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ON A CONJECTURE OF SHOKUROV: CHARACTERIZATION OF TORIC VARIETIES

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Abstract. We verify a special case of V. V. Shokurov's conjecture about characterization of toric varieties. More precisely, we consider three-dimensional log varieties with only purely log terminal singularities and numerically trivial log canonical divisor. In this situation we prove an inequality connecting the rank of the group of Weil divisors modulo algebraic equivalence and the sum of coefficients of the boundary. We describe such varieties for which the equality holds and show that all of them are toric.

1. Introduction. The aim of this note is to discuss the birational characterization of toric varieties. Let X be a normal projective toric variety and let $D=\sum_{i=1}^{r} D_i$ be the sum of invariant divisors. It is well-known that the pair (X, D) has only log canonical singularities (see, e.g., [3, 3.7]), K_X+D is linearly trivial and $r=\operatorname{rank}(\mathrm{Weil}(X)/\approx)+\dim(X)$, where $\mathrm{Weil}(X)$ is the group of Weil divisors and $\approx$ is the algebraic equivalence.

Shokurov observed that this property can characterize toric varieties:

CONJECTURE 1.1 ([12]). *Let $(X, D=\sum d_i D_i)$ be a projective log variety such that (X, D) has only log canonical singularities and numerically trivial. Then*

$$\sum d_i \le \operatorname{rank}(\mathrm{Weil}(X)/\approx)+\dim(X)\,.$$

Moreover, if the equality holds, then $(X, \lfloor D\rfloor)$ is a toric pair.

Shokurov also conjectured the relative version of Conjecture 1.1 (cf. Theorem 2.3) and expects that one can replace the numerical triviality of K_X+D with the nefness of $-(K_X+D)$. We do not discuss these points in detail here.

Conjecture 1.1 was proved in dimension two in [12] (see also [9, Sect. 8] and Proposition 2.1 below). Our main result is the following partial answer to Conjecture 1.1 in dimension three:

THEOREM 1.2. *Let $(X, D=\sum d_i D_i)$ be a three-dimensional projective variety over $\boldsymbol{C}$ such that $K_X+D\equiv 0$ and (X, D) has only purely log terminal singularities. Then*

$$\sum d_i \le \operatorname{rank}(\mathrm{Weil}(X)/\approx)+3\,. \tag{1.3}$$

Moreover, if the equality holds, then up to isomorphisms one of the following holds:

(i) $X\simeq \boldsymbol{P}^3$, $\lfloor D\rfloor=0$ or $\lfloor D\rfloor=\boldsymbol{P}^2$;

(ii) $X\simeq \boldsymbol{P}^1\times\boldsymbol{P}^2$, $\lfloor D\rfloor=0$ *or* $\lfloor D\rfloor=\{\mathrm{pt}\}\times\boldsymbol{P}^2$ *or* $\lfloor D\rfloor=\boldsymbol{P}^1\times\{\mathrm{line}\}$;

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(iii) $X \simeq \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(d))$, $d \geq 1$, $\lfloor D \rfloor$ *is the section corresponding to the surjection* $\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(d) \to \mathcal{O}_{\boldsymbol{P}^1}(d)$;

(iv) $X \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$, $\lfloor D \rfloor = 0$ *or* $\lfloor D \rfloor = \{\mathrm{pt}\} \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$ *or* $\lfloor D \rfloor = \{\mathrm{pt}_1, \mathrm{pt}_2\} \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$;

(v) $X \simeq \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(d))$, $d \geq 1$, $\lfloor D \rfloor$ *is the negative section*, *or a disjoint union of two sections*, *one of them is negative*;

(vi) $X \simeq \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1} \oplus \mathcal{L})$, $\mathcal{L} \in \mathrm{Pic}(\boldsymbol{P}^1 \times \boldsymbol{P}^1)$, $\lfloor D \rfloor$ *is the negative section*, *or a disjoint union of two sections*, *one of them is negative.*

In all cases $(X, \lfloor D \rfloor)$ *is toric.*

Clearly, our theorem is not a characterization of toric varieties, but we hope that Conjecture 1.1 can be proved in a similar way.

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2. Preliminaries.

NOTATION. All varieties are defined over $\boldsymbol{C}$. Basically we employ the standard notation of the Minimal Model Program (MMP, for short). Throughout this paper $\rho(X)$ is the Picard number and $\overline{NE}(X)$ is the Mori cone of X. We call a pair (X, D) consisting of a normal algebraic variety X and a boundary D on X a *log variety* or a *log pair*. Here a *boundary* is a $\boldsymbol{Q}$-Weil divisor $D = \sum d_i D_i$ such that $0 \leq d_i \leq 1$ for all i. A *contraction* is a projective morphism $\varphi\colon X \to Z$ of normal varieties such that $\varphi_* \mathcal{O}_X = \mathcal{O}_Z$. Abbreviations klt, plt, lc are reserved for Kawamata log terminal, purely log terminal and log canonical, respectively (refer to [11], [4] and [3] for the definitions). Let (X, D) be a log pair and let $S := \lfloor D \rfloor$. For simplicity, assume that (X, D) is lc in codimension two. The Adjunction Formula proposed by Shokurov [11, Sect. 3] states that $(K_X + D)|_S = K_S + \mathrm{Diff}_S(D - S)$, where $\mathrm{Diff}_S(D - S)$ is a naturally defined effective $\boldsymbol{Q}$-Weil divisor on S, a so-called *different*. Moreover, $K_X + D$ is plt near S if and only if S is normal and $K_S + \mathrm{Diff}_S(D - S)$ is klt [4, 17.6]. $LCS(X, D)$ denotes the *locus of log canonical singularities* of (X, D) that is the set of all points where (X, D) is not klt [11]. Let $\varphi\colon X \to Z$ be any fiber type contraction and let $D = \sum d_i D_i$ be a $\boldsymbol{Q}$-divisor on X. We will write $D = \sum_{\mathrm{ver}} d_i D_i + \sum_{\mathrm{hor}} d_i D_i = D^{\mathrm{ver}} + D^{\mathrm{hor}}$, where $\sum_{\mathrm{ver}}$ (resp. $\sum_{\mathrm{hor}}$) runs through all components D_i such that $\dim \varphi(D_i) < \dim(Z)$ (resp. $\varphi(D_i) = Z$). We will frequently use the above notation without reference.

In dimension two Conjecture 1.1 is much easier than higher dimensional one. We need only the following weaker version:

PROPOSITION 2.1. *Let* $(X, D = \sum d_i D_i)$ *be a projective log surface such that* $-(K_X + D)$ *is nef and* (X, D) *is lc. Then* $\sum d_i \leq \rho(X) + 2$. *Moreover, if the equality holds and* (X, D) *is klt*, *then* $X \simeq \boldsymbol{P}^2$, *or* $X \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$.

For the general statement we refer to [12], see also [9].

PROOF. Assume that $\sum d_i - \rho(X) - 2 \geq 0$ and run K_X-MMP. According to [1], Log MMP works even in the category of log canonical pairs. On each step, $\sum d_i - \rho(X) - 2$ does not decrease and all assumptions are preserved (see [4, 2.28]). At the end we get one of the following:

Case 1. $\rho(X) = 1$. Then $\sum d_i - \rho(X) - 2 \leq 0$ by [4, 18.24], [1, 5.1].

Case 2. There is an extremal contraction onto a curve $\varphi\colon X \to Z$ (in particular, $\rho(X) = 2$). Let ℓ be a general fiber. Then

$$2 = -K_X \cdot \ell \geq D \cdot \ell = \sum_{\text{hor}} d_i D_i \cdot \ell \geq \sum_{\text{hor}} d_i \,. \tag{2.2}$$

Hence $\sum_{\text{ver}} d_i \geq 2$ and $K_X + D^{\text{hor}}$ is not nef. Let $\phi\colon X \to W$ is a contraction of $(K + D^{\text{hor}})$-negative extremal ray. If ϕ is birational, we replace X with W and obtain Case 1 above. Thus we may assume that W is a curve, so φ and ϕ are symmetric. As above, $\sum_{\text{ver}} d_i \leq 2$, so $\sum d_i = 4$.

Now assume that (X, D) is klt and $\sum d_i - \rho(X) - 2 = 0$. Then after each divisorial contraction $\sum d_i - \rho(X) - 2$ increases. Hence we are only in cases 1 or 2 above. In Case 1, $X \simeq \boldsymbol{P}^2$ by Lemma 3.1 below. In Case 2 the contraction ϕ cannot be divisorial. Hence W is a curve. We have the equality in (2.2), so $\sum_{\text{hor}} d_i = \sum_{\text{ver}} d_i = 2$ and $D_i \cdot \ell = 1$ for any component of D^{hor}. Considering the inequality similar to (2.2) for the contraction ϕ, one can obtain that D^{hor} is vertical with respect to ϕ. In particular, the components of D^{hor} are disjoint sections of φ. Now let ℓ_0 be any fiber of φ. It is known (see, e.g., [9, 7.2]) that φ has at most two singular points on ℓ_0. Since $\sum_{\text{hor}} d_i = 2$, there is a component of D^{hor} intersecting ℓ_0 at a (single) smooth point. Therefore ℓ_0 is not a multiple fiber of φ and X is smooth along ℓ_0. We proved that X is smooth. Taking into account that $\rho(X) = 2$ and that both extremal rays on X are nef, we obtain $X \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$. □

The local version of Conjecture 1.1 was proved in [4, 18.22]:

THEOREM 2.3. *Let $(X, D = \sum d_i D_i)$ be a log pair which is log canonical at a point $P \in \cap D_i$. Assume that K_X and all D_i's are $\boldsymbol{Q}$-Cartier at P. Then $\sum d_i \leq \dim(X)$. Moreover, if the equality holds, then $(X \ni P, D)$ is an abelian quotient of a smooth point and (X, D) is not plt at P.*

Recall that for any plt pair (X, D) of dimension ≤ 3 there is a small birational contraction $q\colon X^q \to X$ such that X^q is $\boldsymbol{Q}$-factorial and $(X^q, D^q := q_*^{-1} D)$ is plt (see [4, 6.11.1], [4, 17.10]). Such q is called a ***Q**-factorialization* of (X, D). Applying a $\boldsymbol{Q}$-factorialization in our situation and taking into account that

$$\text{rank}(\text{Weil}(X)/\!\approx) = (\text{rank}\,\text{Weil}(X^q)/\!\approx) \geq \rho(X^q)\,,$$

we obtain that for Theorem 1.2 it is sufficient to prove the following

PROPOSITION 2.4. *Let $(X, D = \sum d_i D_i)$ be a three-dimensional projective plt pair such that $K_X + D \equiv 0$ and X is $\boldsymbol{Q}$-factorial. Then*

$$\sum d_i \leq \rho(X) + 3\,. \tag{2.5}$$

Moreover, if the equality holds, then for $(X, \lfloor D \rfloor)$ *there are only possibilities* (i)–(iv) *of Theorem* 1.2.

3. Lemmas. In this section we prove several facts related to Conjecture 1.1.

LEMMA 3.1 (cf. [4, 18.24], [1]). *Let* $(X, D = \sum_{i=1}^{r} d_i D_i)$ *be a projective* n*-dimensional log pair such that all* D_i*'s are* $\boldsymbol{Q}$*-Cartier,* $\rho(X) = 1$, (X, D) *is plt and* $-(K_X + D)$ *is nef. Then*

$$\sum d_i \le n + 1\,. \tag{3.2}$$

Moreover, if the equality holds, then $X \simeq \boldsymbol{P}^n$ *and* $D_1, \dots, D_r$ *are hyperplanes.*

Note that in the two-dimensional case any plt pair is automatically $\boldsymbol{Q}$-factorial.

PROOF. We will prove this lemma in the case when $\lfloor D \rfloor = 0$ (i.e., $K_X + D$ is klt). The case when $\lfloor D \rfloor$ is non-trivial (and irreducible) can be treated in a similar way. The inequality (3.2) was proved in [4, 18.24], so we prove the second part of our lemma.

Since $-K_X$ is ample, $\mathrm{Pic}(X) \simeq \boldsymbol{Z}$ (see, e.g., [8, 2.1.2]). Let H be an ample generator of $\mathrm{Pic}(X)$ and let $D_i \equiv a_i H$, $a_i \in \boldsymbol{Q}$, $a_i \ge 0$. Assume that $a_i < a_j$ for $i \ne j$ or $K_X + D \not\equiv 0$. For $0 < \varepsilon \ll 1$, consider

$$D^{(\varepsilon)} := \varepsilon D_i + D - \varepsilon D_j\,.$$

Then $K_X + D^{(\varepsilon)}$ is again klt (because the klt property is an open condition) and $-(K_X + D^{(\varepsilon)})$ is ample. Take $N \in \boldsymbol{N}$ so that $-N(K_X + D^{(\varepsilon)})$ is integral and very ample, and let $M \in |-N(K_X + D^{(\varepsilon)})|$ be a general member. By a Bertini type theorem [3, Sect. 4], $(X, D^{(\varepsilon)} + (1/N)M)$ is klt (and numerically trivial). Moreover, the sum of coefficients of $D^{(\varepsilon)} + (1/N)M$ is equal to $n + 1 + 1/N$. This contradicts (3.2). Hence $K_X + D \equiv 0$ and $D_i \equiv D_j$ for all i, j. Thus, for any pair D_i and D_j there exists $n_{i,j} \in \boldsymbol{N}$ such that $n_{i,j}(D_i - D_j) \sim 0$.

By taking repeated cyclic covers (which are étale in codimension one) $\pi : X' \to \cdots \to X$, we obtain a new plt pair $(X', D' = \sum_{i=1}^{r} d_i D'_i)$ [4, 20.4] such that $D'_i \sim D'_j$, where $D'_i = \pi^* D_i$. On this step, we do not assume that D'_i is irreducible. Then $D'_1, \dots, D'_r$ generate a linear system $\mathcal{M}$ of Weil divisors. If $\mathrm{Bs}(\mathcal{M})$ is not empty, then we pick a point $P' \in D'_1 \cap \cdots \cap D'_r$. By construction, (X', D') is klt at P' and $\sum_{i=1}^{r} d_i \ge n + 1$, a contradiction with Theorem 2.3. Therefore $\mathrm{Bs}(\mathcal{M}) = \varnothing$. In particular, $D'_1, \dots, D'_r$ are ample Cartier divisors and $-K_{X'} \equiv D'$ is ample (i.e., X' is a log Fano variety). This also shows that the Fano index of X' is $r(X') \ge \sum_{i=1}^{r} d_i \ge n + 1$. It is well-known (see, e.g., [8, 3.1.14]) that in this case we have $r(X') = \sum_{i=1}^{r} d_i = n + 1$, $X' \simeq \boldsymbol{P}^n$ and $D'_1, \dots, D'_r$ are hyperplanes. Since $\pi : X' \to X$ is étale outside of $\mathrm{Sing}(X)$ and X' is smooth, the restriction $X' \backslash \pi^{-1}(\mathrm{Sing}(X)) \to X \backslash \mathrm{Sing}(X)$ is the universal covering. This gives us that $\pi : X' \to X$ is Galois. Hence $X = \boldsymbol{P}^n / G$, where $G \subset PGL_{n+1}$ is a finite subgroup. Furthermore, the group G does not permute $D'_1, \dots, D'_r$. Thus G has $r > n + 1$ invariant hyperplanes $D'_1, \dots, D'_r$ in $\boldsymbol{P}^n$. By Theorem 2.3 we have $\bigcap_{i \ne k} D'_i = \varnothing$ for $k = 1, \dots, r$.

Finally, the lemma follows by the following simple fact which can be proved by induction on n. □

SUBLEMMA. *Let $G \subset PGL_{n+1}$ be a finite subgroup. Assume that there are $r \geq n+2$ invariant hyperplanes $H_1, \dots, H_r \subset \boldsymbol{P}^n$ such that $\bigcap_{i \neq k} H_i = \varnothing$ for all $k = 1, \dots, r$. Then $G = \{1\}$.*

LEMMA 3.3. *Let $\varphi\colon X \to Z \ni o$ be a three-dimensional flipping contraction and let $D = \sum d_i D_i$ be a boundary on X such that (X, D) is plt, $\rho(X/Z) = 1$, $-(K_X + D)$ is φ-nef and all D_i's are φ-ample. Assume that X is $\boldsymbol{Q}$-factorial. Then $\sum d_i < 2$.*

PROOF. Let $\chi\colon X \overset{\varphi}{\longrightarrow} Z \overset{\varphi^+}{\longleftarrow} X^+$ be the flip with respect to K_X and let $D^+ = \sum d_i D_i^+$ be the proper transform of D. Then all D_i^+'s are anti-ample over Z. Hence $\varphi^{+-1}(o)$ is contained in $\cap D_i^+$. Consider a general hyperplane section $H \subset X^+$. Then $(H, D|_H)$ is plt [3, Sect. 4]. Applying Theorem 2.3 to H we obtain $\sum d_i < 2$. □

LEMMA 3.4. *Let $\varphi\colon X \to Z$ be a contraction from a projective $\boldsymbol{Q}$-factorial three-fold onto a surface such that $\rho(X/Z) = 1$. Let $D = \sum d_i D_i$ be a boundary on X such that (X, D) is lc, $(X, D - \lfloor D \rfloor)$ is klt and $-(K_X + D)$ is nef. Assume that $\lfloor D \rfloor$ has a component S which is generically a section of φ. Then $\sum_i d_i \leq \rho(X) + 3$. Moreover, if the equality holds and (X, D) is plt, then X is smooth, φ is a $\boldsymbol{P}^1$-bundle, $\varphi|_S\colon S \to Z$ is an isomorphism and $Z \simeq \boldsymbol{P}^2$ or $Z \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$.*

PROOF. Assume that $\sum_i d_i \geq \rho(X) + 3$. Since $-K_X$ is φ-ample, a general fiber ℓ of φ is isomorphic to $\boldsymbol{P}^1$. We have

$$2 = -K_X \cdot \ell = D^{\mathrm{hor}} \cdot \ell \geq \sum_{\mathrm{hor}} d_i\,, \qquad \sum_{\mathrm{ver}} d_i \geq \rho(X) + 1 = \rho(Z) + 2\,. \tag{3.5}$$

Let $\mu := \varphi|_S$. Write $\mathrm{Diff}_S(D - S) = \sum_i \beta_i \Theta_i$. Then

$$\beta_i = 1 - \frac{1}{m_i} + \frac{1}{m_i} \sum_{j \in \mathfrak{M}_i} d_j k_{i,j}\,, \tag{3.6}$$

where $m_i \in \boldsymbol{N} \cup \{\infty\}$, $k_{i,j} \in \boldsymbol{N}$ and the sum runs through the set $\mathfrak{M}_i$ of all components D_j containing Θ_i (see [11, 3.10]). Here $m_i = \infty$ when (X, D) is not plt along Θ_i. It is easy to see that $\beta_i \geq \sum_{j \in \mathfrak{M}_i} d_j$. Put $\Xi := \mu_* \mathrm{Diff}_S(D - S)$ and let $\Xi = \sum \gamma_i \Xi_i$. Since $-(K_S + \Theta)$ is nef, (Z, Ξ) is lc [4, 2.28]. For any component D_i of D^{ver} we have at least one component $\Theta_j \subset D_i \cap S$ such that $\mu(\Theta_j) \neq \mathrm{pt}$. This yields

$$\sum_i \gamma_i = \sum_{\mu(\Theta_i) \neq \mathrm{pt}} \beta_i \geq \sum_{\mathrm{ver}} d_j \geq \rho(Z) + 2\,.$$

Applying Proposition 2.1 to (Z, Ξ), we obtain equalities

$$\sum_i \gamma_i = \sum_{\mu(\Theta_i) \neq \mathrm{pt}} \beta_i = \sum_{\mathrm{ver}} d_j = \rho(Z) + 2\,. \tag{3.7}$$

Hence $\sum_{\mathrm{hor}} d_i = 2$ and $\sum_i d_i = \rho(X) + 3$. This shows the first part of the lemma.

Now assume that (X, D) is plt. By Adjunction [4, 17.6], $(S, \mathrm{Diff}_S(D-S))$ is klt and so is (Z, Ξ). Again, by Proposition 2.1 we have either $Z \simeq \boldsymbol{P}^2$ or $Z \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$. There exists a *standard form* of φ (see [10]), i.e., the commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \dashrightarrow & X \\ \downarrow & & \downarrow \\ \widetilde{Z} & \stackrel{\sigma}{\longrightarrow} & Z \end{array}$$

where $\sigma\colon \widetilde{Z} \to Z$ is a birational morphism of smooth surfaces, $\widetilde{X} \dashrightarrow X$ is a birational map and $\widetilde{\varphi}\colon \widetilde{X} \to \widetilde{Z}$ is a *standard* conic bundle (in particular, $\widetilde{X}$ is smooth and $\rho\left(\widetilde{X}/X\right) = 1$). Take the proper transform $\widetilde{S}$ of S on $\widetilde{X}$. For a general fiber $\widetilde{\ell}$ of $\widetilde{\varphi}$ we have $\widetilde{S} \cdot \widetilde{\ell} = 1$. Since $\rho(\widetilde{X}/\widetilde{Z}) = 1$, $\widetilde{S}$ is $\widetilde{\varphi}$-ample. It gives us that each fiber of $\widetilde{\varphi}$ is reduced and irreducible, i.e., the morphism $\widetilde{\varphi}$ is smooth. By [7], there exists a standard conic bundle $\hat{\varphi}\colon \hat{X} \to Z$ and a birational map $\hat{X} \dashrightarrow X$ over Z. This map indices an isomorphism $(\hat{X}/\hat{\varphi}^{-1}(\mathfrak{M})) \simeq (X/\varphi^{-1}(\mathfrak{M}))$, where $\mathfrak{M} \subset Z$ is a finite number of points. Since both φ, $\hat{\varphi}$ are projective and $\rho(X/Z) = \rho(\hat{X}/Z) = 1$, we have $\hat{X} \simeq X$. But then $\varphi\colon X \to Z$ is smooth, i.e., φ is a $\boldsymbol{P}^1$-bundle.

Now we claim that μ is an isomorphism. Indeed, otherwise S contains a fiber, say ℓ_0. Then S intersects all irreducible components of $D^{\mathrm{hor}} - S$. If some component D_k of $D^{\mathrm{hor}} - S$ does not contain ℓ_0, then $\varphi(S \cap D_k)$ is a component of Ξ. By (3.7) we have

$$\rho(Z) + 2 = \sum_i \gamma_i = \sum_{\mu(\Theta_i) \neq \mathrm{pt}} \beta_i \geq d_k + \sum_{\mathrm{ver}} d_j > \rho(Z) + 2\,,$$

which is impossible. Therefore all components of D^{hor} contain ℓ_0. Taking a general hyperplane section as in the proof of Lemma 3.3, we derive a contradiction. □

COROLLARY 3.7.1. *S does not intersects* $\mathrm{Supp}(D^{\mathrm{hor}} - S)$ *and all components of $D^{\mathrm{hor}} - S$ are sections of φ.*

LEMMA 3.8. *Let $\varphi\colon X \to Z$ be a contraction from a $\boldsymbol{Q}$-factorial three-fold onto a curve and let $D = \sum d_i D_i$ be a boundary on X such that (X, D) is lc, $(X, D - \lfloor D \rfloor)$ is klt. Let F be a general fiber. Assume that $-(K_X + D)$ is φ-nef and $\rho(X/Z) = 1$. Then $\sum_{\mathrm{hor}} d_i \leq 3$. Moreover, if the equality holds and (X, D) is plt, then $F \simeq \boldsymbol{P}^2$ and for any component D_i of D^{hor} the scheme-theoretic restriction $D_i|_F$ is a line.*

PROOF. Put $\Delta := D|_F$. Then (F, Δ) is lc, $(F, \Delta - \lfloor \Delta \rfloor)$ is klt (see [3, Sect. 4]) and $-(K_F + \Delta)$ is nef. Moreover, if (X, D) is plt, then so is (F, Δ). Write $\Delta = \sum \delta_i \Delta_i$, where all Δ_i's are irreducible curves on F. Clearly $D^{\mathrm{ver}}|_F = 0$ and $\sum \delta_i \geq \sum_{\mathrm{hor}} d_i$. If $\rho(F) = 1$, then the assertion of 3.8 follows by Proposition 2.1. Assume that $\rho(F) > 1$. Let C be an extremal K_F-negative curve on F (note that K_F is not nef). Then C intersects all components of Δ (because $\rho(X/Z) = 1$). Let $\upsilon\colon F \to F'$ be the contraction of C. If F' is a curve, then we take C to be a general fiber of υ. By Adjunction, $2 = -\deg K_C \geq \deg \Delta|_C$. This gives us $2 \geq \sum \delta_i \geq \sum_{\mathrm{hor}} d_i$. If υ is birational, then $(F', \upsilon(\Delta))$ is lc and all components of $\upsilon(\Delta)$ pass through the point $\upsilon(C)$. By Theorem 2.3, the sum of coefficients of $\upsilon(\Delta)$ is ≤ 2. Hence $\sum_{\mathrm{hor}} d_i \leq \sum \delta_i \leq 3$. If (F, Δ) is klt, then so is $(F', \upsilon(\Delta))$ and the inequality above is strict. Finally, if (F, Δ) is plt and $\lfloor \Delta \rfloor \neq 0$, then we take C to be $(K_F + \Delta - \lfloor \Delta \rfloor)$-negative extremal curve. Then C is not a component of $\lfloor \Delta \rfloor$. By [3, 3.10], $(F', \upsilon(\Delta))$ is plt. Again, by Theorem 2.3 the sum of coefficients of $\upsilon(\Delta)$ is strictly less than 2. So, $\sum_{\text{hor}} d_i \leq \sum \delta_i < 3$. This proves Lemma 3.8. □

COROLLARY 3.8.1. *Notation being as in Lemma* 3.8, *assume additionally that X is projective, $-(K_X + D)$ is nef* (*not only over Z*), $\sum_{\text{hor}} d_i = 3$, $\sum_{\text{ver}} d_i = 2$ *and* (X, D) *is plt. If $\lfloor D^{\text{hor}} \rfloor \neq \varnothing$, then $\lfloor D^{\text{hor}} \rfloor = \lfloor D \rfloor \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and X is smooth along $\lfloor D \rfloor$. In particular, X has at most isolated singularities.*

PROOF. Put $S := \lfloor D \rfloor$. By [4, 17.5], S is normal. Since $\rho(X/Z) = 1$, S is irreducible and all components of $D - S$ meet S. Let $\operatorname{Diff}_S(D - S) = \sum \beta_i \Theta_i$. Clearly, $-(K_S + \operatorname{Diff}_S(D - S))$ is nef. By [4, 17.6], (S, Θ) is klt. As in the proof of Lemma 3.4, we see $\sum \beta_i \geq \sum d_i - 1 \geq 4$. If $\rho(S) = 2$, then equalities $\sum \beta_i = \sum d_i - 1 = 4$ and Proposition 2.1 give us the assertion. Assume that $\rho(S) > 2$. Then some fiber of $\varphi|_S \colon S \to Z$ is not irreducible. Let Γ be its irreducible component and $\upsilon \colon S \to S'$ be the contraction of Γ. Taking into account that Γ intersects all components of D^{hor}, as in Lemma 3.4 we get a contradiction. □

LEMMA 3.9 (cf. [11, 6.9]). *Let $\varphi \colon X \to Z \ni o$ be a K_X-negative contraction from a $\boldsymbol{Q}$-factorial variety X such that $\rho(X/Z) = 1$ and every fiber has dimension one. Let D be a boundary on X such that $(X, D - \lfloor D \rfloor)$ is klt and $K_X + D$ is φ-numerically trivial. Assume that $\lfloor D \rfloor$ is disconnected near $\varphi^{-1}(o)$. Then $K_X + D$ is plt near $\varphi^{-1}(o)$.*

PROOF. Regard $\varphi \colon X \to Z \ni o$ as a germ near $\varphi^{-1}(o)$. Put $S := \lfloor D \rfloor$. Clearly, for a general fiber ℓ of φ we have $-K_X \cdot \ell = D \cdot \ell = 2$. If S' is an irreducible component of S such that $S' \cdot \ell = 0$, then $S' = \varphi^{-1}(C)$ for a curve $C \subset Z$. In this case, S' contains $\varphi^{-1}(o)$ and S is connected near $\varphi^{-1}(o)$. Therefore S has exactly two connected components S_1 and S_2, which are irreducible and $S_1 \cdot \ell = S_2 \cdot \ell = 1$. Then S_i, $i = 1, 2$, meets all components of $\varphi^{-1}(o)$. Hence $S_i \cap \varphi^{-1}(o)$ is 0-dimensional. Since Z is normal and $\varphi|_{S_i} \colon S_i \to Z$ is birational, $S_i \simeq Z$ and $S_i \cap \varphi^{-1}(o)$ is a single point. In particular, $\varphi^{-1}(o)$ is irreducible. Clearly, $LCS(X, D) \subset S = S_1 \cup S_2$. Assume that (X, D) is not plt. Then there is a divisor $E \neq S_1, S_2$ of the function field $K(X)$ with discrepancy $a(E, D) \leq -1$. Let $V \subset X$ be its center. Then $V \subset S$ and we may assume that $V \subset S_1$ (and $V \neq S_1$). Let $L \subset Z$ be any effective prime divisor containing $\varphi(V)$ and let $F := \varphi^{-1}(L)$. Clearly, $(X, D + F)$ is not lc near V. For sufficiently small positive ε the log pair $(X, D + F - \varepsilon S_1)$ is not lc near V and not klt near S_2. This contradicts Connectedness Lemma [4, 17.4]. □

4. Proof of Theorem 1.2. In this section we prove Proposition 2.4.

4.1. Inductive hypothesis. Notation and assumption in Proposition 2.4 are preserved. Our proof is by induction on $\rho(X)$. In the case $\rho(X) = 1$, the assertion is a consequence of Lemma 3.1. To prove Proposition 2.4 for $\rho(X) \geq 2$ we fix $\rho \in \boldsymbol{N}$, $\rho > 1$. Assume that the inequality (2.5) holds if $\rho(X) < \rho$ and for $\rho(X) = \rho$ we have

$$\sum d_i - \rho(X) - 3 \geq 0\,. \tag{4.2}$$

4.3. If (X, D) is klt, then we run K_X-MMP. On each step $K \equiv -D$ cannot be nef. Obviously, all steps preserve our assumptions (see [4, 2.28]) and the left hand side of (4.2) does not decrease. Moreover, by our assumptions we have no divisorial contractions on X (because after any divisorial contraction the left hand side of (4.2) decreases). Therefore after a number of flips, we obtain a fiber type contraction $\varphi\colon X \to Z$. Since $\rho(X) = \rho \geq 2$, $\dim(Z) = 1$ or 2. Note that all varieties from Theorem 1.2 have no small contractions. Thus, it is sufficient to prove Proposition 2.4 on our new model (X, D).

This procedure does not work if (X, D) is not klt. The difference is that contractions of components of D do not contradict the inductive hypothesis. If (X, D) is not klt, then we run $(K_X + D - \lfloor D \rfloor)$-MMP. Note that $\lfloor D \rfloor$ is normal and irreducible [4, 17.5]. For every extremal ray R we have $\lfloor D \rfloor \cdot R > 0$, so we cannot contract an irreducible component of $\lfloor D \rfloor$. Therefore after every divisorial contraction $\sum d_i - \rho(X)$ decrease, a contradiction with our assumption. Thus, all steps of the MMP are flips. By [4, 2.28], they preserve the plt property of $K + D$. At the end we get a fiber type contraction $\varphi\colon X \to Z$, where $\dim(Z) < 3$ and $\lfloor D \rfloor$ is φ-ample (i.e., $\lfloor D^{\mathrm{hor}} \rfloor \neq 0$). Since $\lfloor D^{\mathrm{hor}} \rfloor$ has a component which intersects all components of D^{ver}, $\lfloor D^{\mathrm{ver}} \rfloor = 0$.

4.4. Case: $\dim(Z) = 1$. Then $\rho(X) = 2$. By Lemma 3.8 and our assumption (4.2), we have $\sum_{\mathrm{hor}} d_i \leq 3$ and $\sum_{\mathrm{ver}} d_i \geq 2$. In particular, $D^{\mathrm{ver}} \neq 0$. Components of D^{ver} are fibers of φ, so they are numerically proportional. Clearly, the log divisor $K_X + D^{\mathrm{hor}} \equiv -D^{\mathrm{ver}}$ is not nef and curves in fibers of φ are trivial with respect to it. Let Q be the extremal $(K_X + D^{\mathrm{hor}})$-negative ray of $\overline{NE}(X) \subset \boldsymbol{R}^2$ and $\phi\colon X \to W$ be its contraction. It follows by Lemma 3.3 that ϕ cannot be a flipping contraction. Let ℓ be a general curve such that $\phi(\ell) = \mathrm{pt}$. Then ℓ dominates Z and $\ell \simeq \boldsymbol{P}^1$. Hence $Z \simeq \boldsymbol{P}^1$.

4.4.1. Subcase: $\lfloor D \rfloor = 0$. We will prove that $X \simeq \boldsymbol{P}^2 \times \boldsymbol{P}^1$. By our inductive hypothesis, ϕ cannot be divisorial. Therefore $\dim(W) = 2$. Further, $D^{\mathrm{ver}} \cdot \ell \leq D \cdot \ell = -K_X \cdot \ell = 2$. Since ℓ intersects all components of D^{ver}, $\sum_{\mathrm{ver}} d_i \leq 2$. This yields $\sum_{\mathrm{ver}} d_i = 2$ and $\sum_{\mathrm{hor}} d_i = 3$. In particular, this proves inequality (2.5). Moreover, $\ell \cdot D^{\mathrm{ver}} = 2$, $\ell \cdot D^{\mathrm{hor}} = 0$ and for any component D_i of D^{ver} we have $D_i \cdot \ell = 1$. Fix two components of D^{ver}, say D_0 and D_1. Then $K_X + D_0 + D_1 + D^{\mathrm{hor}} \equiv K_X + D \equiv 0$, so $(X, D_0 + D_1 + D^{\mathrm{hor}})$ is plt by Lemma 3.9. Applying Lemma 3.4, we obtain $D_0 \simeq D_1 \simeq Z \simeq \boldsymbol{P}^2$, X is smooth and ϕ is a $\boldsymbol{P}^1$-bundle. By [6, 3.5], φ is a $\boldsymbol{P}^2$-bundle. We have a finite morphism $\varphi \times \phi\colon X \to Z \times W = \boldsymbol{P}^1 \times \boldsymbol{P}^2$. Clearly, $\deg(\varphi \times \phi) = \varphi^{-1}(\mathrm{pt}) \cdot \ell = 1$. Hence $\varphi \times \phi$ is an isomorphism.

4.4.2. Subcase: $\lfloor D \rfloor \neq 0$. Since $\rho(X/Z) = 1$, $\lfloor D \rfloor$ is irreducible. Put $S := \lfloor D \rfloor$. Let F be a general fiber of φ. By construction, $-K_X$ is φ-ample. First, assume that $\dim(W) = 2$. Then $D^{\mathrm{ver}} \cdot \ell \leq D \cdot \ell = -K_X \cdot \ell = 2$. Since ℓ intersects all components of D^{ver}, $\sum_{\mathrm{ver}} d_i \leq 2$. This yields $\sum_{\mathrm{ver}} d_i = 2$, $\sum_{\mathrm{hor}} d_i = 3$ and $\sum d_i = 5$. Moreover, $D^{\mathrm{hor}} \cdot \ell = 0$. By Lemma 3.8, $F \simeq \boldsymbol{P}^2$, X is smooth along F and for any component D_i of D^{hor} the scheme-theoretic restriction $D_i|_F$ is a line. Hence components of D^{hor} are numerically equivalent. Let D_1 be a component of $D^{\mathrm{hor}} - S$. Consider the new boundary $D' := D + \varepsilon D_0 - \varepsilon S$. If $0 < \varepsilon \ll 1$, then (X, D') is klt and $K_X + D' \equiv 0$. Applying Case 4.4.1, we get $W \simeq \boldsymbol{P}^2$ and $X \simeq \boldsymbol{P}^2 \times \boldsymbol{P}^1$.

Now assume that ϕ is divisorial. By the inductive hypothesis, ϕ contract S. Since the contraction is extremal, $\phi(S)$ is a curve (otherwise curves $S \cap \varphi^{-1}(\mathrm{pt})$ is contracted by ϕ and φ). All components of $\phi(D^{\mathrm{ver}})$ pass through $\phi(S)$. By taking a general hyperplane section as in the proof of Lemma 3.3, we obtain $\sum_{\mathrm{ver}} d_i \le 2$. By Corollary 3.8.1, we obtain that $S \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$, X has only isolated singularities and X is smooth along S. By Lemma 3.8, $F \simeq \boldsymbol{P}^2$ and X is smooth along F. The curve $F \cap S$ is ample on F, so it is connected and smooth by the Bertini theorem. Therefore $F \cap S$ is a generator of $S = \boldsymbol{P}^1 \times \boldsymbol{P}^1$. Since $\varphi|_S$ is flat, the same holds for arbitrary fiber F_0. Hence all fibers of φ are numerically equivalent and any fiber F_0 contains an ample smooth rational curve. Moreover, this also means that F_0 is not multiple. Thus it is a normal surface. Now as in Case 4.4.1, $K_X + F_0 + F_1 + D^{\mathrm{hor}} \equiv 0$ and by Lemma 3.9, $(X, F_0 + F_1 + D^{\mathrm{hor}})$ is plt for any fibers F_0 and F_1. By Adjunction, $(F_0, D^{\mathrm{hor}}|_{F_0})$ is klt. Clearly, $K_{F_0} \equiv K_X|_{F_0}$ and $S|_{F_0}$ are numerically proportional. Hence F_0 is a log del Pezzo surface of Fano index > 1. Since φ is flat, $(K_{F_0})^2 = (K_F)^2 = 9$. Therefore, $F_0 \simeq \boldsymbol{P}^2$ and X is smooth. By [6, 3.5], φ is a $\boldsymbol{P}^2$-bundle, so $X \simeq \boldsymbol{P}_{\boldsymbol{P}^1}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(a) \oplus \mathcal{O}_{\boldsymbol{P}^1}(b)$, $0 \le a \le b$. The Grothendiek tautological bundle $\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}(1)$ is generated by global sections and not ample. Therefore $\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}(1)$ gives us a supporting function for the extremal ray Q. Since ϕ is birational, $\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}(1)^3 = a + b > 0$. Finally, X contains $S = \boldsymbol{P}^1 \times \boldsymbol{P}^1$. Hence $a = 0$. This proves Proposition 2.4 in the case when Z is a curve.

4.5. Case: $\dim(Z) = 2$. Note that Z has only log terminal singularities (see, e.g., [4, 15.11]). Since $-K_X$ is φ-ample, a general fiber ℓ of φ is $\boldsymbol{P}^1$. Hence $2 = -K_X \cdot \ell = D \cdot \ell = D^{\mathrm{hor}} \cdot \ell \ge \sum_{\mathrm{hor}} d_i$. By our assumption, $\sum_{\mathrm{ver}} d_i \ge \rho(X) + 1$. If (X, D) is not plt, then $\lfloor D \rfloor$ is φ-ample. Clearly, $\lfloor D^{\mathrm{ver}} \rfloor = 0$.

CLAIM 4.5.1. *Notation being as above, K_Z is not nef.*

PROOF. Run $(K_X + D^{\mathrm{ver}})$-MMP. After a number of flips we get either a divisorial contraction (of the proper transform of a component of $\lfloor D \rfloor$), or a fiber type contraction. In both cases Z is dominated by a family of rational curves [2, 5-1-4, 5-1-8]. Therefore K_Z is not nef by [5]. □

CLAIM 4.5.2. *Notation being as in* 4.5, *Z contains no contractible curves. In particular*, $\rho(Z) \le 2$.

PROOF. Assume the converse. Namely, there is an irreducible curve $\Gamma \subset Z$ and a birational contraction $\mu\colon Z \to Z''$ such that $\mu(\Gamma) = \mathrm{pt}$ and $\rho(Z/Z'') = 1$. Denote $F := \varphi^{-1}(\Gamma)$. Since $F \not\subset \lfloor D \rfloor$, $(X, D + \varepsilon F)$ is plt for $0 < \varepsilon \ll 1$ [4, 2.17].

Run $(K + D + \varepsilon F)$-MMP over Z''. By our inductive hypothesis, there are no divisorial contractions (because such a contraction must contract F). At the end we cannot get a fiber type contraction (because $K + D + \varepsilon F \equiv \varepsilon F$ cannot be anti-ample over a lower-dimensional variety). Thus after a number of flips $X \dashrightarrow X'$, we get a model X' over Z'' such that

$K_{X'} + D' + \varepsilon F' \equiv \varepsilon F'$ is nef over Z'', where D' and F' are proper transforms of D and F, respectively. Then $F' \not\equiv 0$ (because $F \not\equiv 0$). Let ℓ' be the proper transform of a general fiber of φ. Since F' is nef over Z'', $F' \cdot \ell' = 0$ and $\rho(X'/Z'') = 2$, we obtain that ℓ' generates an extremal ray of $\overline{NE}(X'/Z'')$. Let $\varphi' \colon X' \to Z'$ be its contraction and $\mu' \colon Z' \to Z''$ be the natural map. Then $\dim(Z') = 2$, $\Gamma' := \varphi'(F')$ is a curve and $\mu'(\Gamma') = \mu(\varphi(F)) = \mu(\Gamma) =$ pt. Therefore $(\Gamma')^2 < 0$. On the other hand, $(\Gamma')^2 \geq 0$, which is a contradiction. Indeed, let $C' \subset F'$ be any curve such that $\varphi'(C') = \Gamma'$. Then $C' \cdot F' \geq 0$. By the projection formula, $(\Gamma')^2 \geq 0$.

Since K_Z is not nef, there is an extremal contraction $\psi \colon Z \to V$. By the above it is not birational. Therefore $\dim(V) = 1$ and $\rho(Z) = 2$. □

COROLLARY 4.5.3. *Notation being as in* 4.5, *one of the following holds*:

(i) $\rho(Z) = 1$ *and* $-K_Z$ *is ample*;

(ii) $\rho(Z) = 2$ *and there is a* K_Z*-negative extremal contraction* $\psi \colon Z \to V$ *onto a curve.*

4.5.4. Subcase: $\lfloor D \rfloor = 0$. Let D_i be a component of D^{ver}. Run $(K + D - d_i D_i)$-MMP:

$$\chi_{(i)} \colon X \dashrightarrow X^{(i)} .$$

As above we get a fiber type contraction $\varphi_{(i)} \colon X^{(i)} \to Z^{(i)}$. Notations D^{ver} and D^{hor} will be fixed with respect to our original φ. If $\dim(Z^{(i)}) = 1$, then replacing X with $X^{(i)}$, we get the case $\dim(Z) = 1$ above. Thus we can assume that $\dim(Z^{(i)}) = 2$ for any choise of D_i. Let $\ell^{(i)} \subset X^{(i)}$ be a general fiber of $\varphi_{(i)}$ and let $L^{(i)} \subset X$ be its proper transform. Clearly, $\chi_{(i)}$ is an isomorphism along $L^{(i)}$. Hence $-K_X \cdot L^{(i)} = 2$, $L^{(i)}$ is nef and $D_i \cdot L^{(i)} > 0$. For $i = 1, \dots, r$ we get rational curves $L^{(1)}, \dots, L^{(r)}$. We shift indexing so that $X = X^{(0)}$ and put $Z = Z^{(0)}$ and $\varphi = \varphi_{(0)}$.

Up to permutations we can take $L^{(0)}, \dots, L^{(s)}$, $s + 1 \leq r$ to be linearly independent in $N_1(X)$. Then for any D_i there exists $L^{(j)}$ such that $D_i \cdot L^{(j)} > 0$. Thus we have

$$2(s+1) = -K_X \cdot \sum_{j=0}^{s} L^{(j)} = D \cdot \sum_{j=0}^{s} L^{(j)} = \sum_{i=1}^{r} d_i \left(D_i \cdot \sum_{j=0}^{s} L^{(j)} \right) \geq \sum_{i=1}^{r} d_i \geq \rho(X) + 3 .$$

Since $3 \geq \rho(Z) + 1 = \rho(X) \geq s + 1$, this yields $\rho(X) = s + 1 = 3$. Thus,

$$D_i \cdot \sum_{j=0}^{2} L^{(j)} = 1 \tag{4.6}$$

holds for all i. Moreover, $L^{(0)}, L^{(1)}, L^{(2)}$ generate $N_1(X)$ and components of D generate $N^1(X)$.

Taking into account that $2 = -K_X \cdot L^{(j)} = D \cdot L^{(j)}$, we decompose D into the sum $D = D^{(0)} + D^{(1)} + D^{(2)}$ of effective $\boldsymbol{Q}$-divisors without common components so that

$$D^{(i)} \cdot L^{(j)} = \begin{cases} 0 & \text{if } i \neq j , \\ 2 & \text{otherwise} . \end{cases} \tag{4.7}$$

Then $D^{(i)} = \varphi^* \Delta^{(i)}$ for $i = 1, 2$, where $\Delta^{(1)}$, $\Delta^{(2)}$ are effective $\boldsymbol{Q}$-divisors on Z. Put $C^{(i)} := \varphi(L^{(i)})$, $i = 1, 2$. Since families $L^{(j)}$ are dense on X, $C^{(j)}$ are nef and $\not\equiv 0$. By the projection formula,

$$\Delta^{(i)} \cdot C^{(j)} \quad \begin{cases} = 0 & \text{if } 1 \leq i \neq j \leq 2\,, \\ > 0 & \text{if } 1 \leq i = j \leq 2\,. \end{cases}$$

Hence $\Delta^{(1)}$ and $\Delta^{(2)}$ generate extremal rays of $\overline{NE}(Z) \subset \boldsymbol{R}^2$. By (4.7), these $\boldsymbol{Q}$-divisors have more than one component, so they are nef and $(\Delta^{(1)})^2 = (\Delta^{(2)})^2 = 0$. This gives us that $C^{(1)}$ and $C^{(2)}$ also generate extremal rays. Therefore $C^{(i)}$ and $\Delta^{(j)}$ are numerically proportional whenever $i \neq j$ and $(C^{(1)})^2 = (C^{(2)})^2 = 0$. In particular, $C^{(i)}$, $i = 1, 2$ generate an one-dimensional base point free linear system which defines a contraction $Z \to \boldsymbol{P}^1$. This also shows that $D^{(i)} = \varphi^* \Delta^{(i)}$, $i = 1, 2$ are nef on X.

Now we claim that $D^{(0)}$ is nef. Assume the opposite. Then for small $\varepsilon > 0$, $(X, D + \varepsilon D^{(0)})$ is klt [4, 2.17]. There is a $(K_X + D + \varepsilon D^{(0)})$-negative extremal ray, say R. By our inductive hypothesis, the contraction of R must be of flipping type. Since $\Delta^{(1)}$, $\Delta^{(2)}$ generate $N^1(Z)$, we have $D^{(i)} \cdot R > 0$ for $i = 1$ or 2. By (4.7), $\sum_j^{(i)} d_j = 2$, where $\sum_j^{(i)}$ runs through all components D_j of $D^{(i)}$. Since components of $D^{(i)} = \varphi^*(\Delta^{(i)})$ are numerically proportional, this contradicts Lemma 3.3. Therefore $D^{(i)}$ are nef for $i = 0, 1, 2$.

We claim that $L^{(i)}$, $i = 0, 1, 2$ generate $\overline{NE}(X)$. Indeed, let $z \in \overline{NE}(X)$ be any element. Then $z \equiv \sum \alpha_i [L^{(i)}]$ for $\alpha_i \in \boldsymbol{R}$. By (4.7), $0 \leq D^{(j)} \cdot z = \alpha_j$. This shows that $L^{(i)}$ generate $\overline{NE}(X)$. From (4.6) we see that components of $D^{(0)}$ are numerically equivalent.

Fix two components D' and D'' of $D^{(0)}$. Then $K_X + D' + D'' + D^{(1)} + D^{(2)} \equiv 0$. By Lemma 3.9, $(X, D' + D'' + D^{(1)} + D^{(2)})$ is plt. Lemma 3.4 implies that $D' \simeq D'' \simeq S \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$, X is smooth and φ is a $\boldsymbol{P}^1$-bundle. As in the case $\dim(Z) = 1$, we have $X \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$.

4.7.5. Subcase: $\lfloor D \rfloor \neq 0$. Let S be a component of $\lfloor D \rfloor$. Clearly, $S \cdot \ell \leq 2$. If S is generically a section of φ, then by Lemma 3.4, X is smooth, φ is a $\boldsymbol{P}^1$-bundle and $S \simeq \boldsymbol{P}^2$ or $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. Therefore $X \simeq \boldsymbol{P}(\mathcal{E})$, where $\mathcal{E}$ is a rank two vector bundle on Z. Since φ has disjoint sections, $\mathcal{E}$ is decomposable. So we may assume that $\mathcal{E} = \mathcal{O}_Z + \mathcal{L}$, where $\mathcal{L}$ is a line bundle. By the projection formula, all components of D^{ver} are nef. Let R be a $(K_X + D^{\text{hor}})$-negative extremal curve and let $\phi\colon X \to W$ be its contraction. Assume that ϕ is of flipping type. By [6], $K_X \cdot R \geq 0$. Hence $D^{\text{hor}} \cdot R < 0$, so R is contained in a section of φ. But all curves on $\boldsymbol{P}^2$ and $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ are movable, a contradiction. If ϕ is of fiber type, then as in the case $\lfloor D \rfloor = 0$ we get $X \simeq Z \times \boldsymbol{P}^1$. Assume that ϕ is of divisorial type. By inductive hypothesis, ϕ contracts a component of $\lfloor D \rfloor$.

Finally, consider the case when $\varphi|_S\colon S \to Z$ is generically finite of degree 2. Obviously, $D^{\text{hor}} = S$. If $\rho(Z) = 1$, then all components of D^{ver} are numerically proportional and $\sum_{\text{ver}} d_i \geq 4$. If additionally $\dim(W) = 2$, then $D^{\text{ver}} \cdot \phi^{-1}(w) \leq 2$ for general $w \in W$. Hence $\sum_{\text{ver}} d_i \leq 2$, a contradiction. Then by Lemmas 3.3 and 3.8, ϕ is divisorial and ϕ must contract S. We derive a contradiction with Theorem 2.3 for $(W, \phi(D^{\text{ver}}))$ near $\phi(S)$.

Therefore $\rho(Z) = 2$ and there is a K_Z-negative extremal contraction $\psi\colon Z \to V$ onto a curve (see Corollary 4.5.3). Let $\pi\colon X \xrightarrow{\varphi} Z \xrightarrow{\psi} V$ be the composition map. Clearly, all fibers of π are irreducible. Write $D = \sum' d_i D_i + \sum'' d_i D_i = D' + D''$, where $\sum'$ (respectively $\sum''$) runs through all components D_i such that $\pi(D_i) = \text{pt}$ (respectively $\phi(D_i) = V$). Thus, S is a component of D'' and components of D' are numerically proportional. Let F be a general fiber. Then $\rho(F) = 2$. Consider the contraction $\varphi|_F\colon F \to \varphi(F)$ and denote $D''|_F = D|_F$ by $\Phi = \sum \alpha_i \Phi_i$. Then (F, Φ) is plt and $K_F + \Phi \equiv 0$. Clearly, the curve $S|_F = \lfloor \Phi \rfloor$ is a 2-section and components of $\Phi - \lfloor \Phi \rfloor$ are fibers of $\varphi|_F$. As in the proof of Lemma 3.8, using the fact that $S|_F$ intersects components of $\Phi - \lfloor \Phi \rfloor$ twice, one can check $\sum \alpha_i < 3$. This yields $\sum'' d_i < 3$ and $\sum' d_i > 2$. Let R be a $(K_X + D'')$-negative extremal ray. Since $\sum' d_i > 2$ and $\rho(X) > 2$, R cannot be an extremal ray of fiber type. According to Lemma 3.3, R also cannot be of flipping type. Therefore R is divisorial and contracts S to a point. Since S intersects all components of D^{ver}, this contradicts Theorem 2.3. The proof of Proposition 2.4 is finished.

CONCLUDING REMARK. In contrast with the purely log terminal case we have no complete results in the log canonical case. The reason is that the steps of MMP are not so simple. In particular, we can have divisorial contractions which contract components of $\lfloor D \rfloor$.

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