q'} \|b - m_S(b)\|_{L^{q'}_\mu} = (\mu(S))^{-1/q'} \sup_{\|h\|_{L^q_\mu} \leq 1} \left| \int_S h(x)[b(x) - m_S(b)]d\mu(x) \right| \leq 2\|l\|.$$

Since the section $S \in \mathcal{F}$ is arbitrary, we may conclude that $b(x) \in BMO_{\mathcal{F}}^{q'}(\mathbf{R}^n)$. This completes the proof of Theorem 1.2.

5. Proof of theorem 1.3. Applying Theorem 1.1, we only have to show that there exists a constant C such that

$$(5.1) \quad \|H(a)\|_{L^1_\mu} \leq C \quad \text{for all } (1, 2)\text{-atom } a.$$

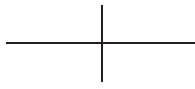
By Definition 1.1, there exists a section $S_0 = S(y_0, t_0) \in \mathcal{F}$ such that $\text{supp}(a) \subset S_0$. Denote $S_0^* = S(y_0, 4\theta^2 t_0)$, where θ is the constant appearing in the property (D) of sections. By the doubling property (1.1) of μ , we have

$$(5.2) \quad \mu(S_0^*) \leq A^{3+2\log_2 \theta} \mu(S_0).$$

Thus

$$(5.3) \quad \int_{\mathbf{R}^n} |H(a)(x)|d\mu(x) = \int_{S_0^*} |H(a)(x)|d\mu(x) + \int_{(S_0^*)^c} |H(a)(x)|d\mu(x) \\ := I_1 + I_2.$$





By the (L^2, L^2) -boundedness of the operator H (see [CG3]) and (5.2), we get

$$(5.4) \quad \begin{aligned} I_1 &\leq [\mu(S_0^*)]^{1/2} \left(\int_{S_0^*} |H(a)(x)|^2 d\mu(x) \right)^{1/2} \\ &\leq (A^{3+2\log_2 \theta})^{1/2} [\mu(S_0)]^{1/2} \|a\|_{L^2_\mu} \leq (A^{3+2\log_2 \theta})^{1/2}. \end{aligned}$$

On the other hand, by the cancellation condition of the atom a , we have

$$\begin{aligned} I_2 &= \int_{(S_0^*)^c} \left| \int_{\mathbf{R}^n} K(x, y) a(y) d\mu(y) \right| d\mu(x) \\ &= \int_{(S_0^*)^c} \left| \sum_i \int_{\mathbf{R}^n} k_i(x, y) a(y) d\mu(y) \right| d\mu(x) \\ &= \int_{(S_0^*)^c} \left| \sum_i \int_{\mathbf{R}^n} [k_i(x, y) - k_i(x, y_0)] a(y) d\mu(y) \right| d\mu(x) \\ &\leq \sum_i \int_{\mathbf{R}^n} |a(y)| \int_{(S_0^*)^c} |k_i(x, y) - k_i(x, y_0)| d\mu(x) d\mu(y) \\ &= \int_{S_0} |a(y)| \sum_i \int_{(S_0^*)^c} |k_i(x, y) - k_i(x, y_0)| d\mu(x) d\mu(y). \end{aligned}$$

By the size condition of the atom a , it suffices to prove that there exists a constant C independent of the atom a such that

$$(5.5) \quad \sum_i \int_{(S_0^*)^c} |k_i(x, y) - k_i(x, y_0)| d\mu(x) \leq C.$$

Indeed, if (5.5) holds, then

$$I_2 \leq C \int_{S_0} |a(y)| d\mu(y) \leq C,$$

which combined with (5.4) implies (5.1).

Therefore, in order to prove Theorem 1.3, it remains only to prove (5.5). By the property (G) of sections, we have

$$(5.6) \quad \rho(y_0, y) < t_0 \quad \text{and} \quad \rho(y_0, x) \geq 4\theta^2 t_0$$

if $y \in S_0$ and $x \in (S_0^*)^c$. So, by (5.6), we see that when $y \in S_0$ and $x \in (S_0^*)^c$,

$$\rho(y_0, x) > 4\theta^2 \rho(y_0, y).$$

Using the conclusion of Lemma 1 in [In], we get (5.5).

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