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HARDY SPACES ASSOCIATED TO THE SECTIONS

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Abstract. In this paper we define the Hardy space $H^1_{\mathcal{F}}(\boldsymbol{R}^n)$ associated with a family $\mathcal{F}$ of sections and a doubling measure μ, where $\mathcal{F}$ is closely related to the Monge-Ampère equation. Furthermore, we show that the dual space of $H^1_{\mathcal{F}}(\boldsymbol{R}^n)$ is just the space $BMO_{\mathcal{F}}(\boldsymbol{R}^n)$, which was first defined by Caffarelli and Gutiérrez. We also prove that the Monge-Ampère singular integral operator is bounded from $H^1_{\mathcal{F}}(\boldsymbol{R}^n)$ to $L^1(\boldsymbol{R}^n, d\mu)$.

1. Introduction. In 1996, Caffarelli and Gutiérrez [CG1] studied real variable theory related to the Monge-Ampère equation. They gave a Besicovitch type covering lemma for a family $\mathcal{F}$ of convex sets in Euclidean n-space $\boldsymbol{R}^n$, where $\mathcal{F} = \{S(x,t)\,;\ x \in \boldsymbol{R}^n \text{ and } t > 0\}$ and $S(x,t)$ is called a *section* (see the definition below) satisfying certain axioms of affine invariance. In terms of the sections, Caffarelli and Gutiérrez set up a variant of the Calderón-Zygmund decomposition by applying this covering lemma and the doubling condition of a Borel measure μ. The decomposition plays an important role in the study of the linearized Monge-Ampère equation [CG2]. As an application of the above decomposition, Caffarelli and Gutiérrez defined the Hardy-Littlewood maximal operator M and $BMO_{\mathcal{F}}(\boldsymbol{R}^n)$ space associated to a family $\mathcal{F}$ of sections and the doubling measure μ, and obtained the weak type (1,1) boundedness of M and the John-Nirenberg inequality for $BMO_{\mathcal{F}}(\boldsymbol{R}^n)$ in [CG1].

Let us recall the definition of sections and the doubling measure listed below. For $x \in \boldsymbol{R}^n$ and $t > 0$, let $S(x,t)$ denote an open and bounded convex set containing x. We call $S(x,t)$ a *section* if the family $\{S(x,t)\,;\ x \in \boldsymbol{R}^n, t > 0\}$ is monotone increasing in t, i.e., $S(x,t) \subset S(x,t')$ for $t \le t'$, and satisfies the following three conditions:

(A) There exist positive constants K_1, K_2, K_3 and ϵ_1, ϵ_2 such that given two sections $S(x_0,t_0)$, $S(x,t)$ with $t \le t_0$ satisfying

$$S(x_0,t_0) \cap S(x,t) \neq \emptyset\,,$$

and an affine transformation T that "normalizes" $S(x_0,t_0)$, that is,

$$B(0,1/n) \subset T(S(x_0,t_0)) \subset B(0,1)\,,$$

there exists $z \in B(0,K_3)$ depending on $S(x_0,t_0)$ and $S(x,t)$, which satisfies

$$B(z, K_2(t/t_0)^{\epsilon_2}) \subset T(S(x,t)) \subset B(z, K_1(t/t_0)^{\epsilon_1})\,,$$

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and

$$T(z) \in B(z, (1/2)K_2(t/t_0)^{\epsilon_2}) .$$

Here and below $B(x,t)$ denotes the Euclidean ball centered at x with radius t.

(B) There exists a constant $\delta > 0$ such that given a section $S(x,t)$ and $y \notin S(x,t)$, if T is an affine transformation that "normalizes" $S(x,t)$, then for any $0 < \epsilon < 1$

$$B(T(y), \epsilon^{\delta}) \cap T(S(x, (1-\epsilon)t)) = \emptyset .$$

(C) $\bigcap_{t>0} S(x,t) = \{x\}$ and $\bigcup_{t>0} S(x,t) = \boldsymbol{R}^n$.

In addition, we also assume that a Borel measure μ which is finite on compact sets is given, $\mu(\boldsymbol{R}^n) = \infty$, and satisfies the following *doubling property* with respect to $\mathcal{F}$, that is, there exists a constant A such that

$$\mu(S(x,2t)) \leq A\mu(S(x,t)) \quad \text{for any section } S(x,t) \in \mathcal{F} . \tag{1.1}$$

An important example of the family $\mathcal{F}$ of sections is given as follows. Let $\phi : \boldsymbol{R}^n \to \boldsymbol{R}$ be a convex smooth function. For any given point $x \in \boldsymbol{R}^n$, let $\mathcal{L}(x)$ be a supporting hyperplane of ϕ at the point $(x, \phi(x))$. For $t > 0$, define the set

$$S_\phi(x,t) = \{y \in \boldsymbol{R}^n \,;\ \phi(y) < \mathcal{L}(x) + t\} .$$

Then

$$\mathcal{F} = \{S_\phi(x,t) \,;\ x \in \boldsymbol{R}^n \text{ and } t > 0\}$$

is a family of sections that satisfies the properties (A), (B) and (C). Moreover, the Monge-Ampère measure generated by the convex function ϕ

$$\det D^2\phi = \mu$$

satisfies the doubling condition (1.1) under certain condition of ϕ. For instance, if the graph of ϕ contains no lines, then μ satisfies the doubling condition (1.1) (see [C, CG1]). The terminology *section* comes from the fact that $S_\phi(x,t)$ is obtained by projecting on $\boldsymbol{R}^n$ the bounded part of the graph of ϕ cut by a hyperplane parallel to the supporting hyperplane at $(x, \phi(x))$.

In [CG1], Caffarelli and Gutiérrez defined the space $BMO_{\mathcal{F}}(\boldsymbol{R}^n)$ associated with the family $\mathcal{F}$ and the Borel measure μ satisfying the doubling condition (1.1). Let f be a real-valued function defined on $\boldsymbol{R}^n$. We say that $f \in BMO_{\mathcal{F}}(\boldsymbol{R}^n)$ if

$$\|f\|_* := \sup_{S\in\mathcal{F}} \frac{1}{\mu(S)} \int_S |f(x) - m_S(f)| d\mu(x) < \infty ,$$

where $m_S(f)$ denotes the mean of f over the section S defined by

$$m_S(f) = \frac{1}{\mu(S)} \int_S f(x) d\mu(x) .$$

Similar to the classic case, Caffarelli and Gutiérrez [CG1] also proved the following John-Nirenberg inequality for $BMO_{\mathcal{F}}$:

There exist positive constants C_1 and C_2 dependent only on the measure μ such that, for every continuous $f \in BMO_{\mathcal{F}}(\boldsymbol{R}^n)$ and every section S,

$$\frac{1}{\mu(S)}\int_S \exp\left(C_1\frac{|f(x)-m_S(f)|}{\|f\|_*}\right)d\mu(x) \le C_2\,.$$

Hence, it is an important and interesting problem to ask whether it is possible to set up a Hardy space with respect to the family of sections $\mathcal{F}$ and a doubling measure. In this paper we are going to construct such a Hardy space. We first introduce $(1,q)$-atoms and the atomic Hardy space $H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n)$ for $q>1$ with respect to the family $\mathcal{F}$. Then we show that the atomic Hardy spaces $H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n)$ are all equivalent for any $q>1$. Thus we may define the Hardy space $H^1_{\mathcal{F}}(\boldsymbol{R}^n)$. We will further prove that the dual space of $H^1_{\mathcal{F}}(\boldsymbol{R}^n)$ is just the space $BMO_{\mathcal{F}}(\boldsymbol{R}^n)$, which was defined by Caffarelli and Gutiérrez in [CG1]. Moreover, as an application of the atomic decomposition, we will also prove that the Monge-Ampère singular integral operator (defined later) is bounded from $H^1_{\mathcal{F}}(\boldsymbol{R}^n)$ to $L^1(\boldsymbol{R}^n, d\mu)$.

We now define a $(1,q)$-atom and the atomic Hardy space with respect to a family $\mathcal{F}$ of sections and a doubling measure μ.

DEFINITION 1.1. Let $1<q\le\infty$. A function $a(x)\in L^q(\boldsymbol{R}^n, d\mu)$ is called a $(1,q)$-atom if there exists a section $S(x_0,t_0)\in\mathcal{F}$ such that

(i) $\operatorname{supp}(a)\subset S(x_0,t_0)$;

(ii) $\int_{\boldsymbol{R}^n} a(x)d\mu(x)=0$;

(iii) $\|a\|_{L^q_\mu}\le[\mu(S(x_0,t_0))]^{-1/q'}$, where $\|a\|_{L^q_\mu}=(\int_{\boldsymbol{R}^n}|a(x)|^q d\mu(x))^{1/q}$ and $1/q+1/q'=1$.

The atomic Hardy space $H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n)$ is defined by

$$\begin{aligned} &H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n)\\ &=\left\{f\in\mathcal{S}'\,;\ f(x)\overset{\mathcal{S}'}{=}\sum_j\lambda_j a_j(x),\ \text{each } a_j \text{ is a } (1,q)\text{-atom and } \sum_j|\lambda_j|<\infty\right\}, \end{aligned} \tag{1.2}$$

where $\mathcal{S}(\boldsymbol{R}^n)$ denotes the space of Schwartz functions and $\mathcal{S}'(\boldsymbol{R}^n)$ is the dual space of $\mathcal{S}(\boldsymbol{R}^n)$. Define the $H^{1,q}_{\mathcal{F}}$ norm of f by

$$\|f\|_{H^{1,q}_{\mathcal{F}}}=\inf\left\{\sum_j|\lambda_j|\right\},$$

where the infimum is taken over all decompositions of $f=\sum_j\lambda_j a_j$ above.

The first result of this paper is

THEOREM 1.1. *For $q>1$, $H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n)=H^{1,\infty}_{\mathcal{F}}(\boldsymbol{R}^n)$.*

By Theorem 1.1, we may take the atomic Hardy space $H^{1,q}_{\mathcal{F}}$ for any $q>1$ as the definition of the Hardy space $H^1_{\mathcal{F}}(\boldsymbol{R}^n)$. Our second task is to show the following duality.

THEOREM 1.2. *The dual space of $H^1_{\mathcal{F}}(\boldsymbol{R}^n)$ is the space $BMO_{\mathcal{F}}(\boldsymbol{R}^n)$.*

In 1997, Caffarelli and Gutiérrez [CG3] defined a class of the Monge-Ampère singular integral operators as follows. Suppose that $0 < \alpha \leq 1$ and $c_1, c_2 > 0$. Let $\{k_i(x, y)\}_{i=1}^{\infty}$ be a sequence of kernels satisfying the following conditions:

(1.3) $\operatorname{supp} k_i(\cdot, y) \subset S(y, 2^i)$ for all $y \in \boldsymbol{R}^n$;

(1.4) $\operatorname{supp} k_i(x, \cdot) \subset S(x, 2^i)$ for all $x \in \boldsymbol{R}^n$;

(1.5) $\int_{\boldsymbol{R}^n} k_i(x, y) d\mu(y) = \int_{\boldsymbol{R}^n} k_i(x, y) d\mu(x) = 0$ for all $x, y \in \boldsymbol{R}^n$;

(1.6) $\sup_i \int_{\boldsymbol{R}^n} |k_i(x, y)| d\mu(y) \leq c_1$ for all $x \in \boldsymbol{R}^n$;

(1.7) $\sup_i \int_{\boldsymbol{R}^n} |k_i(x, y)| d\mu(x) \leq c_2$ for all $y \in \boldsymbol{R}^n$;

(1.8) If T is an affine transformation that normalizes the section $S(y, 2^i)$, then

$$|k_i(u, y) - k_i(v, y)| \leq \frac{c_2}{\mu(S(y, 2^i))} |T(u) - T(v)|^{\alpha};$$

(1.9) If T is an affine transformation that normalizes the section $S(x, 2^i)$, then

$$|k_i(x, u) - k_i(x, v)| \leq \frac{c_2}{\mu(S(x, 2^i))} |T(u) - T(v)|^{\alpha}.$$

Denote $K(x, y) = \sum_i k_i(x, y)$. The *Monge-Ampère singular integral operator* H is defined by

$$H(f)(x) = \int_{\boldsymbol{R}^n} K(x, y) f(y) d\mu(y).$$

Caffarelli and Gutiérrez [CG3] proved that H is bounded from $L^2(\boldsymbol{R}^n, d\mu)$ to $L^2(\boldsymbol{R}^n, d\mu)$. Subsequently, Incognito [In] gave the weak type (1,1) estimate of H. Using the atomic decomposition of $H^1_{\mathcal{F}}(\boldsymbol{R}^n)$, we have the following result for the operator H.

THEOREM 1.3. *The operator H is a bounded operator from $H^1_{\mathcal{F}}(\boldsymbol{R}^n)$ to $L^1(\boldsymbol{R}^n, d\mu)$.*

As an application of Theorem 1.3, we have a different method from [In] to obtain the following corollary.

COROLLARY 1.1. *The operator H is bounded on $L^p(\boldsymbol{R}^n, d\mu)$, $1 < p < \infty$.*

Indeed, it follows from Theorem 1.3 and the $L^2(\boldsymbol{R}^n, d\mu)$ boundedness of H (see [CG3]) that we can easily get the $L^p(\boldsymbol{R}^n, d\mu)$ boundedness of H for $1 < p < 2$ by applying the interpolation theorem. We then use the duality to get the $L^p(\boldsymbol{R}^n, d\mu)$ boundedness of H for $2 < p < \infty$.

The organization of this paper is as follows. In Section 2 we recall some elementary properties of the Hardy-Littlewood maximal operator with respect to sections, and two covering lemmas. The equivalence of all atomic Hardy spaces $H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n)$ will be proved in Section 3. In Section 4, we will show that the dual space of $H^1_{\mathcal{F}}(\boldsymbol{R}^n)$ is $BMO_{\mathcal{F}}(\boldsymbol{R}^n)$. Finally, the $(H^1_{\mathcal{F}}, L^1_{\mu})$ boundedness of the Monge-Ampère singular integral operator H will be proved in Section 5. Finally, we would like to point out that the basic idea of proving our main results in this paper is based on a noted paper [CW2] by Coifman and Weiss.

2. Elementary properties of sections and covering lemmas. From the properties (A) and (B) of sections, Aimar, Forzani, and Toledano [AFT] obtained the following *engulfing property*: There exists a constant $\theta \geq 1$, depending only on δ, K_1, and ϵ_1, such that for each $y \in S(x, t)$,

(D) $S(x, t) \subset S(y, \theta t)$ and $S(y, t) \subset S(x, \theta t)$.

Define a function ρ on $\boldsymbol{R}^n \times \boldsymbol{R}^n$ by

$$\rho(x, y) = \inf\{t > 0\,;\ y \in S(x, t)\}\,.$$

Using the engulfing property (D), Incognito [In] obtained the following conclusions:

(E) $\rho(x, y) \leq \theta\rho(y, x)$ for all $x, y \in \boldsymbol{R}^n$.

(F) $\rho(x, y) \leq \theta^2(\rho(x, z) + \rho(z, y))$ for all $x, y, z \in \boldsymbol{R}^n$.

Obviously, from the definition of ρ, it is easy to see that

(G) for a given section $S(x, t)$, $y \in S(x, t)$ if and only if $\rho(x, y) < t$.

In [CG1], Caffarelli and Gutiérrez defined the Hardy-Littlewood maximal operator M with respect to a family $\mathcal{F}$ of sections and the doubling measure μ by

$$Mf(x) = \sup_{t>0} \frac{1}{\mu(S(x, t))} \int_{S(x,t)} |f(y)|d\mu(y)\,. \tag{2.1}$$

We now give some elementary properties of the operator M.

LEMMA 2.1. *Let M be the Hardy-Littlewood maximal operator defined by* (2.1).

(i) *M is of weak type* $(1, 1)$, *that is, there exists a constant C_0 such that for all $\lambda > 0$ and any $f \in L^1(\boldsymbol{R}^n, d\mu)$*

$$\mu(\{x \in \boldsymbol{R}^n\ ;\ Mf(x) > \lambda\}) \leq \frac{C_0}{\lambda}\|f\|_{L^1_\mu}\,.$$

(ii) *M is of type (p, p) for $1 < p \leq \infty$, that is, there exists a constant C_1 such that for any $f \in L^p(\boldsymbol{R}^n, d\mu)$*

$$\|Mf\|_{L^p_\mu} \leq C_1\|f\|_{L^p_\mu}\,.$$

(iii) *For all $\lambda > 0$, the set $P^\lambda = \{x \in \boldsymbol{R}^n; Mf(x) > \lambda\}$ is a open set in $\boldsymbol{R}^n$.*

(iv) *Let $f \in L^1(\boldsymbol{R}^n, d\mu)$ and $\mathrm{supp}(f) \subset S_0 := S(x_0, t_0) \in \mathcal{F}$. Then there exists a constant $C_2 = C_2(A, \theta)$ such that, when $\lambda > C_2 \cdot m_{S_0}(|f|)$,*

$$P^\lambda = \{x \in \boldsymbol{R}^n; Mf(x) > \lambda\} \subset S(x_0, 2\theta^2(1 + \theta)t_0)\,,$$

where $m_{S_0}(|f|)$ is the mean of $|f|$ over the section S_0.

PROOF. See [CG1] for the proof of conclusion (i). From (i) and the obvious boundedness of M on $L^\infty(\boldsymbol{R}^n, d\mu)$, by applying the Marcinkiewicz interpolation theorem, we get (ii).

Now let us turn to the proof of (iii). Denote by E^c the complement of $E \subset \boldsymbol{R}^n$. It suffices to show that $(P^\lambda)^c = \{x \in \boldsymbol{R}^n\ ;\ Mf(x) \leq \lambda\}$ is a closed set for all $\lambda > 0$. Let $\{x_k\}_{k=1}^\infty \subset (P^\lambda)^c$ be a sequence of points such that $x_k \to x$ as $k \to \infty$. We have to show that, for any $t > 0$ and $S(x,t) \in \mathcal{F}$,

$$\frac{1}{\mu(S(x,t))}\int_{S(x,t)} |f(y)|d\mu(y) \le \lambda\,. \tag{2.2}$$

Denote $S_k = S(x_k, t)$ and $f_k(y) = f(y)\chi_{S(x,t)\triangle S_k}(y)$ for all $k = 1, 2, \ldots,$ where

$$S(x,t)\triangle S_k = (S(x,t)\backslash S_k) \cup (S_k\backslash S(x,t))\,.$$

Thus, $|f_k(y)| \le |f(y)|$ for all k and $\lim_{k\to\infty} f_k(y) = 0$ (μ-a.e.). Applying the Lebesgue dominated convergence theorem, we have

$$\lim_{k\to\infty}\frac{1}{\mu(S(x,t))}\int_{S(x,t)} |f_k(y)|d\mu(y) = 0\,.$$

On the other hand,

$$\frac{1}{\mu(S(x,t))}\int_{S_k} |f(y)|d\mu(y) = \frac{\mu(S_k)}{\mu(S(x,t))}\frac{1}{\mu(S_k)}\int_{S_k} |f(y)|d\mu(y) \le \frac{\mu(S_k)}{\mu(S(x,t))}\cdot\lambda\,.$$

Hence

$$\begin{aligned}\frac{1}{\mu(S(x,t))}\int_{S(x,t)} |f(y)|d\mu(y) &\le \frac{1}{\mu(S(x,t))}\int_{S(x,t)\triangle S_k} |f(y)|d\mu(y)\\ &\quad+\frac{1}{\mu(S(x,t))}\int_{S_k} |f(y)|d\mu(y)\\ &\le \frac{1}{\mu(S(x,t))}\int_{S(x,t)} |f_k(y)|d\mu(y) + \frac{\mu(S_k)}{\mu(S(x,t))}\cdot\lambda\,.\end{aligned}$$

Taking $k \to \infty$, we obtain (2.2).

Finally, we prove the conclusion (iv). Let $x \in \boldsymbol{R}^n$ and suppose $\rho(x_0, x) \ge 2\theta^2(1+\theta)t_0$ (equivalently, $x \notin S(x_0, 2\theta^2(1+\theta)t_0)$ by the property (G) of sections). Then for any $t \le t_0$, $S(x,t) \cap S(x_0,t_0) = \emptyset$. Indeed, if $y \in S(x,t) \cap S(x_0,t_0)$, then by the properties (E), (F) and (G) of sections

$$\begin{aligned}2\theta^2(1+\theta)t_0 \le \rho(x_0,x) \le \theta^2(\rho(x_0,y)+\rho(y,x)) \le \theta^2(\rho(x_0,y)+\theta\rho(x,y))\\ < \theta^2(t_0+\theta t) \le \theta^2(1+\theta)t_0\,.\end{aligned}$$

The contradiction shows that such y cannot exist. Thus $\int_{S(x,t)} |f(y)|d\mu(y) = 0$ for any section $S(x,t)$ with $t \le t_0$. Hence, whenever $x \notin S(x_0, 2\theta^2(1+\theta)t_0)$,

$$Mf(x) = \sup_{t>t_0}\frac{1}{\mu(S(x,t))}\int_{S(x,t)} |f(y)|d\mu(y)\,.$$

On the other hand, for a section $S(x,t)$ with $t > t_0$, we only consider the case that $S(x,t) \cap S(x_0,t_0) \ne \emptyset$. In this case, we take $z \in S(x,t) \cap S(x_0,t_0)$. Using the properties (E) and (F) of sections again, we have

$$S(x_0,t_0) \subset S(z,\theta t_0) \subset S(z,\theta t)\,.$$

On the other hand, by $z \in S(x,t) \subset S(x,\theta t)$ we get $S(z,\theta t) \subset S(x,\theta^2 t)$. Hence

$$S(x_0,t_0) \subset S(x,\theta^2 t)\,. \tag{2.3}$$

By (2.3) and the doubling condition (1.1) of the measure μ,

$$\frac{\mu(S(x_0,t_0))}{\mu(S(x,t))} \le \frac{\mu(S(x,\theta^2 t))}{\mu(S(x,t))} \le A^{1+2\log_2\theta}. \tag{2.4}$$

Denoting $C_2 = A^{1+2\log_2\theta}$, we obtain by (2.4) that for $x \notin S(x_0, 2\theta^2(1+\theta)t_0)$ and $t > t_0$

$$\begin{aligned}\frac{1}{\mu(S(x,t))}\int_{S(x,t)}|f(y)|d\mu(y) &\le \frac{\mu(S(x_0,t_0))}{\mu(S(x,t))}\frac{1}{\mu(S(x_0,t_0))}\int_{S(x_0,t_0)}|f(y)|d\mu(y)\\ &\le C_2\cdot m_{S_0}(|f|).\end{aligned}$$

This shows that whenever $x \notin S(x_0, 2\theta^2(1+\theta)t_0)$, we have $Mf(x) \le C_2 \cdot m_{S_0}(|f|)$. Therefore, if $\lambda > C_2 \cdot m_{S_0}(|f|)$, then $P^\lambda \subset S(x_0, 2\theta^2(1+\theta)t_0)$. This completes the proof of Lemma 2.1.

LEMMA 2.2 (Vitali-Wiener type covering lemma for sections). *Let $E \subset \boldsymbol{R}^n$ be a bounded set. If for each $x \in E$ there exists a section $S(x, t(x)) \subset E$ with $t(x) > 0$, then there exists a sequence $\{x_j\}_{j=1}^\infty \subset E$ such that*

(i) $\{S(x_j, t(x_j))\}_{j=1}^\infty$ *is a disjoint sequence of sections*;
(ii) $\bigcup_{j=1}^\infty S(x_j, 4\theta^3 t(x_j)) \supset E$.

PROOF. Denote $\mathcal{F}_E = \{S(x,t(x))\,;\ x \in E\}$. Since E is a bounded set, we may assume that

$$L = \sup\{t(x)\,;\ S(x,t(x)) \in \mathcal{F}_E\} < \infty.$$

Take $x_1 \in E$ such that $t(x_1) > L/2$. If $E \backslash S(x_1, 4\theta^3 t(x_1)) = \emptyset$, then we stop. Otherwise, we take $x_2 \in E\backslash S(x_1, 4\theta^3 t(x_1))$ such that

$$t(x_2) > \frac{1}{2}\sup\{t(x)\,;\ S(x,t(x)) \in \mathcal{F}_E \text{ and } x \in E\backslash S(x_1, 4\theta^3 t(x_1))\}.$$

If $E\backslash\{S(x_1, 4\theta^3 t(x_1)) \cup S(x_2, 4\theta^3 t(x_2))\} = \emptyset$, then we stop. Otherwise, we take $x_3 \in E\backslash\{S(x_1, 4\theta^3 t(x_1)) \cup S(x_2, 4\theta^3 t(x_2))\}$ such that

$$t(x_3) > \frac{1}{2}\sup\{t(x);\ S(x,t(x)) \in \mathcal{F}_E \text{ and } x \in E\backslash\{S(x_1, 4\theta^3 t(x_1)) \cup S(x_2, 4\theta^3 t(x_2))\}\}.$$

If $E \subset \bigcup_{j=1}^3 S(x_j, 4\theta^3 t(x_j))$, then we stop. Otherwise, we will continue the same process. In general, for the jth-stage we pick $x_j \in E\backslash \bigcup_{i=1}^{j-1} S(x_i, 4\theta^3 t(x_i))$ such that

$$t(x_j) > \frac{1}{2}\sup\left\{t(x)\,;\ S(x,t(x)) \in \mathcal{F}_E \text{ and } x \in E\backslash \bigcup_{i=1}^{j-1} S(x_i, 4\theta^3 t(x_i))\right\}. \tag{2.5}$$

Continuing in this way, we construct a sequence of sections in $\mathcal{F}_E$, possibly infinite and denoted by $\{S(x_j, t(x_j))\}_{j=1}^\infty$, satisfying the following conditions:

(a) For $j > 1$, $x_j \notin \bigcup_{i=1}^{j-1} S(x_i, 4\theta^3 t(x_i))$.
(b) For $i < j$, $t(x_i) > (1/2)t(x_j)$.

We first show that $\{S(x_j, t(x_j))\}$ is disjoint. Suppose that $y \in S(x_i, t(x_i)) \cap S(x_j, t(x_j))$. Without loss of generality, we may assume that $i < j$. Hence $t(x_i) > (1/2)t(x_j)$. By the properties (E), (F) and (G), we have

$$\begin{aligned}\rho(x_i, x_j) &\le \theta^2(\rho(x_i, y) + \rho(y, x_j)) \le \theta^2(\rho(x_i, y) + \theta\rho(x_j, y)) \\ &< \theta^2(t(x_i) + \theta t(x_j)) < \theta^2(1 + 2\theta)t(x_i) \\ &< 4\theta^3 t(x_i)\,.\end{aligned}$$

Using the property (G) again, we get $x_j \in S(x_i, 4\theta^3 t(x_i))$. However, this contradicts the condition (a).

Now we prove that $E \subset \bigcup_{j=1}^{\infty} S(x_j, 4\theta^3 t(x_j))$. If it is not the case, then there exists $x_0 \in E$ such that $x_0 \notin \bigcup_{j=1}^{\infty} S(x_j, 4\theta^3 t(x_j))$. So, there exists a section $S(x_0, t(x_0)) \in \mathcal{F}_E$ with $t(x_0) > 0$. Since $\{S(x_j, t(x_j))\}_{j=1}^{\infty}$ is disjoint and $\bigcup_{j=1}^{\infty} S(x_j, t(x_j)) \subset E$ is bounded, we have

$$\infty > |E| \ge \left|\bigcup_{j=1}^{\infty} S(x_j, t(x_j))\right| = \sum_{j=1}^{\infty} |S(x_j, t(x_j))|\,,$$

where $|E|$ denotes the Lebesgue measure of the set E. From this we get

$$\lim_{j\to\infty} |S(x_j, t(x_j))| = 0\,,$$

and hence

$$\lim_{j\to\infty} t(x_j) = 0\,, \tag{2.6}$$

because, for each j, $S(x_j, t(x_j))$ is a bounded, convex, open set in $\boldsymbol{R}^n$. By (2.6) we may choose j large enough such that $2t(x_j) < t(x_0)$. However, this contradicts $t(x_j) > (1/2)t(x_0)$ by (2.5), because

$$x_0 \in E \setminus \bigcup_{k=1}^{\infty} S(x_k, 4\theta^3 t(x_k)) \subset E \setminus \bigcup_{i=1}^{j-1} S(x_i, 4\theta^3 t(x_i))\,.$$

Thus we finish the proof of Lemma 2.2.

Before proving the following covering lemma, let us recall another property of sections. In [AFT], the authors proved that if a family $\mathcal{F}$ of sections satisfies the properties (A), (B) and (C), then there exists a quasi-metric $d(x, y)$ on $\boldsymbol{R}^n$ with respect to $\mathcal{F}$ defined by

$$d(x, y) = \inf\{r;\, x \in S(y, r) \text{ and } y \in S(x, r)\}\,.$$

The triangular constant of the quasi-metric d is just the θ appeared in the property (D), that is,

$$d(x, y) \le \theta(d(x, z) + d(z, y)) \quad \text{for any } x, y, z \in \boldsymbol{R}^n\,.$$

Moreover, denoting by $B_d(x, r) = \{y \in \boldsymbol{R}^n\,;\ d(x, y) < r\}$ the d-ball of center x with radius r, we have the following facts.

LEMMA 2.3. *Let E be an open set in $\boldsymbol{R}^n$ and E^c denote the complement of E. For any $x \in E$, write $r = d(x, E^c) = \inf\{d(x, y)\ ;\ y \in E^c\}$. Then*

(i) $d(x, E^c) > 0$;

(ii) $B_d(x, r) \subset E$;

(iii) $B_d(x, 2r) \cap E^c \neq \emptyset$.

PROOF. (i) If $d(x, E^c) = 0$, then there exists a sequence $\{y_n\} \in E^c$ such that $d(x, y_n) < 1/n$ for each n. Hence, $y_n \in S(x, 1/n)$ for every n. On the other hand, since E is open, there is an $\varepsilon > 0$ such that $B(x, \varepsilon) = \{y \in \boldsymbol{R}^n\ ;\ |x - y| < \varepsilon\} \subset E$. By the property (C) of sections,

$$y_n \in S(x, 1/n) \subset B(x, \varepsilon) \subset E \quad \text{when } n \text{ is large enough}.$$

But this is impossible because $\{y_n\} \in E^c$ for all n.

(ii) If $B_d(x, r) \cap E^c \neq \emptyset$, then there exists $y_0 \in B_d(x, r) \cap E^c$. Thus

$$r = d(x, E^c) = \inf\{d(x, y)\ ;\ y \in E^c\} \leq d(x, y_0) < r.$$

This contradiction shows that $B_d(x, r) \subset E$.

(iii) If $B_d(x, 2r) \subset E$, then we have $y \in B_d(x, 2r) \subset E$ whenever $d(x, y) < 2r$. On the other hand, there exists a sequence $\{y_n\} \subset E^c$ such that $d(x, y_n) < d(x, E^c) + 1/n = r + 1/n$ for all $n \in \boldsymbol{N}$. Since $r > 0$, we have $r + 1/n < 2r$, when n is large enough. Thus $y_n \in B_d(x, 2r) \subset E$ for n large enough. However, this contradicts $\{y_n\} \subset E^c$ for all n.

The following relationship between a section and a d-ball can be found in [AFT].

(H) For any $x \in \boldsymbol{R}^n$ and any $r > 0$, $S(x, r/2\theta) \subset B_d(x, r) \subset S(x, r)$.

Now let us state and prove the Whitney type covering lemma for sections.

LEMMA 2.4 (Whitney type covering lemma for sections). *Suppose that $E \subset \boldsymbol{R}^n$ is a bounded open set in $\boldsymbol{R}^n$ and $C \geq 1$. Then there exists a sequence of sections $\{S(x_k, t_k)\}_{k=1}^{\infty}$ satisfying the following*:

(i) *Let $S_k = S(x_k, t_k)$. Then $E = \bigcup_{k=1}^{\infty} S_k$.*

(ii) *Let $\tilde{S}_k = S(x_k, 16C\theta^3 t_k)$. Then for each k, $\tilde{S}_k \cap E^c \neq \emptyset$.*

(iii) *Let $\bar{S}_k = S(x_k, 2C\theta t_k)$. Then $\{\bar{S}_k\}_{k=1}^{\infty}$ is a Θ-disjoint collection, that is, there exists a constant $\Theta = \Theta(A, \theta, C)$ such that $\sum_{k=1}^{\infty} \chi_{\bar{S}_k}(x) \leq \Theta$.*

PROOF. Let $r(x) = d(x, E^c)$ for $x \in E$. By property (H), we have

$$\tag{2.7} \begin{aligned} S\left(x, \frac{r(x)}{8\theta^3 C}\right) &\subset B_d\left(x, \frac{r(x)}{4\theta^2}\right) \subset S\left(x, \frac{r(x)}{4\theta^2}\right) \subset B_d\left(x, \frac{r(x)}{2\theta}\right) \\ &\subset S\left(x, \frac{r(x)}{2\theta}\right) \subset B_d(x, r(x)) \subset E. \end{aligned}$$

Therefore, the family of sections $\{S(x, r(x)/4\theta^3 8\theta^3 C)\ ;\ x \in E\}$ satisfies the condition of Lemma 2.2. By the conclusions of Lemma 2.2, there exists a sequence $\{x_k\}_{k=1}^{\infty} \subset E$ such that

(a) $\{S(x_k, r_k/4\theta^3 8\theta^3 C)\}_{k=1}^{\infty}$ is a disjoint sequence of sections,

(b) $\bigcup_{k=1}^{\infty} S(x_k, r_k/8\theta^3 C) \supset E$, where and below we denote $r(x_k)$ by r_k for simplicity. By (2.7) and (b) we obtain

$$E \subset \bigcup_{k=1}^{\infty} S\left(x_k, \frac{r_k}{8\theta^3 C}\right) \subset \bigcup_{k=1}^{\infty} S\left(x_k, \frac{r_k}{4\theta^2}\right) \subset \bigcup_{k=1}^{\infty} B_d\left(x_k, \frac{r_k}{2\theta}\right) \subset E\,. \tag{2.8}$$

We first prove that $\{B_d(x_k, r_k/2\theta)\}_{k=1}^{\infty}$ is a Θ-disjoint collection. Let $z_0 \in B_d(x_k, r_k/2\theta)$ and denote $R_0 = d(z_0, E^c)$. Then

$$r_k = d(z_k, E^c) \le \theta[d(x_k, z_0) + d(z_0, E^c)] \le \theta\left(\frac{r_k}{2\theta} + R_0\right) = \frac{r_k}{2} + \theta R_0\,.$$

Thus $r_k \le 2\theta R_0$. From this, we have

$$B_d(x_k, r_k/2\theta) \subset B_d(z_0, 2\theta R_0) \quad \text{for each } k\,. \tag{2.9}$$

Indeed, for any $y \in B_d(x_k, r_k/2\theta)$,

$$d(z_0, y) \le \theta[d(z_0, x_k) + d(x_k, y)] \le \theta(r_k/2\theta + r_k/2\theta) \le 2\theta R_0\,.$$

On the other hand, we see that

$$\begin{aligned} R_0 = d(z_0, E^c) &\le \theta[d(z_0, x_k) + d(x_k, E^c)] \\ &\le \theta\left(\frac{r_k}{2\theta} + r_k\right) = \left(\frac{1}{2} + \theta\right) r_k = \left(\frac{1}{2} + \theta\right) 4\theta^3 8\theta^3 C \cdot \frac{r_k}{4\theta^3 8\theta^3 C}\,. \end{aligned}$$

Equivalently,

$$\frac{r_k}{4\theta^3 8\theta^3 C} \ge \frac{R_0}{(1/2+\theta)4\theta^3 8\theta^3 C}\,. \tag{2.10}$$

Now we assume that

$$z_0 \in \bigcap_j B_d(x_{k_j}, r_{k_j}/2\theta)\,. \tag{2.11}$$

To simplify the notation we denote $x_j = x_{k_j}$ and $r_j = r_{k_j}$. Then by (2.9), for each j,

$$B_d\left(x_j, \frac{r_j}{4\theta^3 8\theta^3 C}\right) \subset B_d\left(x_j, \frac{r_j}{2\theta}\right) \subset B_d(z_0, 2\theta R_0)\,.$$

Note that for each j, $B_d(x_j, r_j/4\theta^3 8\theta^3 C) \subset S(x_j, r_j/4\theta^3 8\theta^3 C)$ by (H). Hence, the sequence $\{B_d(x_j, r_j/4\theta^3 8\theta^3 C)\}_{j=1}^{\infty}$ is also disjoint by (a). Thus by (2.10)

$$d(x_i, x_j) \ge \min\left\{\frac{r_i}{4\theta^3 8\theta^3 C}, \frac{r_j}{4\theta^3 8\theta^3 C}\right\} \ge \frac{R_0}{(1/2+\theta)4\theta^3 8\theta^3 C}\,.$$

By Lemma 1.1 in [CW1], there exists a constant $\Theta = \Theta(A, \theta, C)$ such that the numbers of j in (2.11) cannot be greater than K. By the Θ-disjointness of $\{B_d(x_k, r_k/2\theta)\}_{k=1}^{\infty}$ and (2.7), we obtain the Θ-disjointness of $\{S(x_k, r_k/4\theta^2)\}_{k=1}^{\infty}$.

Finally, we take $t_k = r_k/8\theta^3 C$. Then by (2.8) we get the conclusions (i) and (iii) of Lemma 2.4. As for the conclusion (ii), it is a direct result of Lemma 2.3 (iii), because

$$\tilde{S}_k = S(x_k, 16C\theta^3 t_k) = S(x_k, 2r_k) \supset B_d(x_k, 2r_k)\,.$$

Therefore we complete the proof of Lemma 2.4.

The following fact is obvious.

LEMMA 2.5. *Suppose that $F_k \subset E_k$ for each k, and $\{E_k\}_{k=1}^{\infty}$ is a Θ-disjoint collection. Then $\{F_k\}_{k=1}^{\infty}$ is also a Θ-disjoint collection.*

REMARK 2.1. By the conclusion (iii) of Lemma 2.4 and Lemma 2.5, $\{S_k\}_{k=1}^{\infty}$ is also a Θ-disjoint collection, since $S_k \subset \bar{S}_k$ for each k.

3. Proof of theorem 1.1. First it is easy to see that for all $q > 1$, $H_{\mathcal{F}}^{1,\infty}(\boldsymbol{R}^n) \subset H_{\mathcal{F}}^{1,q}(\boldsymbol{R}^n)$. We now show that the opposite inclusion also holds. It suffices to prove that every $(1, q)$-atom $a(x)$ has the representation

$$a(x) = \sum_j \alpha_j a_j(x)\,, \tag{3.1}$$

where each $a_j(x)$ is a $(1,\infty)$-atom and $\sum_j |\alpha_j| < \infty$.

Since $a(x)$ is a $(1, q)$-atom, there exists a section $S_0 = S(x_0, t_0) \in \mathcal{F}$ such that $\operatorname{supp}(a) \subset S(x_0, t_0)$. We denote $b(x) = \mu(S_0)a(x)$. Then

$$\text{(i)}\ \operatorname{supp}(b) \subset S_0\,, \quad \text{(ii)} \int b(x)d\mu(x) = 0\,, \quad \text{and} \quad \text{(iii)}\ \|b\|_{L^q(\mu)} \le (\mu(S_0))^{1/q}\,. \tag{3.2}$$

On the other hand, we take the constant $C = \theta(1+\theta)$ in Lemma 2.4. Then by (1.1) we have

$$\frac{\mu(\bar{S}_k)}{\mu(S_k)} \le A^{2+\log_2 \theta^2(1+\theta)} := K_0 \quad \text{for every } k\,. \tag{3.3}$$

For a positive integer m, let $\boldsymbol{N}^m = \boldsymbol{N} \times \boldsymbol{N} \times \cdots \times \boldsymbol{N}$ and $\boldsymbol{N}^0 = \{0\}$. We denote the general element in $\boldsymbol{N}^m$ by j_m. We prove the following proposition by an inductive argument on m.

PROPOSITION 3.1. *There exists a sequence of sections $\{S_{j_\ell}\} \subset \mathcal{F}$, $j_\ell \in \boldsymbol{N}^\ell$, $\ell = 0, 1, \ldots,$ such that for each natural number m*

$$b(x) = D_0 \Theta \alpha \sum_{\ell=0}^{m-1} \alpha^\ell \sum_{j_\ell \in \boldsymbol{N}^\ell} \mu(\bar{S}_{j_\ell}) a_{j_\ell}(x) + \sum_{j_m \in \boldsymbol{N}^m} h_{j_m}(x)\,, \tag{3.4}$$

where $\alpha = \alpha(q, A, \theta)$, $D_0 = D_0(A, \theta)$, and

(I) $a_{j_\ell}(x)$ *is a $(1,\infty)$-atom supported in $\bar{S}_{j_\ell}$, $j_\ell \in \boldsymbol{N}^\ell$, $\ell = 0, 1, \ldots, m-1$;*

(II) $\bigcup_{j_m \in \boldsymbol{N}^m} S_{j_m} \subset \{x \in \boldsymbol{R}^n;\, (M_q b)(x) > \alpha^m/2\}$, *and* $(M_q b)(x) = [M(|b|^q)(x)]^{1/q}$;

(III) $\{\bar{S}_{j_\ell}\}$ *is a Θ^ℓ-disjoint collection*;

(IV) *the functions $h_{j_m}(x)$ are supported in S_{j_m}*;

(V) $\int h_{j_m}(x) d\mu(x) = 0$;

(VI) $|h_{j_m}(x)| \le |b(x)| + D_0 \alpha^m \chi_{S_{j_m}}(x)$;

(VII) $[m_{S_{j_m}}(|h_{j_m}|^q)]^{1/q} \le 2 D_0 \alpha^m$.

We first show that if the properties from (I) to (VII) hold for each $m \in \boldsymbol{N}$, then (3.1) holds. By (3.3), (II), (III), Lemma 2.5 and Lemma 2.1 (i), we have

$$\begin{aligned}\sum_{j_m \in \boldsymbol{N}^m} \mu(\bar{S}_{j_m}) \le K_0 \sum_{j_m \in \boldsymbol{N}^m} \mu(S_{j_m}) &\le K_0 \Theta^m \mu\Big(\bigcup_{j_m \in \boldsymbol{N}^m} S_{j_m}\Big) \\ &\le K_0 \Theta^m \mu(\{x \in \boldsymbol{R}^n ; (M_q b)(x) > \alpha^m/2\}) \\ &\le K_0 \Theta^m C_0 (2/\alpha^m)^q \|b\|^q_{L^q_\mu} .\end{aligned} \tag{3.5}$$

In the last inequality, we use the conclusion (i) of Lemma 2.1. By (iii) in (3.2)

$$\sum_{m=1}^{\infty} \alpha^m \sum_{j_m \in \boldsymbol{N}^m} \mu(\bar{S}_{j_m}) \le C_0 K_0 2^q \sum_{m=1}^{\infty} (\Theta \alpha^{1-q})^m \mu(S_0) .$$

Hence, if we choose α such that $\alpha > \Theta^{1/(q-1)}$, then

$$\sum_{m=1}^{\infty} \alpha^m \sum_{j_m \in \boldsymbol{N}^m} \mu(\bar{S}_{j_m}) \le B \mu(S_0) , \tag{3.6}$$

where $B = B(q, A, \theta, \alpha)$ is independent of $a(x)$.

By (IV) and (VII) we have

$$\int |h_{j_m}(x)| d\mu(x) \le \mu(S_{j_m}) \left(\frac{1}{\mu(S_{j_m})} \int_{S_{j_m}} |h_{j_m}(x)|^q d\mu(x) \right)^{1/q} \le \mu(S_{j_m}) \cdot 2 D_0 \alpha^m . \tag{3.7}$$

Denote $H_m(x) = \sum_{j_m \in \boldsymbol{N}^m} h_{j_m}(x)$. Then (3.5) and (3.7) imply

$$\begin{aligned}\int |H_m(x)| d\mu(x) &\le \sum_{j_m \in \boldsymbol{N}^m} \int |h_{j_m}(x)| d\mu(x) \\ &\le 2 D_0 \alpha^m \sum_{j_m \in \boldsymbol{N}^m} \mu(S_{j_m}) \le 2^{q+1} C_0 K_0 D_0 (\Theta \alpha^{1-q})^m \|b\|^q_{L^q_\mu} .\end{aligned} \tag{3.8}$$

Thus, if $\alpha > \Theta^{1/(q-1)}$, then by (3.8)

$$\lim_{m \to \infty} \int |H_m(x)| d\mu(x) \le C \mu(S_0) \cdot \lim_{m \to \infty} (\Theta \alpha^{1-q})^m = 0 . \tag{3.9}$$

On the other hand, by (I) and (3.6),

$$\begin{aligned}&\int D_0 \Theta \alpha \sum_{i=m}^{\infty} \alpha^i \sum_{j_i \in \boldsymbol{N}^i} \mu(\bar{S}_{j_i}) |a_{j_i}(x)| d\mu(x) \\ &= \int_{\bar{S}_{j_i}} D_0 \Theta \alpha \sum_{i=m}^{\infty} \alpha^i \sum_{j_i \in \boldsymbol{N}^i} \mu(\bar{S}_{j_i}) |a_{j_i}(x)| d\mu(x)\end{aligned} \tag{3.10}$$

$$\leq D_0 \Theta \alpha \sum_{i=m}^{\infty} \alpha^i \sum_{j_i \in \boldsymbol{N}^i} \mu(\bar{S}_{j_i}) \|a_{j_i}\|_{L^\infty_\mu} \cdot \mu(\bar{S}_{j_i})$$

$$\leq D_0 \Theta \alpha \sum_{i=m}^{\infty} \alpha^i \sum_{j_i \in \boldsymbol{N}^i} \mu(\bar{S}_{j_i}) \to 0 \quad (m \to \infty)\,.$$

It follows from (3.9) and (3.10) that, when $m \to \infty$,

$$\frac{D_0 \Theta \alpha}{\mu(S_0)} \sum_{\ell=0}^{m-1} \alpha^\ell \sum_{j_\ell \in \boldsymbol{N}^\ell} \mu(\bar{S}_{j_\ell}) a_{j_\ell}(x) + \frac{1}{\mu(S_0)} \sum_{j_m \in \boldsymbol{N}^m} h_{j_m}(x)$$

converges to $b(x)/\mu(S_0) = a(x)$ in the L^1_μ norm. Thus, in the sense of distribution we have

$$a(x) = \frac{D_0 \Theta \alpha}{\mu(S_0)} \sum_{m=1}^{\infty} \alpha^m \sum_{j_m \in \boldsymbol{N}^m} \mu(\bar{S}_{j_m}) a_{j_m}(x)\,,$$

where each $a_{j_m}(x)$ is a $(1, \infty)$-atom and

$$\frac{D_0 \Theta \alpha}{\mu(S_0)} \sum_{m=0}^{\infty} \alpha^m \sum_{j_m \in \boldsymbol{N}^m} \mu(\bar{S}_{j_m}) \leq B < \infty\,.$$

From this, we obtain $a(x) \in H^{1,\infty}_{\mathcal{F}}(\boldsymbol{R}^n)$. Hence, to prove Theorem 1.1, it remains only to show that the properties from (I) to (VII) hold for each $m \in \boldsymbol{N}$.

PROOF OF PROPOSITION 3.1. We first show that these properties are valid for $m = 1$. Let $E^\alpha = \{x \in \boldsymbol{R}^n;\, (M_q b)(x) > \alpha\}$. By (iii) in (3.2) and Lemma 2.1 (iv), if $\alpha^q > C_2 \geq C_2 \cdot m_{S_0}(|b|^q)$, then

$$E^\alpha \subset S(x_0, 2\theta^2(1+\theta)t_0) := \bar{S}_0\,.$$

From this and Lemma 2.1 (iii), E^α is a bounded open set if $\alpha^q > C_2$. By Lemma 2.1 (i), we have

$$\mu(E^\alpha) \leq C_0(\|b\|_{L^q_\mu}/\alpha)^q \leq C_0 \alpha^{-q} \mu(S_0)\,. \tag{3.11}$$

Applying Lemma 2.4 to E^α with the constant $C = \theta(1+\theta)$, we obtain a sequence of sections $\{S_j = S(x_j, t_j)\}_{j=1}^\infty$ satisfying

(II) $\bigcup_j S_j = E^\alpha \subset \{x \in \boldsymbol{R}^n;\, (M_q b)(x) > \alpha^m/2\}$,

(III) $\{\bar{S}_j = S(x_j, 2\theta^2(1+\theta)t_j)\}$ is a Θ-disjoint collection,

and for each j

$$\tilde{S}_j \cap (E^\alpha)^c \neq \emptyset, \quad \text{where} \quad \tilde{S}_j = S(x_j, 16\theta^4(1+\theta)t_j)\,. \tag{3.12}$$

If we denote by $\chi_j(x)$ the characteristic function of S_j, then $\sum_{j=1}^\infty \chi_j(x) \leq \Theta$ by Remark 2.1. Let

$$\eta_j(x) = \begin{cases} \chi_j(x)/\sum_j \chi_j(x) & \text{if } x \in E^\alpha\,, \\ 0 & \text{if } \ x \notin E^\alpha\,, \end{cases}$$

and

$$g_0(x) = \begin{cases} b(x) & \text{if } x \notin E^\alpha\,, \\ \sum_j m_{S_j}(\eta_j b)\chi_j(x) & \text{if } x \in E^\alpha\,. \end{cases}$$

In addition, $h_j(x) = b(x)\eta_j(x) - m_{S_j}(\eta_j b)\chi_j(x)$ for any $x \in \boldsymbol{R}^n$. Then $b(x) = g_0(x) + \sum_{j=1}^{\infty} h_j(x)$ for any $x \in \boldsymbol{R}^n$.

By the property (C) of sections and the fact that the Hardy-Littlewood maximal operator M related to sections is of weak type (1,1) (see Lemma 2.1 (i)), it is easy to check that the Lebesgue differential theorem holds for the family $\mathcal{F}$ of sections. So, if $x \notin E^\alpha$, we have

$$|g_0(x)| \le |b(x)| \le (M_q b)(x) \le \alpha\,.$$

On the other hand, by (3.12) there exists $z_j \in \tilde{S}_j \cap (E^\alpha)^c$. By the property (D) of sections, we have

$$\tilde{S}_j = S(x_j, 16\theta^4(1+\theta)t_j) \subset S(z_j, 16\theta^5(1+\theta)t_j) \tag{3.13}$$

and

$$S(z_j, 16\theta^4(1+\theta)t_j) \subset S(x_j, 16\theta^5(1+\theta)t_j)\,. \tag{3.14}$$

The above (3.13) yields

$$S(x_j, t_j) \subset \tilde{S}_j \subset S(z_j, 16\theta^5(1+\theta)t_j)\,, \tag{3.15}$$

which implies

$$\begin{aligned}\left(\frac{1}{\mu(S_j)}\int_{S_j}|b(x)|^q d\mu(x)\right)^{1/q} &\le \left(\frac{\mu(S(z_j, 16\theta^5(1+\theta)t_j))}{\mu(S_j)}\right)^{1/q} \\ &\quad\times\left(\frac{1}{\mu(S(z_j, 16\theta^5(1+\theta)t_j))}\int_{S(z_j,16\theta^5(1+\theta)t_j)}|b(x)|^q d\mu(x)\right)^{1/q} \\ &\le \left(\frac{\mu(S(z_j, 16\theta^5(1+\theta)t_j))}{\mu(S_j)}\right)^{1/q}\cdot (M_q b)(z_j)\,.\end{aligned}$$

Using the inclusion relations (3.14) and (3.15) again, we have

$$\begin{aligned}\frac{\mu(S(z_j, 16\theta^5(1+\theta)t_j))}{\mu(S_j)} &= \frac{\mu(S(z_j, 16\theta^5(1+\theta)t_j))}{\mu(S(z_j, 16\theta^4(1+\theta)t_j))} \\ &\quad\times\frac{\mu(S(z_j, 16\theta^4(1+\theta)t_j))}{\mu(S(x_j, 16\theta^5(1+\theta)t_j))}\cdot\frac{\mu(S(x_j, 16\theta^5(1+\theta)t_j))}{\mu(S_j)} \\ &\le A^{1+\log_2\theta}\cdot A^{5+\log_2\theta^5(1+\theta)}\,,\end{aligned}$$

and hence

$$\left(\frac{1}{\mu(S_j)}\int_{S_j}|b(x)|^q d\mu(x)\right)^{1/q} \le (A^{6+\log_2\theta^6(1+\theta)})^{1/q}(M_q b)(z_j)\,. \tag{3.16}$$

Thus, if $x \in E^\alpha$, by Remark 2.1 together with (3.16) and noting that $z_j \in (E^\alpha)^c$, we obtain

$$|g_0(x)| \le \sum_{\substack{\text{at most}\\ \Theta \text{ terms}}} \frac{1}{\mu(S_j)} \int_{S_j} |b(x)\eta_j(x)| d\mu(x)$$
$$\le \sum_{\text{ibid}} \left(\frac{1}{\mu(S_j)} \int_{S_j} |b(x)|^q d\mu(x) \right)^{1/q} \le \Theta D_0 \alpha ,$$

where $D_0 = (A^{6+\log_2 \theta^6(1+\theta)})^{1/q}$. This shows that

(1) $|g_0(x)| \le \Theta D_0 \alpha$ for any $x \in \boldsymbol{R}^n$.

Since $E^\alpha \subset \bar{S}_0$ and $g_0(x) = b(x)$ for $x \notin E^\alpha$, by (i) in (3.2), we have

(2) $\operatorname{supp}(g_0) \subset \bar{S}_0$.

By the definition of $h_j(x)$, we have

(IV) $\operatorname{supp}(h_j) \subset S_j$ for each j,

(V) $\int h_j(x) d\mu(x) = 0$ for each j.

Noting that $\|h_j\|_{L^1_\mu} \le 2\|b\chi_j\|_{L^1_\mu} = 2\int_{S_j} |b(x)| d\mu(x)$ and by Remark 2.1, we have

$$\sum_j \|h_j\|_{L^1_\mu} \le 2 \sum_j \int_{S_j} |b(x)| d\mu(x) \le 2\Theta \int_{\bigcup_j S_j} |b(x)| d\mu(x)$$
$$\le 2\Theta \|b\|_{L^1_\mu} \le 2\Theta \|b\|_{L^q_\mu} (\mu(S_0))^{1/q'} \le 2\Theta \mu(S_0) .$$

Hence $g_0(x) + \sum_{j=1}^\infty h_j(x)$ converges to $b(x)$ in the L^1_μ norm. In fact, it is also convergent almost everywhere, since the sum has at most Θ terms. Thus, by (V) and (ii) in (3.2), we obtain

(3) $\int g_0(x) d\mu(x) = 0$.

Set $a_0(x) = g_0(x)(D_0 \Theta \alpha \mu(\bar{S}_0))^{-1}$. From the facts (1), (2), and (3), we see that $a_0(x)$ is a $(1, \infty)$-atom supported in the section $\bar{S}_0$, which is just (I). So, we have

$$b(x) = D_0 \Theta \alpha \mu(\bar{S}_0) a_0(x) + \sum_{j=1}^\infty h_j(x) ,$$

which is (3.4) for $m = 1$. It follows from (3.16) that

$$m_{S_j}(|b\eta_j|) \le \left(\frac{1}{\mu(S_j)} \int_{S_j} |b(x)|^q d\mu(x) \right)^{1/q} \le D_0 \cdot (M_q b)(z_j) \le D_0 \alpha , \tag{3.17}$$

since $z_j \notin E^\alpha$. Hence $|h_j(x)| \le |b(x)| + m_{S_j}(|b\eta_j|)\chi_j(x) \le |b(x)| + D_0 \alpha \chi_j(x)$ by (3.17), and (VI) holds. Finally, using (3.17) again, it is easy to check that (VII) is also valid. Thus we prove Proposition 3.1 for $m = 1$.

We now assume that Proposition 3.1 holds for m, and show that it is also true for $m + 1$. Let $E_{j_m} = \{x \in \boldsymbol{R}^n \; ; \; (M_q h_{j_m})(x) > \alpha^{m+1}\}$. By the hypothesis (IV), $\operatorname{supp}(h_{j_m}) \subset S_{j_m} = S(x_{j_m}, t_{j_m})$. If $\alpha^q > C_2(2D_0)^q$, then by (VII) we have

$$C_2 m_{S_{j_m}}(|h_{j_m}|^q) \le C_2((2D_0\alpha^m)^q < \alpha^{q(m+1)} .$$

Apply Lemma 2.1 (iv) to get $E_{j_m} \subset \bar{S}_{j_m} := S(x_{j_m}, 2\theta^2(1+\theta)t_{j_m})$. Thus E_{j_m} is a bounded open set if $\alpha^q > C_2(2D_0)^q$ by Lemma 2.1 (iii). Applying Lemma 2.4 for E_{j_m} with the constant $C = \theta(1+\theta)$, we obtain a sequence of sections $\{S^i_{j_m} = S(x^i_{j_m}, t^i_{j_m})\}_{i=1}^{\infty}$ such that

(4) $\bigcup_i S^i_{j_m} = E_{j_m} \subset \{x \in \boldsymbol{R}^n\ ;\ (M_q h_{j_m})(x) > \alpha^{m+1}/2\}$,

(5) $\{\bar{S}^i_{j_m} := S(x^i_{j_m}, 2\theta^2(1+\theta)t^i_{j_m})\}_{i=1}^{\infty}$ is a Θ-disjoint collection,

(6) for each i, $\tilde{S}^i_{j_m} \cap (E_{j_m})^c \neq \emptyset$, where $\tilde{S}^i_{j_m} := S(x^i_{j_m}, 16\theta^4(1+\theta)t^i_{j_m})$.

By the hypothesis (III) for m, we know that $\{\bar{S}_{j_m}\}$ is a Θ^m-disjoint collection, since the totality of sections in the family $\{\bar{S}^i_{j_m}\}$ is Θ^{m+1}-disjoint for all $j_m \in \boldsymbol{N}^m$ and $i \in \boldsymbol{N}$. This shows that (III) holds for $m+1$.

Now denote the characteristic function of section $S^i_{j_m}$ by $\chi^i_{j_m}(x)$. Then it follows from (5) and Lemma 2.5 that $\sum_{i=1}^{\infty} \chi^i_{j_m}(x) \le \Theta$. Let

$$\eta^i_{j_m}(x) = \begin{cases} \chi^i_{j_m}(x)/\sum_\ell \chi^\ell_{j_m}(x) & \text{if } x \in E_{j_m}, \\ 0 & \text{if } x \notin E_{j_m}, \end{cases}$$

and

$$g_{j_m}(x) = \begin{cases} h_{j_m}(x) & \text{if } x \notin E_{j_m}, \\ \sum_i m_{S^i_{j_m}}(h_{j_m}\eta^i_{j_m})\chi^i_{j_m}(x) & \text{if } x \in E_{j_m}. \end{cases}$$

In addition, we have $h^i_{j_m}(x) = h_{j_m}(x)\eta^i_{j_m}(x) - m_{S^i_{j_m}}(h_{j_m}\eta^i_{j_m})\chi^i_{j_m}(x)$ for any $x \in \boldsymbol{R}^n$.

If $x \notin E_{j_m}$, then

$$|g_{j_m}(x)| \le |h_{j_m}(x)| \le (M_q h_{j_m})(x) \le \alpha^{m+1}.$$

On the other hand, by (6) and by making use of the properties of sections and the same idea as in proving (3.16), we may get

$$\left(\frac{1}{\mu(S^i_{j_m})}\int_{S^i_{j_m}} |h_{j_m}(x)|^q d\mu(x)\right)^{1/q} \le (A^{6+\log_2 \theta^6(1+\theta)})^{1/q}(M_q h_{j_m})(z_j) \le D_0\alpha^{m+1}, \tag{3.18}$$

where $z_j \in \tilde{S}^i_{j_m} \cap (E_{j_m})^c$ and $D_0 = (A^{6+\log_2 \theta^6(1+\theta)})^{1/q}$. Hence, if $x \in E_{j_m}$, then by (5), Lemma 2.5 and (3.18) we have

$$\begin{aligned} |g_{j_m}(x)| &\le \sum_{\substack{\text{at most} \\ \Theta \text{ terms}}} \frac{1}{\mu(S^i_{j_m})}\int_{S^i_{j_m}} |h_{j_m}(x)\eta^i_{j_m}(x)| d\mu(x) \\ &\le \sum_{\text{ibid}} \left(\frac{1}{\mu(S^i_{j_m})}\int_{S^i_{j_m}} |h_{j_m}(x)|^q d\mu(x)\right)^{1/q} \\ &\le \Theta D_0 \alpha^{m+1}. \end{aligned}$$

Thus we obtain

(7) $|g_{j_m}(x)| \le \Theta D_0 \alpha^{m+1}$ for any $x \in \boldsymbol{R}^n$.

Since $E_{j_m} \subset \bar{S}_{j_m}$, by the definition of $g_{j_m}(x)$ we have

(8) $\operatorname{supp}(g_{j_m}) \subset \bar{S}_{j_m}$.

In addition, it is obvious that $\mathrm{supp}(h^i_{j_m}) \subset S^i_{j_m}$ and $\int h^i_{j_m}(x)d\mu(x) = 0$ for each j. Thus (IV) and (V) hold for $m+1$. Since $\|h^i_{j_m}\|_{L^1_\mu} \le 2\|h_{j_m}\chi^i_{j_m}\|_{L^1_\mu} = 2\int_{S^i_{j_m}} |h_{j_m}(x)|d\mu(x)$, by (5) together with Lemma 2.5 we have

$$\sum_i \|h^i_{j_m}\|_{L^1_\mu} \le 2\sum_i \int_{S^i_{j_m}} |h_{j_m}(x)|d\mu(x) \le 2\Theta \int_{\bigcup_i S^i_{j_m}} |h_{j_m}(x)|d\mu(x)$$
$$\le 2\Theta\|h_{j_m}\|_{L^1_\mu} \le 2\Theta\|h_{j_m}\|_{L^q_\mu}(\mu(S_{j_m}))^{1/q'} \le 2\Theta\mu(S_{j_m})\,.$$

Hence $g_{j_m}(x)+\sum_{i=1}^\infty h^i_{j_m}(x)$ converges to $h_{j_m}(x)$ in the L^1_μ norm (it is also convergent almost everywhere). Thus, by the cancellation properties of $h_{j_m}(x)$ and $h^i_{j_m}(x)$ for each i, we have

(9) $\int g_{j_m}(x)d\mu(x) = 0.$

If we set $a_{j_m}(x) = g_{j_m}(x)(D_0\Theta\alpha^{m+1}\mu(\bar{S}_{j_m}))^{-1}$, then from (7), (8) and (9) we see that $a_{j_m}(x)$ is a $(1,\infty)$-atom supported in the section $\bar{S}_{j_m}$. This shows that (I) is valid for $m+1$. By the definition of $h^i_{j_m}(x)$, the hypothesis on $h_{j_m}(x)$ for m, and (3.18), we have

$$|h^i_{j_m}(x)| \le \left\{|h_{j_m}(x)| + \frac{1}{\mu(S^i_{j_m})}\int_{S^i_{j_m}} |h_{j_m}(x)|d\mu(x)\right\}\chi^i_{j_m}(x)$$
$$\le \left\{|b(x)| + 2D_0\alpha^m + \left(\frac{1}{\mu(S^i_{j_m})}\int_{S^i_{j_m}} |h_{j_m}(x)|^q d\mu(x)\right)^{1/q}\right\}\chi^i_{j_m}(x)$$
$$\le \{|b(x)| + 2D_0\alpha^m + D_0\alpha^{m+1}\}\chi^i_{j_m}(x)$$
$$\le |b(x)| + 2D_0\alpha^{m+1}\chi^i_{j_m}(x)$$

provided $\alpha > 2$, which means that (VI) holds for $m+1$. By (3.18), we see that (VII) is also valid for $m+1$, since by the definition of $h^i_{j_m}$ we know that $(m_{S^i_{j_m}}(|h^i_{j_m}|^q))^{1/q} \le 2(m_{S^i_{j_m}}(|h_{j_m}\eta^i_{j_m}|^q))^{1/q}$.

Finally, by (VI) we see that

$$(M_q h_{j_m})(x) \le (M_q b)(x) + 2D_0\alpha^m \quad \text{for all } x \in \boldsymbol{R}^n\,.$$

Thus, for any $x \in E_{j_m}$, we have

$$\alpha^{m+1} < (M_q h_{j_m})(x) \le (M_q b)(x) + 2D_0\alpha^m < (M_q b)(x) + \alpha^{m+1}/2 \tag{3.19}$$

as long as $\alpha > 4D_0$. Then, by (4) and (3.19), we obtain

$$\bigcup_{\substack{j_m\in\boldsymbol{N}^m\\ i\in\boldsymbol{N}}} S^i_{j_m} = \bigcup_{j_m\in\boldsymbol{N}^m}\left(\bigcup_{i\in\boldsymbol{N}} S^i_{j_m}\right) \subset \bigcup_{j_m\in\boldsymbol{N}^m} E_{j_m} \subset \{x\in\boldsymbol{R}^n\,;\,(M_q b)(x) > \alpha^{m+1}/2\}\,.$$

So, (II) holds for $m+1$.

In consequence, to complete the proof of Proposition 3.1 we only need to take α to be

$$\alpha > \max\{\Theta^{1/(q-1)},\ C_2^{1/q},\ 2D_0C_2^{1/q},\ 2,\ 4D_0\}\,,$$

since each of these numbers depends only on q, A and θ and is independent of m.

4. Proof of theorem 1.2. We need to give an equivalent definition of $BMO_{\mathcal{F}}(\boldsymbol{R}^n)$ with respect to the family $\mathcal{F}$ and the doubling Borel measure μ. Let f be a real-valued function defined on $\boldsymbol{R}^n$. We say that $f \in BMO^q_{\mathcal{F}}(\boldsymbol{R}^n)$, $1 < q < \infty$, if

$$\|f\|_{q,*} = \sup_{S\in\mathcal{F}} \left(\frac{1}{\mu(S)} \int_S |f(x) - m_S(f)|^q d\mu(x) \right)^{1/q} < \infty .$$

PROPOSITION 4.1. *For any* $1 < q < \infty$, $BMO^q_{\mathcal{F}}(\boldsymbol{R}^n) = BMO_{\mathcal{F}}(\boldsymbol{R}^n)$.

PROOF. By Hölder's inequality, it is easy to get $BMO^q_{\mathcal{F}}(\boldsymbol{R}^n) \subset BMO_{\mathcal{F}}(\boldsymbol{R}^n)$. On the other hand, we assume that $f \in BMO_{\mathcal{F}}(\boldsymbol{R}^n)$ with $\|f\|_* = 1$. Then there exist positive numbers $\varepsilon_0 < 1$ and Γ depending only on A in (1.1) and the constants in the properties (A) and (B) of sections, such that, for any section $S \in \mathcal{F}$ and each $k = 0, 1, 2, \dots$,

$$\mu(\{x \in S\,;\, |f(x) - m_S(f)| > \Gamma + k\Gamma\}) \le \varepsilon_0^k \mu(\{x \in S\,;\, |f(x) - m_S(f)| > \Gamma\}) . \tag{4.1}$$

(See (6-6) in [CG1, p. 1091] for the proof.) Thus

$$\begin{aligned}
\frac{1}{\mu(S)} \int_S |f(x) - m_S(f)|^q d\mu(x) &= \frac{q}{\mu(S)} \int_0^\infty \alpha^{q-1} \mu(\{x \in S\,;\, |f(x) - m_S(f)| > \alpha\}) d\alpha \\
&= \frac{q}{\mu(S)} \int_0^\Gamma \alpha^{q-1} \mu(\{x \in S\,;\, |f(x) - m_S(f)| > \alpha\}) d\alpha \\
&\quad + \frac{q}{\mu(S)} \int_\Gamma^\infty \alpha^{q-1} \mu(\{x \in S\,;\, |f(x) - m_S(f)| > \alpha\}) d\alpha \\
&:= I_1 + I_2 .
\end{aligned}$$

Here we have

$$I_1 \le \frac{q}{\mu(S)} \int_0^\Gamma \alpha^{q-1} \cdot \mu(S) d\alpha \le \Gamma^q < \infty . \tag{4.2}$$

On the other hand, by (4.1) and noting that $\varepsilon_0 < 1$, we get

$$\begin{aligned}
I_2 &= \frac{q}{\mu(S)} \int_0^\infty (\alpha + \Gamma)^{q-1} \mu(\{x \in S\,;\, |f(x) - m_S(f)| > \alpha + \Gamma\}) d\alpha \\
&= \frac{q}{\mu(S)} \sum_{k=0}^\infty \int_{k\Gamma}^{(k+1)\Gamma} (\alpha + \Gamma)^{q-1} \mu(\{x \in S\,;\, |f(x) - m_S(f)| > \alpha + \Gamma\}) d\alpha \\
&\le \frac{q}{\mu(S)} \sum_{k=0}^\infty [(k+1)\Gamma + \Gamma]^{q-1} \mu(\{x \in S\,;\, |f(x) - m_S(f)| > k\Gamma + \Gamma\}) \cdot \Gamma \qquad (4.3) \\
&\le \frac{q}{\mu(S)} \sum_{k=0}^\infty (k+2)^{q-1} \Gamma^q \varepsilon_0^k \mu(\{x \in S\,;\, |f(x) - m_S(f)| > \Gamma\}) \\
&\le q\Gamma^q \sum_{k=0}^\infty (k+2)^{q-1} \varepsilon_0^k \le Cq\Gamma^q .
\end{aligned}$$

From (4.2) and (4.3), we conclude that $BMO^q_{\mathcal{F}}(\boldsymbol{R}^n) \supset BMO_{\mathcal{F}}(\boldsymbol{R}^n)$.

PROOF OF THEOREM 1.2. To prove Theorem 1.2, we need to show that if $g \in BMO_{\mathcal{F}}(\boldsymbol{R}^n)$, then

$$l_g(f) = \int_{\boldsymbol{R}^n} f(x)g(x)d\mu(x) \tag{4.4}$$

is a bounded linear functional on $H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n)$, and conversely that for any bounded linear functional l on $H^1_{\mathcal{F}}(\boldsymbol{R}^n)$, there exists $b \in BMO_{\mathcal{F}}(\boldsymbol{R}^n)$ such that

$$l(f) = \int_{\boldsymbol{R}^n} f(x)b(x)d\mu(x) \quad \text{for all } \ f \in H^1_{\mathcal{F}}(\boldsymbol{R}^n)\,.$$

By the conclusions of Theorem 1.1 and Proposition 4.1, it suffices to show that the dual space of the atomic Hardy space $H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n)$ is $BMO^{q'}_{\mathcal{F}}(\boldsymbol{R}^n)$ for some q with $1 < q < \infty$, that is, $(H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n))' = BMO^{q'}_{\mathcal{F}}(\boldsymbol{R}^n)$, where $1/q + 1/q' = 1$.

We first prove that $BMO^{q'}_{\mathcal{F}}(\boldsymbol{R}^n) \subset (H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n))'$. Write $D = H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n) \cap L^q_c(\boldsymbol{R}^n, d\mu)$, where $L^q_c(\boldsymbol{R}^n, d\mu)$ consists of all functions in $L^q(\boldsymbol{R}^n, d\mu)$ with compact supports. Since the set of all functions with the form $\sum_{k=1}^N \lambda_k a_k(x)$ is dense in $H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n)$, D is a dense subset of $H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n)$. Then we will see that, for any $g \in BMO^{q'}_{\mathcal{F}}(\boldsymbol{R}^n)$, the linear functional l_g defined in (4.4) is bounded on the dense subset D of $H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n)$.

For $N \in \boldsymbol{N}$, we set

$$g_N(x) = \begin{cases} N & \text{if } \ g(x) \ge N\,, \\ g(x) & \text{if } \ |g(x)| < N\,, \\ -N & \text{if } \ g(x) \le -N\,. \end{cases}$$

Then it is easy to verify that $g_N(x) \in BMO^{q'}_{\mathcal{F}}(\boldsymbol{R}^n)$ and $\|g_N\|_{q',*} \le 4\|g\|_{q',*}$.

Set $f(x) = \sum_{k=1}^\infty \lambda_k a_k(x) \in D$, where $a_k(x)$ is a $(1,q)$-atom supported in a section $S_k \in \mathcal{F}$. Thus, by the definition of the $(1,q)$-atom, we have

$$\begin{aligned} \left|\int_{\boldsymbol{R}^n} f(x)g_N(x)d\mu(x)\right| &\le \sum_{k=1}^\infty |\lambda_k| \left|\int_{\boldsymbol{R}^n} a_k(x)g_N(x)d\mu(x)\right| \\ &\le \sum_{k=1}^\infty |\lambda_k| \left|\int_{S_k} a_k(x)[g_N(x) - m_{S_k}(g_N)]d\mu(x)\right| \\ &\le \sum_{k=1}^\infty |\lambda_k|\, \|a_k\|_{L^q_\mu} \left(\int_{S_k} |g_N(x) - m_{S_k}(g_N)|^{q'} d\mu(x)\right)^{1/q'} \\ &\le \sum_{k=1}^\infty |\lambda_k| \left(\frac{1}{\mu(S_k)} \int_{S_k} |g_N(x) - m_{S_k}(g_N)|^{q'} d\mu(x)\right)^{1/q'} \\ &\le \|f\|_{H^{1,q}_{\mathcal{F}}} \cdot 4\|g\|_{q',*}\,. \end{aligned} \tag{4.5}$$

Since $g(x) \in BMO^{q'}_{\mathcal{F}}(\boldsymbol{R}^n)$ is a locally q'-th integrable function on $\boldsymbol{R}^n$,

$$|f(x)g_N(x)| \le |f(x)g(x)| \in L^1(\boldsymbol{R}^n, d\mu)\,.$$

By the Lebesgue dominated convergence theorem and (4.5),

$$\left|\int_{\boldsymbol{R}^n} f(x)g(x)d\mu(x)\right| = \left|\lim_{N\to\infty}\int_{\boldsymbol{R}^n} f(x)g_N(x)d\mu(x)\right| \le \|f\|_{H^{1,q}_{\mathcal{F}}}\cdot 4\|g\|_{q',*}\,.$$

This shows that the linear functional l_g is bounded on D, and $\|l_g\| \le 4\|g\|_{q',*}$. Consequently, l_g has a unique bounded extension on $H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n)$, since D is a dense subset of $H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n)$. In this sense we then have $BMO^{q'}_{\mathcal{F}}(\boldsymbol{R}^n) \subset (H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n))'$.

In order to prove the inverse inclusion $(H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n))' \subset BMO^{q'}_{\mathcal{F}}(\boldsymbol{R}^n)$, we need to show that if l is a bounded linear functional on $H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n)$, then there exists $b(x) \in BMO^{q'}_{\mathcal{F}}(\boldsymbol{R}^n)$ such that for any $f \in H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n)$

$$l(f) = \int_{\boldsymbol{R}^n} f(x)b(x)d\mu(x)\,.$$

The proof will be divided into the following three steps.

Step 1. Let us first prove $(H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n))' \subset (L^q_0(S,d\mu))'$, where $S = S(x,t) \in \mathcal{F}$ is any section in $\boldsymbol{R}^n$ and

$$L^q_0(S,d\mu) = \left\{ f \in L^q(\boldsymbol{R}^n, d\mu)\,;\ f = 0\ \mu\text{-}a.e.\text{ on } S^c \text{ and } \int_S f(x)d\mu(x) = 0 \right\}.$$

Indeed, when $f(x) \in L^q_0(S,d\mu)$, it is easy to check that $a(x) = f(x)(\mu(S))^{-1/q'}\|f\|^{-1}_{L^q_\mu(S)}$ is a $(1,q)$-atom. Thus $f(x) = a(x)(\mu(S))^{1/q'}\|f\|_{L^q_\mu(S)} \in H^{1,q}_{\mathcal{F}}(\boldsymbol{R}^n)$ and $\|f\|_{H^{1,q}_{\mathcal{F}}} \le (\mu(S))^{1/q'}\|f\|_{L^q_\mu(S)}$. Therefore, we have

$$|l(f)| \le \|l\|\cdot(\mu(S))^{1/q'}\|f\|_{L^q_\mu(S)}\,, \tag{4.6}$$

which shows that l is also a bounded linear functional on $L^q_0(S,d\mu)$. Since $L^q_0(S,d\mu) \subset L^q(S,d\mu)$, using the Hahn-Banach extension theorem, we know that l has a unique bounded extension on $L^q(S,d\mu)$. Since $1 < q < \infty$, by the Riesz representation theorem, there exists $b(x) \in L^{q'}(S,d\mu)$ such that

$$l(f) = \int_S f(x)b(x)d\mu(x) \quad \text{for all } f \in L^q_0(S,d\mu)\,. \tag{4.7}$$

Furthermore, we have the following fact:

If $\int_S f(x)b(x)d\mu(x) = 0$ for all $f \in L^q_0(S,d\mu)$, then $b(x)$ is constant for almost every $x \in S$.

Indeed, since S is a bounded convex set, for any $h(x) \in L^q(S,d\mu)$ we have $h(x) - m_S(h) \in L^q_0(S,d\mu)$. Thus

$$0 = \int_S b(x)[h(x) - m_S(h)]d\mu(x) = \int_S h(x)[b(x) - m_S(b)]d\mu(x) \quad \text{for all } h \in L^q(S,d\mu)\,.$$

Hence $b(x) = m_S(b)$ almost every $x \in S$.

Step 2. Fix $x_0 \in \boldsymbol{R}^n$ and choose a sequence of positive increasing numbers $\{t_j\}_{j=1}^\infty$ such that $\lim_{j\to\infty} t_j = \infty$. Then, by the property (C) of sections, $\{S(x_0,t_j)\}_{j=1}^\infty$ is a sequence of sections with $\bigcup_{j=1}^{\infty} S_j = \boldsymbol{R}^n$, where $S_j = S(x_0, t_j)$. By (4.7), for each S_j, there exists $b_j(x) \in L^{q'}(S_j, d\mu)$ satisfying (4.7).

Consider an arbitrary $f \in L_0^q(S_1, d\mu)$. There exists $b_1(x) \in L^{q'}(S_1, d\mu)$ such that

$$l(f) = \int_{S_1} f(x) b_1(x) d\mu(x) . \tag{4.8}$$

By $S_2 \supset S_1$, we have $L_0^q(S_2, d\mu) \supset L_0^q(S_1, d\mu)$ and $f \in L_0^q(S_2, d\mu)$. Therefore, there exists $b_2(x) \in L^{q'}(S_2, d\mu)$ such that

$$l(f) = \int_{S_2} f(x) b_2(x) d\mu(x) = \int_{S_1} f(x) b_2(x) d\mu(x) , \tag{4.9}$$

since $\operatorname{supp}(f) \subset S_1$. From (4.8) and (4.9), we get

$$\int_{S_1} f(x)[b_1(x) - b_2(x)] d\mu(x) = 0 \quad \text{for all } \ f \in L_0^q(S_1, d\mu) . \tag{4.10}$$

Applying the fact shown in Step 1, we have $b_1(x) - b_2(x) = C_1$ for almost every $x \in S_1$. Now we write

$$b(x) = \begin{cases} b_1(x) & \text{if } \ x \in S_1 , \\ b_2(x) + C_1 & \text{if } \ x \in S_2 \backslash S_1 . \end{cases}$$

Then we obtain

$$l(f) = \int_{S_j} f(x) b(x) d\mu(x) \quad \text{for any } \ f \in L_0^q(S_j, d\mu), \ \ j = 1, 2 .$$

By a method quite similar to the above, we may obtain a function $b(x)$ satisfying

$$l(f) = \int_{S_j} f(x) b(x) d\mu(x) \quad \text{for any } \ f \in L_0^q(S_j, d\mu), \ \ j = 1, 2, \ldots . \tag{4.11}$$

Step 3. Now we prove that the above $b(x) \in BMO_{\mathcal{F}}^{q'}(\boldsymbol{R}^n)$ and satisfies

$$l(f) = \int_{\boldsymbol{R}^n} f(x) b(x) d\mu(x) \quad \text{for any } \ f \in H_{\mathcal{F}}^{1,q}(\boldsymbol{R}^n) . \tag{4.12}$$

We need the following fact about sections in $\boldsymbol{R}^n$.

Assume that $S_0 = S(y_0, r) \in \mathcal{F}$ *is an arbitrary section in* $\boldsymbol{R}^n$. *Then there exists* j_0 *such that* $S_{j_0} \supset S_0$, *where* $S_{j_0} = S(x_0, t_{j_0})$ *is the* j_0*-th section of the sequence in Step* 2.

Indeed, by $\bigcup_{j=1}^{\infty} S_j = \boldsymbol{R}^n$, there exists a section $S_i = S(x_0, t_i)$ such that $S(x_0, t_i) \cap S(y_0, r) \neq \emptyset$ with $t_i \geq r$. Then there exists $z \in S(x_0, t_i) \cap S(y_0, r)$. From the property (D) of sections, we have $S(y_0, r) \subset S(z, \theta r) \subset S(z, \theta t_i)$. Since $z \in S(x_0, t_i) \subset S(x_0, \theta t_i)$, using the property (D) again, we know $S(z, \theta t_i) \subset S(x_0, \theta^2 t_i)$ and therefore $S(y_0, r) \subset S(x_0, \theta^2 t_i)$. Now if we take j_0 such that $t_{j_0} \geq \theta^2 t_i$, then $S(y_0, r) \subset S(x_0, t_{j_0})$.

Now, let us return to the proof of (4.12). For any $f \in H_{\mathcal{F}}^{1,q}(\boldsymbol{R}^n)$, we may write $f(x) = \sum_{k=1}^{\infty} \lambda_k a_k(x)$, where $a_k(x)$ is a $(1, q)$-atom supported in the section $S_k \in \mathcal{F}$. By the fact above, for each k there exists j_k such that $S_k \subset S_{j_k} = S(x_0, t_{j_k})$. By the definition of $(1, q)$-atom, we have $a_k(x) \in L_0^q(S_{j_k}, d\mu)$. Thus by (4.11),

$$l(a_k) = \int_{S_{j_k}} a_k(x)b(x)d\mu(x) = \int_{\boldsymbol{R}^n} a_k(x)b(x)d\mu(x)\,. \tag{4.13}$$

Since the functional l is linear, by (4.13) we obtain

$$l(f) = \sum_{k=1}^{\infty} \lambda_k l(a_k) = \sum_{k=1}^{\infty} \lambda_k \int_{\boldsymbol{R}^n} a_k(x)b(x)d\mu(x) = \int_{\boldsymbol{R}^n} f(x)b(x)d\mu(x)\,.$$

Finally, to finish the proof of Step 3, it remains to show that $b(x) \in BMO_{\mathcal{F}}^{q'}(\boldsymbol{R}^n)$. For any section $S \in \mathcal{F}$, let $h(x) \in L^q(S, d\mu)$ with $\mathrm{supp}(h) \subset S$ and $\|h\|_{L^q_\mu} \le 1$. Then $a(x) = (1/2)(\mu(S))^{-1/q'}[h(x) - m_S(h)]\chi_S(x)$ is a $(1, q)$-atom supported in S and $\|a\|_{L^q_\mu} \le 1$. Thus, (4.13) implies that

$$\left| \int_S a(x)b(x)d\mu(x) \right| = |l(a)| \le \|l\|\,.$$

Hence

$$(\mu(S))^{-1/q'} \left| \int_S [h(x) - m_S(h)]b(x)d\mu(x) \right| \le 2\|l\|\,.$$

That is,

$$(\mu(S))^{-1/q'} \left| \int_S h(x)[b(x) - m_S(b)]d\mu(x) \right| \le 2\|l\|\,. \tag{4.14}$$

From (4.14), we have

$$(\mu(S))^{-1/q'} \|b - m_S(b)\|_{L^{q'}_\mu} = (\mu(S))^{-1/q'} \sup_{\|h\|_{L^q_\mu} \le 1} \left| \int_S h(x)[b(x) - m_S(b)]d\mu(x) \right| \le 2\|l\|\,.$$

Since the section $S \in \mathcal{F}$ is arbitrary, we may conclude that $b(x) \in BMO_{\mathcal{F}}^{q'}(\boldsymbol{R}^n)$. This completes the proof of Theorem 1.2.

5. Proof of theorem 1.3. Applying Theorem 1.1, we only have to show that there exists a constant C such that

$$\|H(a)\|_{L^1_\mu} \le C \quad \text{for all } (1, 2)\text{-atom } a\,. \tag{5.1}$$

By Definition 1.1, there exists a section $S_0 = S(y_0, t_0) \in \mathcal{F}$ such that $\mathrm{supp}(a) \subset S_0$. Denote $S_0^* = S(y_0, 4\theta^2 t_0)$, where θ is the constant appearing in the property (D) of sections. By the doubling property (1.1) of μ, we have

$$\mu(S_0^*) \le A^{3+2\log_2\theta} \mu(S_0)\,. \tag{5.2}$$

Thus

$$\begin{aligned} \int_{\boldsymbol{R}^n} |H(a)(x)|d\mu(x) &= \int_{S_0^*} |H(a)(x)|d\mu(x) + \int_{(S_0^*)^c} |H(a)(x)|d\mu(x) \\ &:= I_1 + I_2\,. \end{aligned} \tag{5.3}$$

By the (L^2, L^2)-boundedness of the operator H (see [CG3]) and (5.2), we get

$$
\begin{aligned}
I_1 &\le [\mu(S_0^*)]^{1/2}\left(\int_{S_0^*}|H(a)(x)|^2 d\mu(x)\right)^{1/2} \\
&\le (A^{3+2\log_2\theta})^{1/2}[\mu(S_0)]^{1/2}\|a\|_{L^2_\mu} \le (A^{3+2\log_2\theta})^{1/2}\,.
\end{aligned}
\tag{5.4}
$$

On the other hand, by the cancellation condition of the atom a, we have

$$
\begin{aligned}
I_2 &= \int_{(S_0^*)^c}\left|\int_{\boldsymbol{R}^n} K(x,y)a(y)d\mu(y)\right| d\mu(x) \\
&= \int_{(S_0^*)^c}\left|\sum_i \int_{\boldsymbol{R}^n} k_i(x,y)a(y)d\mu(y)\right| d\mu(x) \\
&= \int_{(S_0^*)^c}\left|\sum_i \int_{\boldsymbol{R}^n} [k_i(x,y)-k_i(x,y_0)]a(y)d\mu(y)\right| d\mu(x) \\
&\le \sum_i \int_{\boldsymbol{R}^n}|a(y)|\int_{(S_0^*)^c}|k_i(x,y)-k_i(x,y_0)|d\mu(x)d\mu(y) \\
&= \int_{S_0}|a(y)|\sum_i\int_{(S_0^*)^c}|k_i(x,y)-k_i(x,y_0)|d\mu(x)d\mu(y)\,.
\end{aligned}
$$

By the size condition of the atom a, it suffices to prove that there exists a constant C independent of the atom a such that

$$
\sum_i \int_{(S_0^*)^c}|k_i(x,y)-k_i(x,y_0)|d\mu(x) \le C\,. \tag{5.5}
$$

Indeed, if (5.5) holds, then

$$
I_2 \le C\int_{S_0}|a(y)|d\mu(y) \le C\,,
$$

which combined with (5.4) implies (5.1).

Therefore, in order to prove Theorem 1.3, it remains only to prove (5.5). By the property (G) of sections, we have

$$
\rho(y_0, y) < t_0 \quad \text{and} \quad \rho(y_0, x) \ge 4\theta^2 t_0 \tag{5.6}
$$

if $y \in S_0$ and $x \in (S_0^*)^c$. So, by (5.6), we see that when $y \in S_0$ and $x \in (S_0^*)^c$,

$$
\rho(y_0, x) > 4\theta^2\rho(y_0, y)\,.
$$

Using the conclusion of Lemma 1 in [In], we get (5.5).

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