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THE MEAN VALUE OF THE PRODUCT OF CLASS NUMBERS OF PAIRED QUADRATIC FIELDS I

ANTHONY C. KABLE AND AKIHIKO YUKIE

The second author dedicates this series of papers to Professor Tetsuji Shioda
on his sixtieth birthday

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Abstract. Let k be a number field and $\tilde{k}$ a fixed quadratic extension of k. In this paper and its companions, we find the mean value of the product of class numbers and regulators of two quadratic extensions $F, F^* \neq \tilde{k}$ contained in the biquadratic extensions of k containing $\tilde{k}$.

1. Introduction. This is the first part of a series of three papers. Part III deals with uniquely dyadic phenomena, and so is naturally a unit. We had originally intended to publish Parts I and II together, but reconsidered on account of their combined length.

If k is a number field, then let Δ_k, h_k and R_k be the absolute discriminant (which is an integer), the class number and the regulator, respectively. We fix a number field k and a quadratic extension $\tilde{k}$ of k. If $F \neq \tilde{k}$ is another quadratic extension of k, let $\tilde{F}$ be the composite of F and $\tilde{k}$. Then $\tilde{F}$ is a biquadratic extension of k, and so contains precisely three quadratic extensions, $\tilde{k}$, F and, say, F^* of k. We say that F and F^* are *paired*. In this paper and its companions [17], [18], we shall find the mean value of $h_F R_F h_{F^*} R_{F^*}$ or, equivalently, the mean value of $h_{\tilde{F}} R_{\tilde{F}}$ with respect to $|\Delta_F|$.

Our main results are Theorem 7.12 and Corollaries 7.17 and 7.18 in which k is an arbitrary number field and F runs through quadratic extensions with given local behaviors at a fixed finite number of places. However, for the sake of simplicity, we state our results here assuming that $k = \boldsymbol{Q}$ and that F runs through either real or imaginary quadratic extensions of $\boldsymbol{Q}$ without any further local conditions.

Let $\tilde{k} = \boldsymbol{Q}(\sqrt{d_0})$, where $d_0 \neq 1$ is a square free integer. Suppose $|\Delta_{\boldsymbol{Q}(\sqrt{d_0})}| = \prod_p p^{\tilde{\delta}_p(d_0)}$ is the prime decomposition. Note that $\tilde{\delta}_p(d_0) > 0$ if and only if p is ramified in $\boldsymbol{Q}(\sqrt{d_0})$. Moreover, if $p \neq 2$ is ramified in $\boldsymbol{Q}(\sqrt{d_0})$, then $\tilde{\delta}_p(d_0) = 1$, and if $p = 2$, then $\tilde{\delta}_p(d_0) = 2$ when $d_0 \equiv 3$ (4), and $\tilde{\delta}_p(d_0) = 3$ when d_0 is an even number. Note that if $d_0 \equiv 1, 5$ (8), then the prime 2 is split or inert in $\boldsymbol{Q}(\sqrt{d_0})$, respectively.

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For any prime number p, we put

$$E'_p(d_0) = \begin{cases} 1 - 3p^{-3} + 2p^{-4} + p^{-5} - p^{-6} & \text{if } p \text{ is split in } \tilde{k}\,, \\ (1+p^{-2})(1-p^{-2}-p^{-3}+p^{-4}) & \text{if } p \text{ is inert in } \tilde{k}\,, \\ (1-p^{-1})(1+p^{-2}-p^{-3}+p^{-2\tilde{\delta}_p(d_0)-2\lfloor\tilde{\delta}_p(d_0)/2\rfloor-1}) & \text{if } p \text{ is ramified in } \tilde{k}\,, \end{cases}$$

where $\lfloor\tilde{\delta}_p(d_0)/2\rfloor$ is the largest integer less than or equal to $\tilde{\delta}_p(d_0)/2$.

We define

$$c_+(d_0) = \begin{cases} 16 & d_0 > 0\,, \\ 8\pi & d_0 < 0\,, \end{cases} \qquad c_-(d_0) = \begin{cases} 4\pi^2 & d_0 > 0\,, \\ 8\pi & d_0 < 0\,, \end{cases}$$

$$M(d_0) = |\Delta_{\mathbf{Q}(\sqrt{d_0})}|^{1/2}\zeta_{\mathbf{Q}(\sqrt{d_0})}(2)\prod_p E'_p(d_0)\,.$$

The following theorems are special cases of Corollaries 7.17 and 7.18.

THEOREM 1.1. *With either choice of sign we have*

$$\lim_{X\to\infty} X^{-2} \sum_{\substack{[F:\mathbf{Q}]=2,\\ 0<\pm\Delta_F<X}} h_F R_F h_{F^*} R_{F^*} = c_\pm(d_0)^{-1} M(d_0)\,.$$

THEOREM 1.2. *With either choice of sign we have*

$$\lim_{X\to\infty} X^{-2} \sum_{\substack{[F:\mathbf{Q}]=2,\\ 0<\pm\Delta_F<X}} h_{F(\sqrt{d_0})} R_{F(\sqrt{d_0})} = c_\pm(d_0)^{-1} h_{\mathbf{Q}(\sqrt{d_0})} R_{\mathbf{Q}(\sqrt{d_0})} M(d_0)\,.$$

Note that in Theorem 1.1 if $d_0 > 0$ and $\Delta_F < 0$, then both F and F^* are imaginary quadratic fields, and so Theorem 1.1 states that

$$\lim_{X\to\infty} X^{-2} \sum_{\substack{[F:\mathbf{Q}]=2,\\ 0<-\Delta_F<X}} h_F h_{F^*} = \frac{1}{4\pi^2} M(d_0)\,,$$

which reflects the titles of this series of papers.

Theorems of this kind are called *density theorems*. Many density theorems are known in number theory including, for example, the prime number theorem, the theorem of Davenport-Heilbronn [6], [7] on the density of the number of cubic fields and the theorem of Goldfeld-Hoffstein [9] on the density of class number times regulator of quadratic fields.

Among the three density theorems we quoted above, the prime number theorem, which is probably the best known density theorem, is of a more multiplicative nature than the other two theorems, and our result has more similarities to these. We would like to point out that the Euler factor $1-p^{-2}-p^{-3}+p^{-4}$, which appears in $E'_p(d_0)$ in our result when p is inert, also occurred in the Goldfeld-Hoffstein theorem at every odd prime. We do not as yet understand the significance of this coincidence.

The original proof of the Davenport-Heilbronn theorem used the "fundamental domain method" and the original proof of the Goldfeld-Hoffstein theorem used Eisenstein series of half-integral weight. However, we can also prove these two theorems by using the zeta function theory of prehomogeneous vector spaces. The Davenport-Heilbronn theorem corresponds to the space of binary cubic forms and the Goldfeld-Hoffstein theorem corresponds to the space of binary quadratic forms. The global theory of these two cases was investigated extensively by Shintani in [22], [23]. The local theory and the proof of the density theorem, which use the global theory carried out by Shintani, were done by Datskovsky and Wright [4], [5] in the first case and by Datskovsky [3] in the second (also correcting a minor error in the constant appearing in the Goldfeld-Hoffstein theorem). This zeta function theory of prehomogeneous vector spaces is the approach we take to prove Theorems 1.1 and 1.2.

We now recall the definition of prehomogeneous vector spaces. Let G be a reductive group and V a representation of G both of which are defined over an arbitrary field k of characteristic zero. For simplicity, we assume that V is an irreducible representation of G.

DEFINITION 1.3. The pair (G, V) is called a *prehomogeneous vector space* if

(1) there exists a Zariski open G-orbit in V and

(2) there exists a non-constant polynomial $P(x) \in k[V]$ and a rational character $\chi(g)$ of G such that $P(gx) = \chi(g)P(x)$ for all $g \in G$ and $x \in V$.

Any polynomial $P(x)$ in the above definition is called a relative invariant polynomial. It is known that if $P(x)$ is the relative invariant polynomial of the lowest degree, then any other relative invariant polynomial is a constant multiple of a power of $P(x)$. So, if we put $V^{\mathrm{ss}} = \{x \in V \mid P(x) \neq 0\}$, then this definition does not depend on the choice of $P(x)$.

The notion of prehomogeneous vector spaces was introduced by Mikio Sato in the early 1960's. The principal parts of global zeta functions for some prehomogeneous vector spaces have been determined by Shintani [22], [23], and the second author [28], [29]. Roughly speaking, the global zeta function is a counting function for the unnormalized Tamagawa numbers of the stabilizers of points in V_k^{ss}. This interpretation of expected density theorems for prehomogeneous vector spaces is discussed in the introduction to [26] and in Section 5 of [16], p. 342, in some cases including those we will consider in this paper. Unfortunately, the global zeta function is not exactly this counting function, and Datskovsky and Wright formulated in [5] what we call the filtering process to deal with this difficulty.

To explain the need for the filtering process we consider the space of binary quadratic forms. Gauss made a conjecture in [8] on the density of class number times regulator of orders in quadratic fields. This conjecture was proved by Lipschutz [20] in the case of imaginary quadratic fields and by Siegel [24] in the case of real quadratic fields, and much work has been done on the error term estimate also (see Shintani [23], pp. 44, 45 and Chamizo-Iwaniec [2], for example). However, each quadratic field has infinitely many orders, and so we must filter out this repetition in order to obtain the density of class number times regulator for quadratic fields.

In order to apply the filtering process it is necessary to carry out at least the following steps:

(1) Find the principal part of the global zeta function at its rightmost pole.

(2) Find a uniform estimate for the standard local zeta functions.

(3) Find the local densities.

Note that, despite Tauberian theory, (1) is necessary even to show the existence of the density. The standard local zeta functions will be defined in Section 6. If we apply the filtering process the constant in the density theorem will have an Euler product and we call the Euler factor the *local density*. Also, we must point out that the present formulation of the filtering process does not allow us to use the poles of the global zeta function other than the rightmost pole, as can be done in the case of integral equivalence classes. It is an important problem in the future to improve the filtering process so that we can get error term estimates. However, although it does not, in its current form, yield an error term, our approach does appear to be the only one presently available that allows the field k to be a general number field rather than just $\boldsymbol{Q}$.

Let Aff^n be n-dimensional affine space regarded as a variety over the ground field k. Let $\tilde{k}$ be a fixed quadratic extension of k, W the space of binary $\tilde{k}$-Hermitian forms and $\mathrm{M}(2,2)$ the space of 2×2 matrices. We regard $\mathrm{GL}(2)_{\tilde{k}}$ as a group over k. In this series of papers, we consider the following two prehomogeneous vector spaces:

(1) $G = \mathrm{GL}(2) \times \mathrm{GL}(2) \times \mathrm{GL}(2), \quad V = \mathrm{M}(2,2) \otimes \mathrm{Aff}^2$,

(2) $G = \mathrm{GL}(2)_{\tilde{k}} \times \mathrm{GL}(2), \quad V = W \otimes \mathrm{Aff}^2$.

Case (2) is a k-form of case (1). We gave an interpretation for the expected density theorem for Case (2) in Section 5 of [16]. Let k be a number field and $\tilde{G}$ the image of G in $\mathrm{GL}(V)$. For $x \in V_k^{\mathrm{ss}}$, let $\tilde{G}_x^{\circ}$ be the identity component of the stabilizer. In Case (2), the orbit space $G_k \backslash V_k^{\mathrm{ss}}$ corresponds bijectively with quadratic extensions of k and, if x corresponds to fields other than k and $\tilde{k}$, the weighting factor in the density theorem is the unnormalized Tamagawa number of $\tilde{G}_x^{\circ}$, which is more or less $h_F R_F h_{F^*} R_{F^*}$ or $h_{\tilde{F}} R_{\tilde{F}}$. The principal part at the rightmost pole of the global zeta function for this case was obtained in [27], Corollary 8.16. Therefore it remains to carry out Steps (2) and (3) of the filtering process.

In order to carry out the filtering process, we first have to express the global zeta function as a Dirichlet series with appropriate weighting factors. This requires an extensive preparation including the task of defining a measure on the stabilizer of each point. The main purpose of this part is to carry out the necessary preparation to use the filtering process, to deduce the final form of the density theorem assuming properties of the Dirichlet series in question, and to prove a uniform estimate for the standard local zeta functions. We shall compute the local densities in Parts II and III.

Let v be a finite place of a number field k and k_v its completion at v. The local zeta functions we consider are certain integrals over G_{k_v}-orbits in $V_{k_v}^{\mathrm{ss}}$. The analogous integral over the set $V_{k_v}^{\mathrm{ss}}$ is called the *Igusa zeta function*. Igusa has made significant contributions to the computation of this type of integral (see [10], [11], [12] [13], [14], [15]), and the explicit form of the Igusa zeta function is known in many cases. However, we need information on integrals over orbits and we cannot deduce a uniform estimate from the present knowledge of Igusa zeta functions. Datskovsky and Wright [4] and Datskovsky [3] accomplished the uniform estimate for the standard local zeta functions by explicitly computing them at all finite places. However, as the rank of the group grows, it becomes increasingly difficult to compute the explicit forms of the standard local zeta functions, especially at special places such as dyadic places, and we have to be abstemious with our labor. So we shall only prove a uniform estimate for the standard local zeta functions at all but finitely many places, without finding their explicit forms.

We follow Datskovsky's approach in [3] (which can also be seen implicitly in [7]) to find the local densities. We must consider biquadratic extensions and consequently the dyadic places of k are difficult and technical to handle, given the possible appearance of wild ramification. We devote Part III [18] to consideration of biquadratic extensions generated by two ramified quadratic extensions over a dyadic field. However, the reader should be able to find all the main ingredients for proving Theorems 1.1 and 1.2 in this part and Part II [17].

For the rest of this introduction we discuss the organization of this paper. Throughout, except in Section 3, k is a fixed number field and $\tilde{k}$ is a fixed quadratic extension of k. In Section 3, k is an arbitrary field of characteristic zero and $\tilde{k}$ is a quadratic extension of it. In Section 2 we describe notation we use throughout the paper. In Section 3 we review from [16] the interpretation of the orbit space $G_k\backslash V_k^{\mathrm{ss}}$ for the prehomogeneous vector spaces (1) and (2) above and fix parametrizations of the stabilizers of certain points in V_k^{ss}. In Section 4 we fix various normalizations regarding the invariant measure on $\mathrm{GL}(2)$ both locally and globally. In Section 5 we define a measure on the stabilizer of each point in V^{ss}, both locally and globally, that is in some sense canonical and prove that the volume of $\tilde{G}^{\circ}_{x\,\boldsymbol{A}}/\tilde{G}^{\circ}_{x\,k}$ is the unnormalized Tamagawa number of $\tilde{G}^{\circ}_x$. As we mentioned above, this volume is the weighting factor in the density theorem. We also introduce the local zeta functions. In Section 6 we first define and review the analytic properties of the global zeta function. Then we define the standard local zeta functions and express the global zeta function in terms of them, thus making it more or less a counting function for $h_F R_F h_{F^*} R_{F^*}$. The final and most important purpose of this section is to review the filtering process and to identify the conditions under which it works. Assuming these conditions, we then deduce a preliminary density theorem involving certain as yet unevaluated constants. In Section 7 we list the values of those constants from later parts and state the final form of the density theorem. Therefore, Sections 6 and 7 are the heart of this series of papers. After finishing these sections, the reader should understand the outline of the proof of our result. Later sections and Parts II and III are devoted to verifying the conditions mentioned above and to evaluating the constants involved. In Section 8 we define the notion of omega sets, and prove that the omega sets exist for most orbits at finite places. In Section 9 we prove a uniform estimate for the standard local zeta functions.

2. Notation. This section is confined to establishing our basic notational conventions. Additional notation required throughout the paper will be introduced and explained in the next three sections. More specialized notation will be introduced in the section where it is required.

If X is a finite set, then $\#X$ will denote its cardinality. The standard symbols $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{Z}$ will denote respectively the set of rational, real and complex numbers and the rational integers. If $a \in \boldsymbol{R}$, then the largest integer z such that $z \le a$ is denoted $\lfloor a \rfloor$ and the smallest integer z such that $z \ge a$ by $\lceil a \rceil$. The set of positive real numbers is denoted $\boldsymbol{R}_+$. If R is any ring, then $R^{\times}$ is the set of invertible elements of R and if V is a variety defined over R, then V_R denotes its R-points. If G is an algebraic group, then G° denotes its identity component.

Both k and $\tilde{k}$ are number fields, and so each number-theoretic object we introduce for k has its counterpart for $\tilde{k}$. Generally the notation for the $\tilde{k}$ object will be derived from that of the k object by adding a tilde. Let $\mathfrak{M}$, $\mathfrak{M}_{\infty}$, $\mathfrak{M}_{\mathrm{f}}$, $\mathfrak{M}_{\mathrm{dy}}$, $\mathfrak{M}_{\boldsymbol{R}}$ and $\mathfrak{M}_{\boldsymbol{C}}$ denote respectively the set of all places of k, all infinite places, all finite places, all dyadic places (those dividing the place of $\boldsymbol{Q}$ at 2), all real places and all complex places. (Correspondingly we have $\widetilde{\mathfrak{M}}$ and so on.) Let $\mathfrak{M}_{\mathrm{rm}}$, $\mathfrak{M}_{\mathrm{in}}$ and $\mathfrak{M}_{\mathrm{sp}}$ be the sets of places of k which are respectively ramified, inert and split on extension to $\tilde{k}$. Recall that a real place of k which lies under a complex place of $\tilde{k}$ is regarded as ramified.

Let $\mathcal{O}$ be the ring of integers of k. If $v \in \mathfrak{M}$, then k_v denotes the completion of k at v and $|\ |_v$ denotes the normalized absolute value on k_v. If $v \in \mathfrak{M}_{\mathrm{f}}$, then $\mathcal{O}_v$ denotes the ring of integers of k_v, π_v a uniformizer in $\mathcal{O}_v$, $\mathfrak{p}_v$ the maximal ideal of $\mathcal{O}_v$ and q_v the cardinality of $\mathcal{O}_v/\mathfrak{p}_v$. If $a \in k_v$ and $(a) = \mathfrak{p}_v^i$, then we write $\mathrm{ord}_{k_v}(a) = i$. If $\mathfrak{i}$ is a fractional ideal in k_v and $a - b \in \mathfrak{i}$, then we write $a \equiv b\ (\mathfrak{i})$ or $a \equiv b\ (c)$ if c generates $\mathfrak{i}$.

If k_1/k_2 is a finite extension either of local fields or of number fields, then we shall write Δ_{k_1/k_2} for the relative discriminant of the extension; it is an ideal in the ring of integers of k_2. The symbol Δ_{k_1} will stand for $\Delta_{k_1/\boldsymbol{Q}_p}$ or $\Delta_{k_1/\boldsymbol{Q}}$ according as the situation is local or global. To ease the notational burden we shall use the same symbol, Δ_{k_1}, for the classical absolute discriminant of k_1 over $\boldsymbol{Q}$. Since this number generates the ideal Δ_{k_1}, the resulting notational identification is harmless. If $\mathfrak{i}$ is a fractional ideal in the number field k_1 and v is a finite place of k_1, then we write $\mathfrak{i}_v$ for the closure of $\mathfrak{i}$ in $k_{1,v}$. It is a fractional ideal in $k_{1,v}$. If $\mathfrak{i}$ is integral, then we put $\mathcal{N}(\mathfrak{i}) = \#(\mathcal{O}_{k_1}/\mathfrak{i})$. Note that $\mathcal{N}(\mathfrak{i}) = \prod_v \mathcal{N}_v(\mathfrak{i}_v)$, where the product is over all finite places of k_1 and $\mathcal{N}_v(\mathfrak{p}_v^a) = q_v^a$ for $a \in \boldsymbol{Z}$. This formula serves to extend the domain of $\mathcal{N}$ to all fractional ideals in k_1. We shall use the notation Tr_{k_1/k_2} and N_{k_1/k_2} for the trace and the norm in the extension k_1/k_2.

Returning to k, we let r_1, r_2, h_k, R_k and e_k be, respectively, the number of real places, the number of complex places, the class number, the regulator and the number of roots of unity contained in k. It will be convenient to set

$$\mathfrak{C}_k = 2^{r_1}(2\pi)^{r_2} h_k R_k e_k^{-1}\,. \tag{2.1}$$

We assume that the reader is familiar with the basic definitions and facts concerning adèles and idèles. These may be found in [25]. The ring of adèles, the group of idèles and the adèlic absolute value of k are denoted by $\boldsymbol{A}$, $\boldsymbol{A}^{\times}$ and $|\ |$, respectively. When we have to show the number field or the local field on which we consider the absolute value, we may use notation such as $|\ |_F$. There is a natural inclusion $\boldsymbol{A} \to \tilde{\boldsymbol{A}}$, under which an adèle $(a_v)_v$ corresponds to the adèle $(b_w)_w$ such that $b_w = a_v$ if $w|v$. Let $\boldsymbol{A}^1 = \{t \in \boldsymbol{A}^{\times} \mid |t| = 1\}$. Using the identification $\tilde{k} \otimes_k \boldsymbol{A} \cong \tilde{\boldsymbol{A}}$, the norm map $\mathrm{N}_{\tilde{k}/k}$ can be extended to a map from $\tilde{\boldsymbol{A}}$ to $\boldsymbol{A}$. It is known (see [25], p. 139) that $|\mathrm{N}_{\tilde{k}/k}(t)| = |t|_{\tilde{\boldsymbol{A}}}$ for $t \in \tilde{\boldsymbol{A}}$. Suppose $[k : \boldsymbol{Q}] = n$. Then $[\tilde{k} : \boldsymbol{Q}] = 2n$. For $\lambda \in \boldsymbol{R}_+$, $\underline{\lambda} \in \boldsymbol{A}^{\times}$ is the idèle whose component at any infinite place is $\lambda^{1/n}$ and whose component at any finite place is 1. Also $\underline{\tilde{\lambda}} \in \tilde{\boldsymbol{A}}^{\times}$ is the idèle whose component at any infinite place is $\lambda^{1/2n}$ and whose component at any finite place is 1. Clearly $\underline{\lambda} = \tilde{\underline{\lambda}}^2$. Since $|\underline{\lambda}| = \lambda$ and $|\tilde{\underline{\lambda}}|_{\tilde{\boldsymbol{A}}} = \lambda$ we conclude that $|\underline{\lambda}|_{\tilde{\boldsymbol{A}}} = \lambda^2$. When we have to show the number field on which we consider $\underline{\lambda}$, we use the notation such as $\underline{\lambda}_F$.

If V is a vector space over k we let $V_{\boldsymbol{A}}$ be its adèlization and V_∞ and V_{f} its infinite and finite parts. Let $\mathcal{S}(V_{\boldsymbol{A}})$, $\mathcal{S}(V_\infty)$, $\mathcal{S}(V_{\mathrm{f}})$ and $\mathcal{S}(V_{k_v})$ be the spaces of Schwartz-Bruhat functions on each of the indicated domains.

We choose a Haar measure dx on $\boldsymbol{A}$ so that $\int_{\boldsymbol{A}/k} dx = 1$. For any $v \in \mathfrak{M}_{\mathrm{f}}$, we choose a Haar measure dx_v on k_v so that $\int_{\mathcal{O}_v} dx_v = 1$. We use the ordinary Lebesgue measure dx_v for v real, and $dx_v \wedge d\bar{x}_v$ for v imaginary. Then $dx = |\Delta_k|^{-1/2} \prod_v dx_v$ (see [25], p. 91).

We define a Haar measure $d^\times t^1$ on $\boldsymbol{A}^1$ so that $\int_{\boldsymbol{A}^1/k^\times} d^\times t^1 = 1$. Using this measure, we choose a Haar measure $d^\times t$ on $\boldsymbol{A}^\times$ so that

$$\int_{\boldsymbol{A}^\times} f(t) d^\times t = \int_0^\infty \int_{\boldsymbol{A}^1} f(\underline{\lambda} t^1) d^\times \lambda d^\times t^1 ,$$

where $d^\times \lambda = \lambda^{-1} d\lambda$. For any $v \in \mathfrak{M}_{\mathrm{f}}$, we choose a Haar measure $d^\times t_v$ on $k_v^\times$ so that $\int_{\mathcal{O}_v^\times} d^\times t_v = 1$. Let $d^\times t_v(x) = |x|_v^{-1} dx_v$ if v is real, and $d^\times t_v(x) = |x|_v^{-1} dx_v \wedge d\bar{x}_v$ if v is imaginary. Then $d^\times t = \mathfrak{C}_k^{-1} \prod_v d^\times t_v$ (see [25], p. 95). We later have to compare the global measure and the product of local measures, and for that purpose it is convenient to denote the product of local measures on $\boldsymbol{A}$, $\boldsymbol{A}^\times$ as follows:

$$d_{\mathrm{pr}} x = \prod_v dx_v , \quad d_{\mathrm{pr}}^\times t = \prod_v d^\times t_v . \tag{2.2}$$

Let $\zeta_k(s)$ be the Dedekind zeta function of k. We define

$$Z_k(s) = |\Delta_k|^{s/2} \left(\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \right)^{r_1} ((2\pi)^{1-s} \Gamma(s))^{r_2} \zeta_k(s) . \tag{2.3}$$

This definition differs from that in [25], p. 129 by the inclusion of the $|\Delta_k|^{s/2}$ factor and from that in [28] by a factor of $(2\pi)^{r_2}$. It is adopted here as the most convenient for our purposes. We note that it was the quotient $Z_k(s)/Z_k(s+1)$ rather than $Z_k(s)$ itself which played a significant role in [28] and this quotient is unchanged here. It is known ([25], p. 129) that

$$\operatorname{Res}_{s=1} \zeta_k(s) = |\Delta_k|^{-1/2} \mathfrak{C}_k , \quad \text{and so} \quad \operatorname{Res}_{s=1} Z_k(s) = \mathfrak{C}_k . \tag{2.4}$$

Finally, we introduce the following notation:

$$a(t_1, t_2) = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} , \quad n(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} . \tag{2.5}$$

3. A review of the orbit space. This section is devoted to defining the prehomogeneous vector spaces which are at the heart of this work and reviewing their fundamental properties. Arithmetic plays no role here, so in this section k may be any field of characteristic zero and $\tilde{k}$ any quadratic extension of k. We denote the non-identity element of $\mathrm{Gal}(\tilde{k}/k)$ by σ.

A matrix $x \in \mathrm{M}(2,2)_{\tilde{k}}$ is said to be *Hermitian* if ${}^t x = x^\sigma$. The set of all Hermitian matrices in $\mathrm{M}(2,2)_{\tilde{k}}$ forms a k-vector space which we shall denote by W. The elements of W are also referred to as binary Hermitian forms.

We define and discuss the two spaces we require in parallel as far as possible; they will be distinguished as Cases (1) and (2). Let

$$V = \begin{cases} \mathrm{M}(2,2) \otimes \mathrm{Aff}^2 & \text{in Case (1)}, \\ W \otimes \mathrm{Aff}^2 & \text{in Case (2)}, \end{cases} \tag{3.1}$$

where Aff^n is the n-dimensional affine space regarded as a variety over k. Let

$$G = \begin{cases} \mathrm{GL}(2) \times \mathrm{GL}(2) \times \mathrm{GL}(2) & \text{in Case (1)}, \\ \mathrm{GL}(2)_{\tilde{k}} \times \mathrm{GL}(2) & \text{in Case (2)}, \end{cases} \tag{3.2}$$

where $\mathrm{GL}(2)_{\tilde{k}}$ is regarded as an algebraic group over k by restriction of scalars. If $g \in G$, then we shall write $g = (g_1, g_2, g_3)$ in Case (1) and $g = (g_1, g_2)$ in Case (2). It will be convenient to identify $x = (x_1, x_2) \in V$ with the 2×2-matrix $M_x(v) = v_1 x_1 + v_2 x_2$ of linear forms in the variables v_1 and v_2, which we collect into the row vector $v = (v_1, v_2)$. With this identification, we define a rational action of G on V via

$$M_{gx}(v) = \begin{cases} g_1 M_x(v g_3)\, {}^t g_2 & \text{in Case (1)}, \\ g_1 M_x(v g_2)\, {}^t g_1^\sigma & \text{in Case (2)}. \end{cases} \tag{3.3}$$

In both cases we define $F_x(v) = -\det M_x(v)$. Then

$$F_{gx}(v) = \begin{cases} \det g_1 \det g_2 F_x(v g_3) & \text{in Case (1)}, \\ \mathrm{N}_{\tilde{k}/k}(\det g_1) F_x(v g_2) & \text{in Case (2)}. \end{cases} \tag{3.4}$$

We let $P(x)$ be the discriminant of the binary quadratic form $F_x(v)$. Then $P(x) \in k[V]$ and $P(gx) = \chi(g) P(x)$, where

$$\chi(g) = \begin{cases} (\det g_1 \det g_2 \det g_3)^2 & \text{in Case (1)}, \\ (\mathrm{N}_{\tilde{k}/k}(\det g_1) \det g_3)^2 & \text{in Case (2)}. \end{cases} \tag{3.5}$$

A calculation shows that $P(x)$ is not identically zero, and so it is a relatively invariant polynomial for (G, V) in each case. We let V^{ss} denote the complement of the hypersurface defined by $P(x) = 0$ in V.

We define $\tilde{T} = \ker(G \to \mathrm{GL}(V))$; in Case (1)

$$\tilde{T} = \{(t_1 I_2, t_2 I_2, t_3 I_2) \mid t_1, t_2, t_3 \in \mathrm{GL}(1),\ t_1 t_2 t_3 = 1\} \tag{3.6}$$

and in Case (2)

$$\tilde{T} = \{(t_1 I_2, t_2 I_2) \mid t_1 \in \mathrm{GL}(1)_{\tilde{k}}, t_2 \in \mathrm{GL}(1),\ \mathrm{N}_{\tilde{k}/k}(t_1) t_2 = 1\}. \tag{3.7}$$

It will be convenient to introduce standard coordinates on G and V. Elements of G have the form $g = (g_1, g_2, g_3)$ or $g = (g_1, g_2)$. In either case we shall write

$$g_i = \begin{pmatrix} g_{i11} & g_{i12} \\ g_{i21} & g_{i22} \end{pmatrix} \tag{3.8}$$

for each i. Elements of V are vectors $x = (x_1, x_2)$. We shall put

$$x_i = \begin{pmatrix} x_{i11} & x_{i12} \\ x_{i21} & x_{i22} \end{pmatrix} \tag{3.9}$$

in Case (1) and

$$x_i = \begin{pmatrix} x_{i0} & x_{i1} \\ x_{i1}^{\sigma} & x_{i2} \end{pmatrix} \tag{3.10}$$

in Case (2).

In the language of Galois descent, Case (2) is a k-form of Case (1); they become isomorphic on extension of scalars from k to $\tilde{k}$. Indeed, it is well known that, as $\tilde{k}$-varieties,

$$G \times \tilde{k} \cong \mathrm{GL}(2) \times \mathrm{GL}(2) \times \mathrm{GL}(2) \tag{3.11}$$

and

$$W \times \tilde{k} \cong \mathrm{M}(2, 2) \tag{3.12}$$

so that

$$V \times \tilde{k} \cong \mathrm{M}(2, 2) \otimes \mathrm{Aff}^2, \tag{3.13}$$

and a calculation shows that the induced action of $G \times \tilde{k}$ on $V \times \tilde{k}$ is that of Case (1). The Galois automorphism σ induces a k-automorphism of the k-varieties G and V which we denote by $i(\sigma)$. If $(g_1, g_2, g_3) \in G_{\tilde{k}}$, then $i(\sigma)(g_1, g_2, g_3) = (g_2^{\sigma}, g_1^{\sigma}, g_3^{\sigma})$ and if $x \in W_{\tilde{k}}$, then $i(\sigma)x = {}^t x^{\sigma}$, where σ as a superscript denotes the entry-by-entry action of σ. In particular, G_k is embedded in $G_{\tilde{k}} \cong (G \times \tilde{k})_{\tilde{k}}$ via the map $(g_1, g_2) \mapsto (g_1, g_1^{\sigma}, g_2)$.

We are now ready to recall the description of the space of non-singular orbits in V_k.

DEFINITION 3.14. Let $\mathfrak{E}\mathfrak{x}_2$ be the set of isomorphism classes of extensions of k of degree at most two.

It is proved in [26], pp. 305–310 and [16], p. 324 that $G_k \backslash V_k^{\mathrm{ss}}$ corresponds bijectively with $\mathfrak{E}\mathfrak{x}_2$. Moreover, if $x \in V$, then the corresponding field is generated by the roots of $F_x(v) = 0$. We denote this field by $k(x)$.

Suppose that $p(z) = z^2 + a_1 z + a_2 \in k[z]$ has distinct roots α_1 and α_2. We collect these into a set $\alpha = \{\alpha_1, \alpha_2\}$ since the numbering is arbitrary. Define $w_p \in V_k$ by

$$w_p = \left(\begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix}, \begin{pmatrix} 1 & a_1 \\ a_1 & a_1^2 - a_2 \end{pmatrix} \right) ; \tag{3.15}$$

a computation shows that $F_{w_p}(z, 1) = p(z)$, and so $w_p \in V_k^{\mathrm{ss}}$ and $k(w_p) = k(\alpha)$ is the splitting field of p. Let

$$w = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) , \tag{3.16}$$

$$h_{\alpha} = \begin{pmatrix} 1 & -1 \\ -\alpha_1 & \alpha_2 \end{pmatrix} \tag{3.17}$$

and then define $g_p \in G_{k(w_p)}$ by

$$g_p = \begin{cases} (h_\alpha, h_\alpha, (\alpha_2 - \alpha_1)^{-1}h_\alpha) & \text{in Case (1) or when } k(w_p) = \tilde{k}\,, \\ (h_\alpha, (\alpha_2 - \alpha_1)^{-1}h_\alpha) & \text{otherwise}\,. \end{cases} \tag{3.18}$$

With these definitions it is easy to check that $w_p = g_p w$.

We close this section with a detailed description of the k-rational points of the stabilizer G_{w_p}. Similar descriptions were derived in [16] and [26] and, although we are using different orbital representatives here, the arguments are so similar that they will only be sketched. The method is as follows: We begin with a description of G_w as a k-variety; this is given in Section 3 of [26] for Case (1) and in Section 2 of [16] for Case (2). Then we find, by direct calculation, the k-rational points in $g_p G_{w\,k(w_p)} g_p^{-1}$ and this gives us $G_{w_p\,k}$.

If we let

$$t = \begin{cases} (a(t_{11}, t_{12}), a(t_{21}, t_{22}), a(t_{31}, t_{32})) & \text{in Case (1)}\,, \\ (a(t_{11}, t_{12}), a(t_{21}, t_{22})) & \text{in Case (2)}\,, \end{cases} \tag{3.19}$$

then

$$G^\circ_{w\,k} = \begin{cases} \{t \mid t_{ij} \in k^\times, t_{1j}t_{2j}t_{3j} = 1 \text{ for all } i, j\} & \text{in Case (1)}\,, \\ \{t \mid t_{1j} \in \tilde{k}^\times, t_{2j} \in k^\times, \mathrm{N}_{\tilde{k}/k}(t_{1j})t_{2j} = 1 \text{ for all } j\} & \text{in Case (2)}\,, \end{cases} \tag{3.20}$$

and so $G^\circ_{w\,k} \cong \mathrm{GL}(1)^4_k$ in Case (1) and $G^\circ_{w\,k} \cong \mathrm{GL}(1)^2_{\tilde{k}}$ in Case (2). If we let

$$\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \tag{3.21}$$

then the class of (τ, τ, τ) in Case (1) or of (τ, τ) in Case (2) generates $G_{w\,k}/G^\circ_{w\,k}$.

Now let

$$t = \begin{cases} (a(t_{11}, t_{12}), a(t_{21}, t_{22})) & \text{in Case (2) when } k(w_p) \neq \tilde{k}\,, \\ (a(t_{11}, t_{12}), a(t_{21}, t_{22}), a(t_{31}, t_{32})) & \text{otherwise}\,. \end{cases} \tag{3.22}$$

We assume that $k(w_p)/k$ is quadratic, since if $k(w_p) = k$, then $G_{w_p\,k}$ is conjugate to $G_{w\,k}$ over k. Let ν be the non-trivial element of $\mathrm{Gal}(k(w_p)/k)$, which may also be thought of as an element of $\mathrm{Gal}(\tilde{k}(w_p)/\tilde{k})$ when $k(w_p) \neq \tilde{k}$. Here $\tilde{k}(w_p)$ denotes the composite of $\tilde{k}$ and $k(w_p)$.

In Case (1), $G^\circ_{w_p\,k}$ is

$$\{g_p t g_p^{-1} \mid t_{ij} \in k(w_p)^\times, t_{i1} = t_{i2}^\nu, t_{1j}t_{2j}t_{3j} = 1 \text{ for all } i, j\}\,, \tag{3.23}$$

and so $G^\circ_{w_p\,k} \cong \mathrm{GL}(1)_{k(w_p)} \times \mathrm{GL}(1)_{k(w_p)}$. In Case (2) when $k(w_p) = \tilde{k}$, $G^\circ_{w_p\,k}$ is

$$\{g_p t g_p^{-1} \mid t_{ij} \in \tilde{k}^\times, t_{12}^\sigma = t_{21}, t_{11}^\sigma = t_{22}, t_{1j}t_{2j}t_{3j} = 1 \text{ for all } i, j\}\,, \tag{3.24}$$

and so $G^\circ_{w_p\,k} \cong \mathrm{GL}(1)_{\tilde{k}} \times \mathrm{GL}(1)_{\tilde{k}}$. In Case (2) when $k(w_p) \neq \tilde{k}$, $G^\circ_{w_p\,k}$ is

$$\begin{aligned} \{g_p t g_p^{-1} \mid t_{1j} \in \tilde{k}(w_p)^\times, t_{2j} \in k(w_p)^\times, t_{11}^\nu = t_{12}, \\ \mathrm{N}_{\tilde{k}(w_p)/k(w_p)}(t_{1j})t_{2j} = 1 \text{ for all } j\}\,, \end{aligned} \tag{3.25}$$

and so $G^\circ_{w_p k} \cong \mathrm{GL}(1)_{\tilde{k}(w_p)}$. In every instance, $G_{w_p k}/G^\circ_{w_p k}$ is generated by $g_p(\tau,\tau,\tau)g_p^{-1}$ or $g_p(\tau,\tau)g_p^{-1}$ as the case may be.

It will be convenient to have an explicit description of how $G_{w_p k}$ is embedded in G_k in each case. To this end, define

$$A_p(c,d) = \begin{pmatrix} c & -d \\ a_2 d & c - a_1 d \end{pmatrix} \quad \text{and} \quad \tau_p = \begin{pmatrix} -1 & 0 \\ -a_1 & 1 \end{pmatrix}. \tag{3.26}$$

It is easy to check that any matrix which has both ${}^t(1 \ \ -\alpha_1)$ and ${}^t(1 \ \ -\alpha_2)$ as eigenvectors must equal $A_p(c,d)$ for some c and d. Consequently, the set of all such matrices is closed under multiplication, any two such matrices commute and if such a matrix is invertible, then its inverse lies in the same set.

LEMMA 3.27. *In Case* (1), $G^\circ_{w_p k}$ *consists of elements of* G_k *of the form*

$$(A_p(c_1,d_1), A_p(c_2,d_2), A_p(c_3,d_3)), \tag{3.28}$$

where $c_i, d_i \in k$, $\det(A_p(c_i,d_i)) \neq 0$ *for* $i = 1, 2$ *and* (c_3,d_3) *is related to* (c_1,d_1,c_2,d_2) *by the equation*

$$A_p(c_3,d_3) = A_p(c_1,d_1)^{-1}A_p(c_2,d_2)^{-1}. \tag{3.29}$$

Moreover $[G_{w_p k} : G^\circ_{w_p k}] = 2$ *and* $G_{w_p k}/G^\circ_{w_p k}$ *is generated by the class of* (τ_p,τ_p,τ_p).

PROOF. Suppose first that $k(w_p) = k$. Then $G^\circ_{w_p k} = g_p G^\circ_{w k} g_p^{-1}$ and, by (3.20), the elements of $G^\circ_{w k}$ may be characterized as those $(g_1,g_2,g_3) \in G_k$ such that ${}^t(1 \ \ 0)$ and ${}^t(0 \ \ -1)$ are both eigenvectors for each g_i and $g_1g_2g_3 = I_2$. Since $h_\alpha {}^t(1 \ \ 0) = {}^t(1 \ \ -\alpha_1)$ and $h_\alpha {}^t(0 \ \ -1) = {}^t(1 \ \ -\alpha_2)$, the first claim follows. If $k(w_p) \neq k$, then calculation gives $h_\alpha a(t,t^\nu)h_\alpha^{-1} = A_p(c,d)$ where $t = c + d\alpha_1 \in k(w_p)$. With this observation, the first claim follows in this case from (3.23). Finally, $h_\alpha \tau h_\alpha^{-1} = \tau_p$ and the second claim is established. □

LEMMA 3.30. *In Case* (2), $G^\circ_{w_p k}$ *consists of elements of* G_k *of the form*

$$(A_p(c_1,d_1), A_p(c_2,d_2)), \tag{3.31}$$

where $c_1, d_1 \in \tilde{k}$, $c_2, d_2 \in k$, $\det(A_p(c_1,d_1)) \neq 0$ *and* (c_2,d_2) *is related to* (c_1,d_1) *by the equation*

$$A_p(c_2,d_2) = A_p(c_1,d_1)^{-1}A_p(c_1^\sigma,d_1^\sigma)^{-1}. \tag{3.32}$$

Moreover, $[G_{w_p k} : G^\circ_{w_p k}] = 2$ *and* $G_{w_p k}/G^\circ_{w_p k}$ *is generated by the class of* (τ_p,τ_p).

PROOF. If $k(w_p) = k$, then, by (3.20), $G^\circ_{w k}$ may be characterized as the set of (g_1,g_2) in G_k such that ${}^t(1 \ \ 0)$ and ${}^t(0 \ \ -1)$ are eigenvectors of g_1 and $g_1 g_1^\sigma g_2 = I_2$. Since $G^\circ_{w_p k} = g_p G^\circ_{w k} g_p^{-1}$ and $h_\alpha^\sigma = h_\alpha$, the claim follows. If $k(w_p) \neq k, \tilde{k}$, then $h_\alpha^\sigma = h_\alpha$ and a similar argument works on setting $A_p(c_1,d_1) = h_\alpha a(t_{11}, t_{11}^\nu)h_\alpha^{-1}$ in the notation of (3.25).

This leaves the case where $k(w_p) = \tilde{k}$. We use the notation of (3.24). If we set $g_1 = h_\alpha a(t_{11},t_{12})h_\alpha^{-1} = A_p(c_1,d_1)$ for some $c_1, d_1 \in \tilde{k}$, then, using the equation $h_\alpha^\sigma = -h_\alpha\tau$, we have $g_1^\sigma = h_\alpha a(t_{12}^\sigma, t_{11}^\sigma)h_\alpha^{-1}$, and so $g_2 = g_1^{-1}g_1^{-\sigma}$ is $h_\alpha a((t_{11}t_{12}^\sigma)^{-1}, (t_{11}^\sigma t_{12})^{-1})h_\alpha^{-1}$. Thus $(g_1, g_2) \in G^\circ_{w_p\,k}$. Finally, we have $\tau_p = h_\alpha \tau h_\alpha^{-1}$ and the last claim follows from this. □

4. An invariant measure on $\mathrm{GL}(2)$**.** Assume now that k is a number field. In this section we choose an invariant measure on $\mathrm{GL}(2)$ in both the local and adèlic situations.

Let $T \subseteq \mathrm{GL}(2)$ be the set of diagonal matrices and $N \subseteq \mathrm{GL}(2)$ the set of lower triangular matrices whose diagonal entries are 1. Then $B = TN$ is a Borel subgroup of $\mathrm{GL}(2)$. Let $T_+ = \{\lambda = a(\underline{\lambda}_1, \underline{\lambda}_2) \mid \lambda_1, \lambda_2 \in \boldsymbol{R}_+\}$ and $K = \prod_{v\in\mathfrak{M}} K_v$, where $K_v = \mathrm{O}(2)$ if $v \in \mathfrak{M}_{\boldsymbol{R}}$, $K_v = \mathrm{U}(2)$ if $v \in \mathfrak{M}_{\boldsymbol{C}}$ and $K_v = \mathrm{GL}(2)_{\mathcal{O}_v}$ if $v \in \mathfrak{M}_{\mathrm{f}}$. The group $\mathrm{GL}(2)_{\boldsymbol{A}}$ has the Iwasawa decomposition $\mathrm{GL}(2)_{\boldsymbol{A}} = KT_{\boldsymbol{A}}N_{\boldsymbol{A}}$, and so any element $g \in \mathrm{GL}(2)_{\boldsymbol{A}}$ can be expressed as $g = \kappa(g)t(g)n(u(g))$, where $\kappa(g) \in K$, $t(g) = a(t_1(g), t_2(g))$ and $u(g) \in \boldsymbol{A}$.

The measure du on $\boldsymbol{A}$ defined in Section 2 induces an invariant measure on $N_{\boldsymbol{A}}$. Since K is compact we can choose an invariant measure $d\kappa$ on it so that the total volume of K is 1. On $T_{\boldsymbol{A}}$ we put $d^\times t = d^\times t_1\, d^\times t_2$ for $t = a(t_1, t_2)$, where $d^\times t_j$ is the measure on $\boldsymbol{A}^\times$ defined in Section 2. Then $db = |t_1t_2^{-1}|^{-1}\, d^\times t\, du$ defines an invariant measure on $B_{\boldsymbol{A}}$ and $dg = d\kappa\, db$ defines an invariant measure on $\mathrm{GL}(2)_{\boldsymbol{A}}$.

We make parallel definitions of invariant measures on $\mathrm{GL}(2)_{k_v}$, K_v, B_{k_v}, N_{k_v} and T_{k_v}, which we denote by dg_v, $d\kappa_v$, db_v, du_v and $d^\times t_v$, respectively. As in Section 2, we denote the product of local measures on $G_{\boldsymbol{A}}$ as

$$d_{\mathrm{pr}}g = \prod_v dg_v\,. \tag{4.1}$$

Then (see Section 2) we have

$$du = |\Delta_k|^{-1/2}\prod_v du_v, \quad d^\times t = \mathfrak{C}_k^{-2}\prod_v d^\times t_v \quad \text{and so} \quad dg = |\Delta_k|^{-1/2}\mathfrak{C}_k^{-2}d_{\mathrm{pr}}g\,. \tag{4.2}$$

Let $\mathrm{GL}(2)^0_{\boldsymbol{A}} = \{g \in \mathrm{GL}(2)_{\boldsymbol{A}} \mid |\det(g)| = 1\}$. If, for $\lambda \in \boldsymbol{R}_+$, we define $c(\lambda) = a(\underline{\lambda}, \underline{\lambda})$, then any element of $\mathrm{GL}(2)_{\boldsymbol{A}}$ may be written uniquely as $g = c(\lambda)g^0$ with $g^0 \in \mathrm{GL}(2)^0_{\boldsymbol{A}}$. We choose a Haar measure on $\mathrm{GL}(2)^0_{\boldsymbol{A}}$ so that $dg = 2d^\times\lambda\, dg^0$. It is well-known that the volume of $\mathrm{GL}(2)^0_{\boldsymbol{A}}/\mathrm{GL}(2)_k$ with respect to dg^0 is

$$\mathfrak{V}_k = 1/\operatorname{Res}_{s=1}(Z_k(s)/Z_k(s+1)) = \mathfrak{C}_k^{-1}Z_k(2)\,. \tag{4.3}$$

As in Section 2, we note that all these definitions apply equally well to the number field $\tilde{k}$ and yield a measure on $\mathrm{GL}(2)_{\tilde{\boldsymbol{A}}}$ and so on. Having chosen an invariant measure on $\mathrm{GL}(2)$ both locally and adèlically, we also get local and adèlic invariant measures on G by taking the relevant product measures in each case.

5. The canonical measure on the stabilizer. In this section we shall define a measure on $G^\circ_{x\,\boldsymbol{A}}$ for $x \in V_k^{\mathrm{ss}}$ which is canonical (in a sense made precise by Proposition 5.16) and compute the volume of $G^\circ_{x\,\boldsymbol{A}}/\tilde{T}_{\boldsymbol{A}}G^\circ_{x\,k}$ under this measure. We also make a canonical choice of measure on the stabilizer quotient $G_{\boldsymbol{A}}/G^\circ_{x\,\boldsymbol{A}}$ and define constants $b_{x,v}$ which will play an essential role in what follows.

Let $v \in \mathfrak{M}$ and $x \in V_{k_v}^{\mathrm{ss}}$. If $v \notin \mathfrak{M}_{\mathrm{sp}}$, then v extends uniquely to a place of $\tilde{k}$ which we also denote by v. In this case $\tilde{k}_v \cong k_v \otimes_k \tilde{k}$. We denote by $\tilde{k}_v(x)$ the composite of $\tilde{k}_v$ and $k_v(x)$.

Before we begin this task it will be convenient for bookkeeping purposes to attach to each orbit in $V_{k_v}^{\mathrm{ss}}$, where $v \in \mathfrak{M}$, an index which records the arithmetic properties of v and of the extension of k_v corresponding to the orbit. The orbit corresponding to k_v itself will have index (sp), (in) or (rm) according as v is in $\mathfrak{M}_{\mathrm{sp}}$, $\mathfrak{M}_{\mathrm{in}}$ or $\mathfrak{M}_{\mathrm{rm}}$. The orbit corresponding to the unique unramified quadratic extension of k_v will have index (sp ur), (in ur) and (rm ur) for $v \in \mathfrak{M}_{\mathrm{sp}}$, $v \in \mathfrak{M}_{\mathrm{in}}$ and $v \in \mathfrak{M}_{\mathrm{rm}}$, respectively. An orbit corresponding to a ramified quadratic extension of k_v will have index (sp rm) if $v \in \mathfrak{M}_{\mathrm{sp}}$ and (in rm) if $v \in \mathfrak{M}_{\mathrm{in}}$. If $v \in \mathfrak{M}_{\mathrm{rm}}$, then the orbits corresponding to ramified quadratic extensions of k_v are subdivided into three types; the one corresponding to $\tilde{k}_v$ has index (rm rm)*, those corresponding to quadratic extensions $k_v(x)/k_v$ such that $k_v(x) \neq \tilde{k}_v$ and $\tilde{k}_v(x)/\tilde{k}_v$ is unramified have index (rm rm ur) and those corresponding to quadratic extensions $k_v(x)/k_v$ such that $k_v(x) \neq \tilde{k}_v$ and $\tilde{k}_v(x)/\tilde{k}_v$ is ramified have index (rm rm rm). This last index can occur only if $v \in \mathfrak{M}_{\mathrm{dy}}$.

From Section 3 we know that the group $G^{\circ}_{x\,k_v}$ may be determined up to isomorphism solely from the index of the orbit of x. In fact, if we define

$$H_{x\,k_v} = \begin{cases} (k_v^{\times})^4 & \text{(sp)}\,, \\ (k_v(x)^{\times})^2 & \text{(sp ur), (sp rm)}\,, \\ (\tilde{k}_v^{\times})^2 & \text{(in), (rm), (in ur), (rm rm)*}\,, \\ \tilde{k}_v(x)^{\times} & \text{otherwise}\,, \end{cases} \tag{5.1}$$

for each of the various indices, then $G^{\circ}_{x\,k_v} \cong H_{x\,k_v}$ in all cases. We may regard $H_{x\,k_v}$ as the k_v-points of an algebraic group H_x defined over $\mathcal{O}_v$ and we shall do so below.

As in Section 3, if $k_v(x)/k_v$ is quadratic, then we shall write ν for the generator of $\mathrm{Gal}(k_v(x)/k_v)$. If $\tilde{k}_v(x) \neq \tilde{k}_v$, then ν may also be regarded as the generator of $\mathrm{Gal}(\tilde{k}_v(x)/\tilde{k}_v)$. Also the *type* of $x \in V_{k_v}^{\mathrm{ss}}$ will be the index attached to the orbit $G_{k_v}x$.

We wish to introduce parameterizations for the elements of the stabilizer in the various cases. If x is a point of type (sp), we write

$$s_x(t_x) = (a(t_{11}, t_{12}), a(t_{21}, t_{22}), a((t_{11}t_{21})^{-1}, (t_{12}t_{22})^{-1}))\,, \tag{5.2}$$

where $t_x = (t_{11}, \ldots, t_{22}) \in (k_v^{\times})^4$. Let $s_{x1}(t_x), s_{x2}(t_x), s_{x3}(t_x)$ be the three components of $s_x(t_x)$. If x is a point of type (sp ur) or (sp rm), we write

$$s_x(t_x) = (a(t_{11}, t_{11}^{\nu}), a(t_{21}, t_{21}^{\nu}), a((t_{11}t_{21})^{-1}, (t_{11}^{\nu}t_{21}^{\nu})^{-1}))\,, \tag{5.3}$$

where $t_x = (t_{11}, t_{21}) \in (k(x)_v^{\times})^2$. We use the notation $s_{x1}(t_x)$ et cetera in this case also. If x is a point of type (in) or (rm), then we write

$$s_x(t_x) = (a(t_{11}, t_{12}), a(\mathrm{N}_{\tilde{k}_v/k_v}(t_{11}^{-1}), \mathrm{N}_{\tilde{k}_v/k_v}(t_{12}^{-1}))\,, \tag{5.4}$$

where $t_x = (t_{11}, t_{12}) \in (\tilde{k}_v^\times)^2$. We use the notation $s_{x1}(t_x)$ et cetera in this case also. If x is a point of type (in ur) or (rm rm)*, then we write

$$s_x(t_x) = (a(t_{11}, t_{12}), a(t_{12}^\sigma, t_{11}^\sigma), a((t_{11}t_{12}^\sigma)^{-1}, (t_{11}^\sigma t_{12})^{-1}))\,, \tag{5.5}$$

where $t_x = (t_{11}, t_{12}) \in (\tilde{k}_v^\times)^2$. We use the notation $s_{x1}(t_x)$ et cetera in this case also. Finally, if x is a point of type (in rm), (rm ur), (rm rm ur) or (rm rm rm), then we write

$$s_x(t_x) = (a(t_{11}, t_{11}^\nu), a(\mathrm{N}_{\tilde{k}_v(x)/k_v(x)}(t_{11}^{-1}), \mathrm{N}_{\tilde{k}_v(x)/k_v(x)}(t_{11}^{-1})^\nu))\,, \tag{5.6}$$

where $t_x = t_{11} \in \tilde{k}_v(x)^\times$. We use the notation $s_{x1}(t_x)$ et cetera in this case also. On $H_{x\,k_v}$ we define an invariant measure $dt_{x,v}$ as follows:

$$dt_{x,v} = \begin{cases} d^\times t_{11v}\, d^\times t_{12v}\, d^\times t_{21v}\, d^\times t_{22v} & \text{(sp)}\,, \\ d^\times t_{11v}\, d^\times t_{21v} & \text{(sp ur), (sp rm)}\,, \\ d^\times t_{11v}\, d^\times t_{12v} & \text{(in), (rm), (in ur), (rm rm)*}\,, \\ d^\times t_{11v} & \text{otherwise.} \end{cases} \tag{5.7}$$

We note that if $v \in \mathfrak{M}_\mathrm{f}$, then the volume of $H_{x\,\mathcal{O}_v}$ under this measure is 1 in every case.

Suppose that $x \in V_{k_v}^{\mathrm{ss}}$ corresponds to a quadratic extension of k_v. Then it is possible to choose an element $g_x \in G_{k_v(x)}$ such that $x = g_x w$. Consider the following condition on such an element.

CONDITION 5.8. $g_x^{-1} g_x^\nu = (-\tau, -\tau, \tau)$ or $(-\tau, \tau)$.

It is possible to find g_x satisfying this condition for any x. Indeed, $x = g_{xw_p} w_p$ for some $g_{xw_p} \in G_{k_v}$ and some choice of p. Then $x = g_{xw_p} g_p w$ and $g_x = g_{xw_p} g_p \in G_{k_v(x)}$ satisfies the condition.

PROPOSITION 5.9. *If* g_x *satisfies Condition* 5.8, *then*

$$G_{x\,k_v}^\circ = g_x\{s_x(t_x) \mid t_x \in H_{x\,k_v}\} g_x^{-1}\,. \tag{5.10}$$

PROOF. We have $k_v(x) = k_v(w_p)$ for some p. Since g_x and g_p both satisfy Condition 5.8, $g_x g_p^{-1} \in G_{k_v}$ and if we put $h = g_x g_p^{-1}$, then $hw_p = x$, and so $G_{x\,k_v}^\circ = hG_{w_p\,k_v}^\circ h^{-1}$. From Section 3,

$$G_{w_p\,k_v}^\circ = g_p\{s_x(t_x) \mid t_x \in H_{x\,k_v}\} g_p^{-1} \tag{5.11}$$

and the conclusion follows. □

If $x = g_x w$ with $g_x \in G_{k_v}$, then we need not impose any condition on g_x.

Suppose now that $g_x \in G_{k_v(x)}$, $x = g_x w$ and g_x satisfies Condition 5.8 if $k_v(x) \neq k_v$. Then we can define an isomorphism $\theta_{g_x} : G_{x\,k_v}^\circ \to H_{x\,k_v}$ by setting $\theta_{g_x}(g_x s_x(t_x) g_x^{-1}) = t_x$. If g_{x1} and g_{x2} are two such elements, then let $h = g_{x2} g_{x1}^{-1}$. From the condition, we see that

$h \in G_{x\,k_v}$. Also

$$\begin{aligned} \theta_{g_{x1}}(g) &= s_x^{-1}(g_{x1}^{-1} g\, g_{x1}) \\ &= s_x^{-1}(g_{x2}^{-1} h g h^{-1} g_{x2}) \\ &= \theta_{g_{x2}}(hgh^{-1})\,, \end{aligned} \tag{5.12}$$

and so $\theta_{g_{x1}}$ and $\theta_{g_{x2}}$ differ by the automorphism $g \mapsto hgh^{-1}$ of $G^\circ_{x\,k_v}$. Since $G^\circ_{x\,k_v}$ is abelian and $G_{x\,k_v}/G^\circ_{x\,k_v}$ has order two, the automorphism $g \mapsto hgh^{-1}$ depends only on the class of h in $G_{x\,k_v}/G^\circ_{x\,k_v}$ and either is the identity (if $h \in G^\circ_{x\,k_v}$) or squares to the identity (if $h \notin G^\circ_{x\,k_v}$). In either case, this automorphism is measure preserving and hence we may make the following definition without ambiguity.

DEFINITION 5.13. Let $dg''_{x,v} = \theta^*_{g_x}(dt_{x,v})$ for any choice of $g_x \in G_{k_v(x)}$ such that $g_x w = x$ and g_x satisfies Condition 5.8 if $k_v(x) \neq k_v$.

This establishes a choice of invariant measure on $G^\circ_{x\,k_v}$ for each $x \in V^{\mathrm{ss}}_{k_v}$.

We have

$$\tilde{T}_{k_v} = \begin{cases} \{(t_1 I_2, t_2 I_2, (t_1 t_2)^{-1} I_2)\} & \text{in case (1)}\,, \\ \{(t_1 I_2, \mathrm{N}_{\tilde{k}_v/k_v}(t_1)^{-1} I_2)\} & \text{in case (2)}\,, \end{cases} \tag{5.14}$$

and so $\tilde{T}_{k_v} \cong (k_v^\times)^2$ in case (1) and $\tilde{T}_{k_v} \cong \tilde{k}_v^\times$ in case (2). We use the measure

$$d^\times \tilde{t}_v = \begin{cases} d^\times t_{1v}\, d^\times t_{2v} & \text{in case (1)}\,, \\ d^\times t_{1v} & \text{in case (2)}\,, \end{cases} \tag{5.15}$$

on this group. We let $d\tilde{g}''_{x,v}$ be the measure on $G^\circ_{x\,k_v}/\tilde{T}_{k_v}$ such that $dg''_{x,v} = d\tilde{g}''_{x,v}\, d^\times \tilde{t}_v$.

It is to achieve the following result that we have taken such pains with the definition of the measures.

PROPOSITION 5.16. *Suppose that $x, y \in V^{\mathrm{ss}}_{k_v}$ and that $y = g_{xy} x$ for some $g_{xy} \in G_{k_v}$. Let $i_{g_{xy}} : G^\circ_{y\,k_v} \to G^\circ_{x\,k_v}$ be the isomorphism $i_{g_{xy}}(g) = g_{xy}^{-1} g\, g_{xy}$. Then*

$$dg''_{y,v} = i^*_{g_{xy}}(dg''_{x,v}) \quad \textit{and} \quad d\tilde{g}''_{y,v} = i^*_{g_{xy}}(d\tilde{g}''_{x,v})\,. \tag{5.17}$$

PROOF. Let g_x be chosen as above and put $g_y = g_{xy} g_x$. Then $g_y \in G_{k_v(y)} = G_{k_v(x)}$, $g_y w = y$ and if $k_v(y) \neq k$, then $g_y^{-1} g_y^{v} = g_x^{-1} g_x^{v}$, so that g_y satisfies Condition 5.8 in this case. It follows that

$$\begin{aligned} i^*_{g_{xy}}(dg''_{x,v}) &= i^*_{g_{xy}} \theta^*_{g_x}(dt_{x,v}) \\ &= (\theta_{g_x} i_{g_{xy}})^*(dt_{x,v}) \\ &= \theta^*_{g_y}(dt_{y,v}) \\ &= dg''_{y,v} \end{aligned} \tag{5.18}$$

because $H_{x\,k_v} = H_{y\,k_v}$ and $dt_{x,v} = dt_{y,v}$. This establishes the first claim and the second then follows from the observation that $i_{g_{xy}}|_{\tilde{T}_{k_v}}$ is the identity map. □

We choose a left invariant measure $dg'_{x,v}$ on $G_{k_v}/G^\circ_{x\,k_v}$ so that if $\Phi\in\mathcal{S}(V_{k_v})$, then

$$\int_{G_{k_v}/G^\circ_{x\,k_v}} |P(g'_{x,v}x)|_v^s\Phi(g'_{x,v}x)dg'_{x,v}=\int_{G_{k_v}x}|P(y)|_v^{s-2}\Phi(y)dy\,, \tag{5.19}$$

where dy is the Haar measure such that the volume of $V_{\mathcal{O}_v}$ is one if $v\in\mathfrak{M}_{\mathrm{f}}$, Lebesgue measure if $v\in\mathfrak{M}_{\boldsymbol{R}}$ and 2^8 times Lebesgue measure if $v\in\mathfrak{M}_{\boldsymbol{C}}$. This is possible because $|P(y)|_v^{-2}\,dy$ is a G_{k_v}-invariant measure on $V^{\mathrm{ss}}_{k_v}$ and each of the orbits $G_{k_v}x$ is an open set in $V^{\mathrm{ss}}_{k_v}$. Note that

$$\begin{aligned}&\int_{G_{k_v}/G^\circ_{x\,k_v}}|\chi(g'_{x,v})|_v^s\Phi(g'_{x,v}x)dg'_{x,v}\\&\qquad=|P(x)|_v^{-s}\int_{G_{k_v}/G^\circ_{k_v}}|P(g'_{x,v}x)|_v^s\Phi(g'_{x,v}x)dg'_{x,v}\end{aligned}\tag{5.20}$$

and so, from (5.19), this integral converges absolutely at least when $\mathrm{Re}(s)>2$. If $g_{xy}\in G_{k_v}$ satisfies $y=g_{xy}x$ and $i_{g_{xy}}$ is the inner automorphism $g\mapsto g_{xy}^{-1}gg_{xy}$ of G_{k_v}, then $i_{g_{xy}}(G^\circ_{y\,k_v})=G^\circ_{x\,k_v}$, and so $i_{g_{xy}}$ induces a map $i_{g_{xy}}:G_{k_v}/G^\circ_{y\,k_v}\to G_{k_v}/G^\circ_{x\,k_v}$. Since the integral on the right hand side of (5.19) depends only on the orbit of x, it follows that $i^*_{g_{xy}}(dg'_{x,v})=dg'_{y,v}$.

DEFINITION 5.21. For $v\in\mathfrak{M}$ and $x\in V^{\mathrm{ss}}_{k_v}$ we let $b_{x,v}>0$ be the constant verifying $dg_v=b_{x,v}\,dg'_{x,v}\,dg''_{x,v}$, where dg_v is the measure on G_{k_v} chosen at the end of Section 4.

DEFINITION 5.22. For $\Phi\in\mathcal{S}(V_{k_v})$ and $s\in\boldsymbol{C}$ we define

$$\begin{aligned}Z_{x,v}(\Phi,s)&=b_{x,v}\int_{G_{k_v}/G^\circ_{x\,k_v}}|\chi(g'_{x,v})|_v^s\Phi(g'_{x,v}x)dg'_{x,v}\\&=b_{x,v}|P(x)|_v^{-s}\int_{G_{k_v}x}|P(y)|_v^{s-2}\Phi(y)dy\,.\end{aligned}$$

PROPOSITION 5.23. *If $x,y\in V^{\mathrm{ss}}_{k_v}$ and $G_{k_v}x=G_{k_v}y$, then $b_{x,v}=b_{y,v}$.*

PROOF. Since the group G_{k_v} is unimodular $i^*_{g_{x,y}}dg_v=dg_v$. So

$$\begin{aligned}dg_v&=b_{y,v}dg'_{y,v}dg''_{y,v}\\&=b_{y,v}i^*_{g_{x,y}}dg'_{x,v}i^*_{g_{x,y}}dg''_{x,v}=b_{y,v}b^{-1}_{x,v}i^*_{g_{x,y}}dg_v\\&=b_{y,v}b^{-1}_{x,v}dg_v\,.\end{aligned}$$

Therefore $b_{x,v}=b_{y,v}$. □

Let $d_{\mathrm{pr}}g''_x=\prod_v dg''_{x,v}$, $d_{\mathrm{pr}}\tilde{g}''_x=\prod_v d\tilde{g}''_{x,v}$ and $d^\times_{\mathrm{pr}}\tilde{t}=\prod_v d^\times\tilde{t}_v$, where $d^\times\tilde{t}_v$ is defined in (5.15).

PROPOSITION 5.24. *Suppose $x\in V^{\mathrm{ss}}_k$ and $k(x)\neq k,\tilde{k}$. Then, with respect to the measure $d_{\mathrm{pr}}\tilde{g}''_x$, the volume of $G^\circ_{x\,\boldsymbol{A}}/\tilde{T}_{\boldsymbol{A}}G^\circ_{x\,k}$ is $2\mathfrak{C}_{\tilde{k}(x)}/\mathfrak{C}_{\tilde{k}}$.*

PROOF. Identifying $\tilde{T}$ with $\mathrm{GL}(1)_{\tilde{k}}$ and G_x° with $\mathrm{GL}(1)_{\tilde{k}(x)}$, we define $\tilde{T}_{\boldsymbol{A}}^1$ (resp. $G_{x\boldsymbol{A}}^{\circ 1}$) to be the set of idèles of $\tilde{k}$ (resp. $\tilde{k}(x)$) with absolute value one. Let $d_{\mathrm{pr}}^\times \tilde{t}^1$ and $d_{\mathrm{pr}} g_x''^1$ be the measures on $\tilde{T}_{\boldsymbol{A}}^1$ and $G_{x\boldsymbol{A}}^{\circ 1}$, such that $d_{\mathrm{pr}} g_x'' = d^\times \lambda d_{\mathrm{pr}} g_x''^1$, $d_{\mathrm{pr}}^\times \tilde{t} = d^\times \lambda d_{\mathrm{pr}}^\times \tilde{t}^1$ for

$$g_x'' = \underline{\lambda}_{\tilde{k}(x)} g_x''^1 , \quad \tilde{t} = \underline{\lambda}_{\tilde{k}} \tilde{t}^1 .$$

Note that if $\lambda \in \boldsymbol{R}_+$, then the absolute value of $\underline{\lambda}_{\tilde{k}}$ as an idèle of $\tilde{k}(x)$ is λ^2. Therefore, $d_{\mathrm{pr}} g_x'' = 2 d^\times \lambda d_{\mathrm{pr}} g_x''^1$ for $g_x'' = \underline{\lambda}_{\tilde{k}} g_x''^1$. Since $d_{\mathrm{pr}} g_x'' = d_{\mathrm{pr}}^\times \tilde{t} d_{\mathrm{pr}} \tilde{g}_x''$, this implies that $2 d_{\mathrm{pr}} g_x''^1 = d_{\mathrm{pr}}^\times \tilde{t}^1 d_{\mathrm{pr}} \tilde{g}_x''$. So

$$\begin{aligned} 2 \int_{G_{x\boldsymbol{A}}^{\circ 1}/G_{x\,k}^\circ} d_{\mathrm{pr}} g_x''^1 &= \int_{G_{x\boldsymbol{A}}^{\circ 1}/G_{x\,k}^\circ \tilde{T}_{\boldsymbol{A}}^1} d_{\mathrm{pr}} \tilde{g}_x'' \int_{\tilde{T}_{\boldsymbol{A}}^1/\tilde{T}_k} d_{\mathrm{pr}}^\times \tilde{t}^1 \\ &= \mathrm{vol}(G_{x\boldsymbol{A}}^{\circ 1}/\tilde{T}_{\boldsymbol{A}}^1 G_{x\,k}^\circ) \int_{\tilde{T}_{\boldsymbol{A}}^1/\tilde{T}_k} d_{\mathrm{pr}}^\times \tilde{t}^1 . \end{aligned}$$

Since

$$\int_{G_{x\boldsymbol{A}}^{\circ 1}/G_{x\,k}^\circ} d_{\mathrm{pr}} \tilde{g}_x''^1 = \mathfrak{C}_{\tilde{k}(x)} \quad \text{and} \quad \int_{\tilde{T}_{\boldsymbol{A}}^1/\tilde{T}_k} d_{\mathrm{pr}}^\times \tilde{t}^1 = \mathfrak{C}_{\tilde{k}}$$

this proves the proposition. □

For the rest of this paper we consider Case (2) in both the global and the local situations and Case (1) only in the local situation, where it arises as a localization of Case (2).

6. A preliminary mean value theorem and the formulation of its proof. In this section we introduce the global zeta function of the prehomogeneous vector space (G, V) and recall from [27] its most basic analytic properties. The zeta function is approximately the Dirichlet generating series for the sequence $\mathrm{vol}(G_{x\boldsymbol{A}}^\circ/\tilde{T}_{\boldsymbol{A}} G_{x\,k}^\circ)$. If it were exactly this generating series, then our work would be almost complete, since Tauberian theory would allow us to extract the mean value of the coefficients from the analytic behavior of the series. Unfortunately, the actual zeta function contains an additional factor in each term and we proceed to explain the filtering process by which this difficulty may be surmounted. This leads us, on the basis of a number of assumptions, to a preliminary form of the mean value theorem that is our goal. The validity of these assumptions is demonstrated in later sections. The final form of the theorem, which differs from the preliminary form mostly in being more explicit, is given in the next section.

We put $G_1 = \mathrm{GL}(2)_{\tilde{k}}$ and $G_2 = \mathrm{GL}(2)$. Let $G_{\boldsymbol{A}} = G_{1\boldsymbol{A}} \times G_{2\boldsymbol{A}}$, let dg_1 and dg_2 be the measures on $G_{1\boldsymbol{A}}$ and $G_{2\boldsymbol{A}}$ which were defined in Section 4 and put $dg = dg_1 dg_2$ for $g = (g_1, g_2)$; this is a Haar measure on $G_{\boldsymbol{A}}$. Write $\tilde{G} = G/\tilde{T}$, so that V is a faithful representation of $\tilde{G}$. Since $\tilde{T} \cong \mathrm{GL}(1)_{\tilde{k}}$ as groups over k, the first Galois cohomology group of $\tilde{T}$ is trivial, and it follows that $\tilde{G}_F \cong G_F/\tilde{T}_F$ for any field $F \supseteq k$. Thus $\tilde{G}_{\boldsymbol{A}} \cong G_{\boldsymbol{A}}/\tilde{T}_{\boldsymbol{A}}$ and $\tilde{G}_{\boldsymbol{A}}/\tilde{G}_k \cong G_{\boldsymbol{A}}/\tilde{T}_{\boldsymbol{A}} G_k$. Let $d_{\mathrm{pr}}^\times \tilde{t}$ be the measure on $\tilde{T}_{\boldsymbol{A}}$ defined immediately before Proposition 5.24. Then $d^\times \tilde{t} = \mathfrak{C}_{\tilde{k}}^{-1} d_{\mathrm{pr}}^\times \tilde{t}$ is the measure on $\tilde{T}_{\boldsymbol{A}}$ compatible under the isomorphism $\tilde{T}_{\boldsymbol{A}} \cong \tilde{\boldsymbol{A}}^\times$ with the measure defined on $\tilde{\boldsymbol{A}}^\times$ in Section 2. We choose the measure $d\tilde{g}$ on $\tilde{G}_{\boldsymbol{A}}$ which satisfies $dg = d\tilde{g}\, d^{\times}\tilde{t}$. Similarly, we choose the measure $d\tilde{g}_v$ on $\tilde{G}_{k_v}$ which satisfies $dg_v = d\tilde{g}_v\, d^{\times}\tilde{t}_v$. Let $d_{\mathrm{pr}}\tilde{g} = \prod_v d\tilde{g}_v$. From (4.2), we obtain

$$d\tilde{g} = |\Delta_k \Delta_{\tilde{k}}|^{-1/2} \mathfrak{C}_k^{-2} \mathfrak{C}_{\tilde{k}}^{-1} d_{\mathrm{pr}}\tilde{g}\,. \tag{6.1}$$

DEFINITION 6.2. Let $L_0 = \{x \in V_k^{\mathrm{ss}} \mid k(x) \neq k, \tilde{k}\}$. For $\Phi \in \mathcal{S}(V_{\boldsymbol{A}})$ and $s \in \boldsymbol{C}$ we define

$$Z(\Phi, s) = \int_{G_{\boldsymbol{A}}/\tilde{T}_{\boldsymbol{A}} G_k} |\chi(\tilde{g})|^s \sum_{x \in L_0} \Phi(\tilde{g}x) d\tilde{g}\,.$$

The integral $Z(\Phi, s)$ is called the *global zeta function* of (G, V). It was proved in [27] that the integral converges (absolutely and uniformly on compacta) if $\mathrm{Re}(s)$ is sufficiently large. However, a slightly different formulation was used in [27] and it is necessary to say a few words about the translation from that paper to this.

The definition of the zeta function used in [27] is stated in Definition (2.10) of that paper. For our purposes we shall always take the character ω appearing there to be the trivial character. The domain of integration used in [27] is $\boldsymbol{R}_+ \times G_{\boldsymbol{A}}^0/G_k$, where $G_{\boldsymbol{A}}^0 = G_{1\boldsymbol{A}}^0 \times G_{2\boldsymbol{A}}^0$ is the set of elements of $G_{\boldsymbol{A}}$ both of whose entries have determinant of idèle norm 1. We have $(\boldsymbol{R}_+ \times G_{\boldsymbol{A}}^0)/\tilde{T}_{\boldsymbol{A}}^1 \cong \tilde{G}_{\boldsymbol{A}}$ via the map which sends the class of (λ, g^0) to the class of $(1, c(\underline{\lambda}))g^0$. In [27], $\boldsymbol{R}_+ \times G_{\boldsymbol{A}}^0$ is made to act on $V_{\boldsymbol{A}}$ by requiring that $(\lambda, 1)$ acts by multiplication by $\underline{\lambda}$ and the above isomorphism is compatible with this.

We must compare the measure $d\tilde{g}$ on $\tilde{G}_{\boldsymbol{A}}$ with the measure $d^{\times}\lambda\, dg^0$ which was used in [27]. We have $\tilde{G}_{\boldsymbol{A}} \cong (\boldsymbol{R}_+^2 \times G_{\boldsymbol{A}}^0)/(\boldsymbol{R}_+ \times \tilde{T}_{\boldsymbol{A}}^1)$ where $\boldsymbol{R}_+ \times \tilde{T}_{\boldsymbol{A}}^1$ is included in $\boldsymbol{R}_+^2 \times G_{\boldsymbol{A}}^0$ via $(\lambda, \tilde{t}) \mapsto (\lambda, \lambda^{-1}, \tilde{t})$ and $\boldsymbol{R}_+^2 \times G_{\boldsymbol{A}}^0$ maps onto $\tilde{G}_{\boldsymbol{A}}$ via $(\lambda_1, \lambda_2, g^0) \mapsto (c(\underline{\tilde{\lambda}}_1), c(\underline{\lambda}_2))g^0 \cdot \tilde{T}_{\boldsymbol{A}}$ (recall that $\underline{\tilde{\lambda}}_1 \in \tilde{\boldsymbol{A}}$ and $\underline{\lambda}_2 \in \boldsymbol{A}$). In this quotient we have chosen the measure $d\tilde{g}$ to be compatible with the measures $4d^{\times}\lambda_1\, d^{\times}\lambda_2\, dg^0$ on $\boldsymbol{R}_+^2 \times G_{\boldsymbol{A}}^0$ and $d^{\times}\lambda\, d^{\times}\tilde{t}^1$ on $\boldsymbol{R}_+ \times \tilde{T}_{\boldsymbol{A}}^1$, where the volume of $\tilde{T}_{\boldsymbol{A}}^1/\tilde{T}_k$ under $d^{\times}\tilde{t}^1$ is 1 (as in Section 2). From this it follows that the measures $4d^{\times}\lambda\, dg^0$ and $d^{\times}\tilde{t}^1$ are compatible with the measure $d\tilde{g}$ in the quotient $(\boldsymbol{R}_+ \times G_{\boldsymbol{A}}^0)/\tilde{T}_{\boldsymbol{A}}^1 \cong \tilde{G}_{\boldsymbol{A}}$.

Furthermore, $|\chi(1, c(\underline{\lambda}))| = \lambda^4$, and so if $Z^*(\Phi, s)$ denotes the zeta function studied in [27], then we have $Z(\Phi, s) = 4Z^*(\Phi, 4s)$. In [27], Corollary 8.16 it is shown that $Z^*(\Phi, s)$ has a meromorphic continuation to the region $\mathrm{Re}(s) > 6$ with a simple pole at $s = 8$ with residue $\mathfrak{V}_k \mathfrak{V}_{\tilde{k}} \hat{\Phi}(0)$. Thus we arrive at:

THEOREM 6.3. *The zeta function $Z(\Phi, s)$ has a meromorphic continuation to the region* $\mathrm{Re}(s) > 3/2$ *with a simple pole at $s = 2$ with residue* $\mathfrak{V}_k \mathfrak{V}_{\tilde{k}} \hat{\Phi}(0)$.

Note that $\hat{\Phi}(0)$ is the Fourier transform of Φ evaluated at the origin, and so is simply the integral of Φ over the $V_{\boldsymbol{A}}$. We define $\Sigma(\Phi) = \hat{\Phi}(0)$ for $\Phi \in \mathcal{S}(V_{\boldsymbol{A}})$. For $v \in \mathfrak{M}$ and $\Phi_v \in \mathcal{S}(V_{k_v})$ we can define the local version of the distribution $\Sigma(\Phi)$ by

$$\Sigma_v(\Phi_v) = \int_{V_{k_v}} \Phi_v(y) dy\,. \tag{6.4}$$

Since the coordinate system of V consists of four coordinates in k and two coordinates in $\tilde{k}$, if $\Phi = \bigotimes_v \Phi_v$, then

$$\Sigma(\Phi) = |\Delta_k|^{-2}|\Delta_{\tilde{k}}|^{-1} \prod_v \Sigma_v(\Phi_v)\,. \tag{6.5}$$

This completes our review of the analytic properties of the global zeta function. Before we can rewrite $Z(\Phi, s)$ in a form which makes this analytic information bear on the problem at hand we must return briefly to the local situation.

Let $v \in \mathfrak{M}_{\mathrm{f}}$. If F/k_v is a quadratic extension, then F is generated over k_v by either of the roots of some irreducible polynomial $p(z) = z^2 + a_1 z + a_2 \in k_v[z]$. In fact, this polynomial may always be chosen to satisfy the more stringent condition that $\mathcal{O}_F$ is generated over $\mathcal{O}_v$ by either of the roots of $p(z)$. If this condition is satisfied, then the discriminant of $p(z)$ generates the ideal Δ_{F/k_v}. We wish to recall how this may be achieved in each case.

Recall that $p(z) \in k_v[z]$ is called an *Eisenstein polynomial* if $a_1 \in \mathfrak{p}_v$ and $a_2 \in \mathfrak{p}_v \setminus \mathfrak{p}_v^2$. If F/k_v is a ramified extension, then there is always an Eisenstein polynomial whose roots generate F over k_v and any such polynomial will satisfy the stronger condition stated above. For each $v \in \mathfrak{M}_{\mathrm{f}}$, k_v has a unique unramified quadratic extension. If F is this extension and $v \notin \mathfrak{M}_{\mathrm{dy}}$, then we may satisfy the stronger condition simply by taking $p(z)$ with $a_1 = 0$ and $-a_2$ any non-square unit in k_v. If $v \in \mathfrak{M}_{\mathrm{dy}}$, then we must instead take $p(z)$ to be an *Artin-Schreier polynomial*, which means, by definition, that $p(z)$ is irreducible in $k_v[z]$, $a_1 = -1$ and a_2 is a unit. Note that p stays irreducible modulo $\mathfrak{p}_v$ in this case by Hensel's lemma.

For each $v \in \mathfrak{M}_{\mathrm{f}}$ we choose a list of representatives $w_{v,1}, \ldots, w_{v,N_v}$, one for each of the G_{k_v}-orbits in $V_{k_v}^{\mathrm{ss}}$, in such a way that $P(w_{v,i})$ generates the ideal $\Delta_{k(w_{v,i})/k_v}$ for $i = 1, \ldots, N_v$. This is possible, in light of the previous paragraph, if we take each $w_{v,i}$ to equal w_p for a suitable $p(z) \in k_v[z]$. In the special case where $k(w_{v,i}) = k_v$ we take $w_{v,i} = w_p$ for $p(z) = z^2 - z$. For $v \in \mathfrak{M}_\infty$ we require instead that $|P(w_{v,i})|_v = 1$ for $i = 1, \ldots, N_v$, which is clearly possible. In both cases we assume for convenience that $w_{v,1}$ represents the orbit corresponding to k_v itself. This done, if F/k is a quadratic extension, then let $w_{v,i_v(F)}$ represent the orbit corresponding to F_v/k_v (with $i_v(F) = 1$ if v splits in F). Then we have

$$\mathcal{N}(\Delta_{F/k})^{-1} = \prod_{v\in\mathfrak{M}_{\mathrm{f}}} \mathcal{N}_v(\Delta_{F/k,v})^{-1} = \prod_{v\in\mathfrak{M}_{\mathrm{f}}} |P(w_{v,i_v(F)})|_v = \prod_{v\in\mathfrak{M}} |P(w_{v,i_v(F)})|_v\,. \tag{6.6}$$

For $x \in L_0$ and $\Phi = \bigotimes \Phi_v \in \mathcal{S}(V_{\mathbf{A}})$ we define the *orbital zeta function* of x to be $Z_x(\Phi, s) = \prod_{v\in\mathfrak{M}} Z_{x,v}(\Phi_v, s)$. If x lies in the orbit of $w_{v,i}$ in $V_{k_v}^{\mathrm{ss}}$, then we shall write $\Xi_{x,v}(\Phi_v, s) = Z_{w_{v,i},v}(\Phi_v, s)$ and $\Xi_x(\Phi, s) = \prod_{v\in\mathfrak{M}} \Xi_{x,v}(\Phi_v, s)$. We call $\Xi_{x,v}(\Phi_v, s)$ the *standard local zeta function* and $\Xi_x(\Phi_v, s)$ the *standard orbital zeta function.*

PROPOSITION 6.7. *For $x \in L_0$ and $\Phi = \bigotimes \Phi_v \in \mathcal{S}(V_{\mathbf{A}})$ we have*

$$Z_x(\Phi, s) = \mathcal{N}(\Delta_{k(x)/k})^{-s}\,\Xi_x(\Phi, s)\,.$$

PROOF. For each $v \in \mathfrak{M}$ let $i_v(x)$ be such that $x \in G_{k_v} w_{v,i_v(x)}$. Then, from (5.22),

$$
\begin{aligned}
Z_{x,v}(\Phi_v, s) &= b_{x,v}|P(x)|_v^{-s} \int_{G_{k_v}x} |P(y)|_v^{s-2}\Phi_v(y)dy \\
&= \frac{|P(w_{v,i_v(x)})|_v^s}{|P(x)|_v^s} \cdot \frac{b_{w_{v,i_v(x)},v}}{|P(w_{v,i_v(x)})|_v^s} \int_{G_{k_v}w_{v,i_v(x)}} |P(y)|_v^{s-2}\Phi_v(y)dy \\
&= \frac{|P(w_{v,i_v(x)})|_v^s}{|P(x)|_v^s} \cdot Z_{w_{v,i_v(x)},v}(\Phi_v, s) \\
&= \frac{|P(w_{v,i_v(x)})|_v^s}{|P(x)|_v^s} \cdot \Xi_{x,v}(\Phi_v, s)\,,
\end{aligned}
\tag{6.8}
$$

where we have used Proposition 5.23 in passing from the first line to the second. Applying (6.6) to $F = k(x)$, we find that $\prod_{v\in\mathfrak{M}} |P(w_{v,i_v(x)})|_v^s = \mathcal{N}(\Delta_{k(x)/k})^{-s}$. Since $x \in V_k^{\mathrm{ss}}$, $P(x) \in k^\times$, and so the Artin product formula implies that $\prod_{v\in\mathfrak{M}} |P(x)|_v = 1$. Now taking the product over all $v \in \mathfrak{M}$ on both sides of (6.8) proves the identity. □

For convenience, we introduce the abbreviation

$$
\mathcal{R}_1 = |\Delta_k|^{-1/2}|\Delta_{\tilde{k}}|^{-1/2}\mathfrak{C}_k^{-2}\mathfrak{C}_{\tilde{k}}^{-2}\,. \tag{6.9}
$$

PROPOSITION 6.10. *If $\Phi = \bigotimes \Phi_v \in \mathcal{S}(V_{\mathbf{A}})$, then we have*

$$
Z(\Phi, s) = \mathcal{R}_1 \sum_{x\in G_k\backslash L_0} \mathcal{N}(\Delta_{k(x)/k})^{-s}\mathfrak{C}_{\tilde{k}(x)}\Xi_x(\Phi, s)\,.
$$

PROOF. From Definition 6.2 we have

$$
\begin{aligned}
Z(\Phi, s) &= \sum_{x\in G_k\backslash L_0} \int_{G_{\mathbf{A}}/\tilde{T}_{\mathbf{A}}G_k} |\chi(\tilde{g})|^s \sum_{\gamma\in G_k/G_{x\,k}} \Phi(\tilde{g}\gamma x)d\tilde{g} \\
&= \sum_{x\in G_k\backslash L_0} \int_{G_{\mathbf{A}}/\tilde{T}_{\mathbf{A}}G_{x\,k}} |\chi(\tilde{g})|^s\Phi(\tilde{g}x)d\tilde{g} \\
&= \frac{1}{2}\sum_{x\in G_k\backslash L_0} \int_{G_{\mathbf{A}}/\tilde{T}_{\mathbf{A}}G^\circ_{x\,k}} |\chi(\tilde{g})|^s\Phi(\tilde{g}x)d\tilde{g} \quad \text{since } [G_{x\,k} : G^\circ_{x\,k}] = 2 \\
&= \frac{1}{2}\mathcal{R}_1\mathfrak{C}_{\tilde{k}} \sum_{x\in G_k\backslash L_0} \int_{G_{\mathbf{A}}/\tilde{T}_{\mathbf{A}}G^\circ_{x\,k}} |\chi(\tilde{g})|^s\Phi(\tilde{g}x)d_{\mathrm{pr}}\tilde{g} \quad \text{by (6.1)} \\
&= \frac{1}{2}\mathcal{R}_1\mathfrak{C}_{\tilde{k}} \sum_{x\in G_k\backslash L_0} \left(\prod_v b_{x,v}\right) \int_{G_{\mathbf{A}}/G^\circ_{x\,\mathbf{A}}} |\chi(\tilde{g}')|^s\Phi(\tilde{g}'x)d_{\mathrm{pr}}\tilde{g}' \\
&\qquad\qquad \cdot \int_{G^\circ_{x\,\mathbf{A}}/\tilde{T}_{\mathbf{A}}G^\circ_{x\,k}} d_{\mathrm{pr}}\tilde{g}'' \quad \text{by Definition 5.21} \\
&= \frac{1}{2}\mathcal{R}_1\mathfrak{C}_{\tilde{k}} \sum_{x\in G_k\backslash L_0} \left(\prod_v Z_{x,v}(\Phi_v, s)\right) \cdot \mathrm{vol}(G^\circ_{x\,\mathbf{A}}/\tilde{T}_{\mathbf{A}}G^\circ_{x\,k}) \quad \text{by Definition 5.22}
\end{aligned}
$$

$$= \mathcal{R}_1 \sum_{x \in G_k \backslash L_0} Z_x(\Phi, s)\mathfrak{C}_{\tilde{k}(x)} \quad \text{by Proposition 5.24}$$
$$= \mathcal{R}_1 \sum_{x \in G_k \backslash L_0} \mathcal{N}(\Delta_{k(x)/k})^{-s}\mathfrak{C}_{\tilde{k}(x)}\Xi_x(\Phi, s) \quad \text{by Proposition 6.7.}$$

□

We are now ready to describe the filtering process. This process was originally used in [5] and is described in a general setting in [28], §0.5. Our discussion will follow this latter reference, but with simplifications arising from the fact that we know the residue of the global zeta function explicitly (by Theorem 6.3).

We set $S_0 = \mathfrak{M}_\infty \cup \mathfrak{M}_{\mathrm{rm}} \cup \mathfrak{M}_{\mathrm{dy}}$ and fix a finite set $S \supseteq S_0$ of places of k. For each finite subset $T \supseteq S$ of $\mathfrak{M}$ we consider T-tuples $\omega_T = (\omega_v)_{v \in T}$ where each ω_v is one of the standard orbital representatives, $w_{v,i}$, for the orbits in $V_{k_v}^{\mathrm{ss}}$ chosen above. If $x \in V_k^{\mathrm{ss}}$ and $x \in G_{k_v}\omega_v$, then we write $x \approx \omega_v$ and if $x \approx \omega_v$ for all $v \in T$, then we write $x \approx \omega_T$.

For later purposes, it is convenient to make the following definition.

DEFINITION 6.11. For any $v \in \mathfrak{M}_{\mathrm{f}}$, $\Phi_{v,0}$ is the characteristic function of $V_{\mathcal{O}_v}$.

Let $\Xi_{x,v}(s) = \Xi_{x,v}(\Phi_{v,0}, s)$ and $\Xi_{x,T}(s) = \prod_{v \notin T} \Xi_{x,v}(s)$. From the integral defining $\Xi_{x,v}(s)$ it follows that for $v \notin S_0$ this function may be expressed as $\Xi_{x,v}(s) = \sum_{n=-\infty}^{\infty} a_{x,v,n} q_v^{-ns}$ for certain numerical coefficients $a_{x,v,n}$. In Section 8 we shall establish the following condition.

CONDITION 6.12. For all $v \notin S_0$ and all $x \in V_{k_v}^{\mathrm{ss}}$ we have $a_{x,v,n} = 0$ for $n < 0$, $a_{x,v,0} = 1$ and $a_{x,v,n} \geq 0$ for all n.

Suppose that we have Dirichlet series $L_i(s) = \sum_{m=1}^{\infty} l_{i,m} m^{-s}$ for $i = 1, 2$. If $l_{1,m} \leq l_{2,m}$ for all $m \geq 1$, then we shall write $L_1(s) \preccurlyeq L_2(s)$. In Section 9 we shall establish that for every $v \notin S_0$ there exists a Dirichlet series $L_v(s) = \sum_{n=0}^{\infty} l_{v,n} q_v^{-ns}$ which satisfies the following condition.

CONDITION 6.13. (1) For all $v \notin S_0$ and $x \in V_{k_v}^{\mathrm{ss}}$, $\Xi_{x,v}(s) \preccurlyeq L_v(s)$.

(2) The series defining $L_v(s)$ converges to a holomorphic function in the region $\mathrm{Re}(s) > 1$ and the product $\prod_{v \notin S_0} L_v(s)$ converges absolutely and locally uniformly in the region $\mathrm{Re}(s) > 3/2$.

(3) For all $v \notin S_0$, $l_{v,0} = 1$ and $l_{v,n} \geq 0$ for all n.

For any $T \supseteq S$ we define $L_T(s) = \prod_{v \notin T} L_v(s)$. Both $\Xi_{x,T}(s)$ and $L_T(s)$ are Dirichlet series and if we let

$$\Xi_{x,T}(s) = \sum_{m=1}^{\infty} a^*_{x,T,m} m^{-s} \quad \text{and} \quad L_T(s) = \sum_{m=1}^{\infty} l^*_{T,m} m^{-s}, \tag{6.14}$$

then $a^*_{x,T,m}$ (resp. $l^*_{T,m}$) is the sum of the terms $\prod_{v \notin T} a_{x,v,n_v}$ (resp. $\prod_{v \notin T} l_{v,n_v}$) over all possible factorizations $m = \prod_{v \notin T} q_v^{n_v}$. Since only finitely-many places, v, of k can have q_v equal to a power of a particular prime, the number of such factorizations is finite. Also, in any such factorization, $n_v = 0$ for all but finitely-many v, and so this sum is well-defined. It follows from Conditions 6.12 and 6.13 that $0 \le a^*_{x,T,m} \le l^*_{T,m}$ and $a^*_{x,T,1} = 1$ for all $x \in V_k^{\mathrm{ss}}$, all $T \supseteq S$ and all $m \ge 1$. We shall use these observations in the proof of Theorem 6.22 below.

We define

$$\xi_{\omega_T}(s) = \sum_{x \in G_k \backslash L_0, x \approx \omega_T} \mathcal{N}(\Delta_{k(x)/k})^{-s} \mathfrak{C}_{\tilde{k}(x)} \varXi_{x,T}(s) \tag{6.15}$$

and

$$\xi_{\omega_S,T}(s) = \sum_{x \in G_k \backslash L_0, x \approx \omega_S} \mathcal{N}(\Delta_{k(x)/k})^{-s} \mathfrak{C}_{\tilde{k}(x)} \varXi_{x,T}(s)\,, \tag{6.16}$$

which is the sum of $\xi_{\omega_T}(s)$ over all $\omega_T = (\omega_v)_{v \in T}$ which extend the fixed S-tuple ω_S. In order to determine the analytic properties of these Dirichlet series we require the following result.

LEMMA 6.17. *Let $v \in \mathfrak{M}$, $x \in V_{k_v}^{\mathrm{ss}}$ and $r \in \boldsymbol{C}$. Then there exists $\Phi_v \in \mathcal{S}(V_{k_v})$ such that the support of Φ_v is contained in $G_{k_v}x$, $Z_{x,v}(\Phi_v, s)$ is an entire function and $Z_{x,v}(\Phi_v, r) \neq 0$.*

PROOF. The set $G_{k_v}x$ is open and $y \mapsto |P(y)|_v^{r-2}$ is a continuous function on it. We may therefore find an open set U containing x, having compact closure $\bar{U} \subseteq G_{k_v}x$ and such that

$$\big|\,|P(y)|_v^{r-2} - |P(x)|_v^{r-2}\big| < \frac{1}{2}|P(x)|_v^{r-2} \tag{6.18}$$

for $y \in \bar{U}$. We can then choose $\Phi_v \in \mathcal{S}(V_{k_v})$ in such a way that $\mathrm{supp}(\Phi_v) \subseteq \bar{U}$ and $\int_{\bar{U}} \Phi_v(y)\,dy = 1$. Now (6.18) implies that $|P(y)|_v$ does not vanish on $\bar{U}$ and hence it is bounded both above and below by positive constants on this compactum. Thus $Z_{x,v}(\Phi_v, s)$ is entire. The inequality (6.18) also implies that

$$|Z_{x,v}(\Phi_v, r) - b_{x,v}|P(x)|_v^{-2}| \le \frac{1}{2} b_{x,v}|P(x)|_v^{-2}$$

and hence $Z_{x,v}(\Phi_v, r) \neq 0$. □

PROPOSITION 6.19. *Let $T \supseteq S$ be a finite set of places of k and ω_T be a T-tuple, as above. The Dirichlet series $\xi_{\omega_T}(s)$ has a meromorphic continuation to the region* $\mathrm{Re}(s) > 3/2$. *Its only possible singularity in this region is a simple pole at $s = 2$ with residue*

$$\mathcal{R}_2 \prod_{v \in T} b^{-1}_{\omega_v,v} |P(\omega_v)|_v^2\,,$$

where

$$\mathcal{R}_2 = \mathrm{Res}_{s=1}\,\zeta_k(s) \cdot \mathrm{Res}_{s=1}\,\zeta_{\tilde{k}}(s) \cdot Z_k(2) Z_{\tilde{k}}(2)/|\Delta_k|\,.$$

PROOF. For each $v \in T$ we choose $\Phi_v \in \mathcal{S}(V_{k_v})$ such that $\mathrm{supp}(\Phi_v) \subseteq G_{k_v}\omega_v$. Let $\Phi = \bigotimes_{v \in T} \Phi_v \otimes \bigotimes_{v \notin T} \Phi_{v,0} \in \mathcal{S}(V_{\boldsymbol{A}})$. For $v \in T$ we have $\varXi_{x,v}(\Phi_v, s) = 0$ unless $x \approx \omega_v$

and hence

$$Z(\Phi, s) = \mathcal{R}_1\biggl(\prod_{v\in T} \Xi_{\omega_v,v}(\Phi_v, s)\biggr) \sum_{x\in G_k\backslash L_0, x\approx\omega_T} \mathcal{N}(\Delta_{k(x)/k})^{-s}\mathfrak{C}_{\tilde{k}(x)}\Xi_{x,T}(s)$$
$$= \mathcal{R}_1\biggl(\prod_{v\in T} \Xi_{\omega_v,v}(\Phi_v, s)\biggr)\xi_{\omega_T}(s)$$

by Proposition 6.10. By Lemma 6.17 and Theorem 6.3, this formula implies the first statement.

Now choose Φ_v for $v \in T$ so that $\Xi_{\omega_v,v}(\Phi_v, 2) \neq 0$. It follows directly from the definition that $\Xi_{\omega_v,v}(\Phi_v, 2) = b_{\omega_v,v}|P(\omega_v)|_v^{-2}\Sigma_v(\Phi_v)$ for all $v \in T$, and so the residue of $\xi_{\omega_T}(s)$ at $s = 2$ is

$$\mathcal{R}_1^{-1}\biggl(\prod_{v\in T} b_{\omega_v,v}^{-1}|P(\omega_v)|_v^2\biggr)\biggl(\prod_{v\in T}\Sigma_v(\Phi_v)\biggr)^{-1}\operatorname{Res}_{s=2} Z(\Phi, s)\,.$$

We have $\Sigma_v(\Phi_{v,0}) = 1$ for $v \notin T$ and hence

$$\operatorname{Res}_{s=2} Z(\Phi, s) = \mathfrak{V}_k\mathfrak{V}_{\tilde{k}}|\Delta_k|^{-2}|\Delta_{\tilde{k}}|^{-1}\prod_{v\in T}\Sigma_v(\Phi_v)\,.$$

Combining the last two equations shows that the residue of $\xi_{\omega_T}(s)$ at $s = 2$ is

$$\mathcal{R}_1^{-1}\mathfrak{V}_k\mathfrak{V}_{\tilde{k}}|\Delta_k|^{-2}|\Delta_{\tilde{k}}|^{-1}\biggl(\prod_{v\in T} b_{\omega_v,v}^{-1}|P(\omega_v)|_v^2\biggr),$$

and using the definition of $\mathcal{R}_1$ and the values of $\mathfrak{V}_k$ and $\mathfrak{V}_{\tilde{k}}$ (see the end of Section 4) gives the second claim. □

COROLLARY 6.20. *The Dirichlet series $\xi_{\omega_S,T}(s)$ has a meromorphic continuation to the region* $\operatorname{Re}(s) > 3/2$. *Its only possible singularity in this region is a simple pole at $s = 2$ with residue*

$$\mathcal{R}_2\biggl(\prod_{v\in S} b_{\omega_v,v}^{-1}|P(\omega_v)|_v^2\biggr)\cdot\prod_{v\in T\setminus S}\sum_x\bigl(b_{x,v}^{-1}|P(x)|_v^2\bigr),$$

where the sum is over the complete set, $\{x\}$, *of standard orbit representatives for $G_{k_v}\backslash V_{k_v}^{\mathrm{ss}}$.*

PROOF. We have $\xi_{\omega_S,T}(s) = \sum_{\omega_T}\xi_{\omega_T}(s)$ where the sum is over all T-tuples ω_T which extend the S-tuple ω_S. The claim follows immediately. □

We let $E_v = \sum_x b_{x,v}^{-1}|P(x)|_v^2$ for $v \notin S_0$, where the sum is over all standard representatives, x, for orbits in $G_{k_v}\backslash V_{k_v}^{\mathrm{ss}}$. In Section 7 we shall prove that the following condition holds.

CONDITION 6.21. The product $\prod_{v\notin S_0} E_v$ converges to a positive number.

We are now ready to state and prove, subject to Conditions 6.12, 6.13 and 6.21, the theorem which is the goal of this section.

THEOREM 6.22. *Let $S \supseteq S_0$ be a finite set of places of k and ω_S an S-tuple of standard orbital representatives. Then*

$$\lim_{X\to\infty} X^{-2} \sum_{\substack{x\in G_k\backslash L_0, x\approx\omega_S \\ \mathcal{N}(\Delta_{k(x)/k})\le X}} \mathfrak{C}_{\tilde{k}(x)} = \frac{1}{2}\mathcal{R}_2 \prod_{v\in S}(b_{\omega_v,v}^{-1}|P(\omega_v)|_v^2) \cdot \prod_{v\notin S} E_v \,.$$

PROOF. In the following, sums over x will be understood to include the conditions $x \in G_k\backslash L_0$ and $x \approx \omega_S$ as well as any further conditions which may be explicitly imposed. We have $\xi_{\omega_S,T}(s) = \sum_{m=1}^{\infty} c_m m^{-s}$ where

$$c_m = \sum_{x,n,\mathcal{N}(\Delta_{k(x)/k})n=m} \mathfrak{C}_{\tilde{k}(x)} a_{x,T,n}^* \,.$$

Applying the Tauberian theorem ([21], p. 464, Theorem I) to $\xi_{\omega_S,T}(s)$, we obtain, in light of Corollary 6.20,

$$\lim_{X\to\infty} X^{-2} \sum_{x,n,\mathcal{N}(\Delta_{k(x)/k})n\le X} \mathfrak{C}_{\tilde{k}(x)} a_{x,T,n}^* = \frac{1}{2}\mathcal{R}_2 \left(\prod_{v\in S} b_{\omega_v,v}^{-1}|P(\omega_v)|_v^2\right) \cdot \prod_{v\in T\backslash S} E_v \,.$$

We shall denote the right hand side of this equation by $\mathcal{L}_T$. Note that $\mathcal{L} = \lim_{T\to\mathfrak{M}} \mathcal{L}_T$ is the right hand side of the equation in the statement. Since $a_{x,T,n}^* \ge 0$ for all n and $a_{x,T,1}^* = 1$ we obtain

$$\limsup_{X\to\infty} X^{-2} \sum_{\mathcal{N}(\Delta_{k(x)/k})\le X} \mathfrak{C}_{\tilde{k}(x)} \le \mathcal{L}_T$$

for all T, and so $\limsup_{X\to\infty} X^{-2} \sum_{\mathcal{N}(\Delta_{k(x)/k})\le X} \mathfrak{C}_{\tilde{k}(x)} \le \mathcal{L}$. It follows that there is a constant C such that $\sum_{\mathcal{N}(\Delta_{k(x)/k})\le X} \mathfrak{C}_{\tilde{k}(x)} \le CX^2$ for all $X > 0$ (note that if $X < 1$, then the sum is 0). Furthermore,

$$\begin{aligned}
\sum_{\mathcal{N}(\Delta_{k(x)/k})\le X} \mathfrak{C}_{\tilde{k}(x)} &= \sum_{\mathcal{N}(\Delta_{k(x)/k})n\le X} \mathfrak{C}_{\tilde{k}(x)} a_{x,T,n}^* - \sum_{\mathcal{N}(\Delta_{k(x)/k})n\le X, n\ge 2} \mathfrak{C}_{\tilde{k}(x)} a_{x,T,n}^* \\
&\ge \sum_{\mathcal{N}(\Delta_{k(x)/k})n\le X} \mathfrak{C}_{\tilde{k}(x)} a_{x,T,n}^* - \sum_{\mathcal{N}(\Delta_{k(x)/k})n\le X, n\ge 2} \mathfrak{C}_{\tilde{k}(x)} l_{T,n}^* \\
&= \sum_{\mathcal{N}(\Delta_{k(x)/k})n\le X} \mathfrak{C}_{\tilde{k}(x)} a_{x,T,n}^* - \sum_{n=2}^{\infty} l_{T,n}^* \sum_{\mathcal{N}(\Delta_{k(x)/k})\le X/n} \mathfrak{C}_{\tilde{k}(x)} \\
&\ge \sum_{\mathcal{N}(\Delta_{k(x)/k})n\le X} \mathfrak{C}_{\tilde{k}(x)} a_{x,T,n}^* - CX^2 \sum_{n=2}^{\infty} l_{T,n}^* n^{-2} \\
&= \sum_{\mathcal{N}(\Delta_{k(x)/k})n\le X} \mathfrak{C}_{\tilde{k}(x)} a_{x,T,n}^* - CX^2(L_T(2)-1)\,.
\end{aligned}$$

It follows that, for all $T \supseteq S$,

$$\liminf_{X\to\infty} X^{-2} \sum_{\mathcal{N}(\Delta_{k(x)/k})\le X} \mathfrak{C}_{\tilde{k}(x)} \ge \mathcal{L}_T - C(L_T(2)-1)$$

and letting $T \to \mathfrak{M}$ we obtain

$$\liminf_{X\to\infty} X^{-2} \sum_{\mathcal{N}(\Delta_{k(x)/k})\le X} \mathfrak{C}_{\tilde{k}(x)} \ge \mathcal{L}$$

since $\lim_{T\to\mathfrak{M}} L_T(2) = 1$. □

The remainder of this paper and its companions [17], [18] are devoted to verifying the conditions enunciated in this section and to evaluating the constants which appear in Theorem 6.22. In the next section we make use of the results of this work to state the theorem in a more explicit form.

7. The mean value theorem. In this section we shall derive a more explicit and convenient mean value theorem from Theorem 6.22. Throughout, k will be a number field and $\tilde{k}$ a fixed quadratic extension of k. If F_1 and F_2 are distinct quadratic extensions of k, neither equal to $\tilde{k}$, then we shall say that F_1 and F_2 are *paired* (with respect to $\tilde{k}$) if $F_2 \subseteq F_1 \cdot \tilde{k}$. Since this condition uniquely determines F_2 from F_1, we may write $F_2 = F_1^*$ if F_2 and F_1 are paired. Our first result will be used below to express $\mathfrak{C}_{\tilde{k}(x)}$ in terms of $\mathfrak{C}_{k(x)}$ and $\mathfrak{C}_{k(x)^*}$ for $x \in L_0$.

PROPOSITION 7.1. *Suppose that L/k is a biquadratic extension of number fields and that k_1, k_2 and k_3 are the quadratic extensions of k contained in L. Then $\mathfrak{C}_L = \mathfrak{C}_k^{-2}\mathfrak{C}_{k_1}\mathfrak{C}_{k_2}\mathfrak{C}_{k_3}$.*

PROOF. This identity is perhaps the simplest instance of what is known as a Brauer relation (see [1], p. 162, for instance). For the reader's convenience we sketch the proof from the theory of the Dedekind zeta function. Using Theorem 1.1, Chapter XII of [19], p. 230 we have the factorization

$$\zeta_L(s) = \zeta_k(s)L(s,\chi_1)L(s,\chi_2)L(s,\chi_3)\,,$$

where χ_j is the idèle class character of k corresponding by class field theory to k_j. Multiplying both sides of this identity by $\zeta_k(s)^2$ we obtain

$$\zeta_L(s)\zeta_k(s)^2 = \zeta_{k_1}(s)\zeta_{k_2}(s)\zeta_{k_3}(s)\,. \tag{7.2}$$

Since $\operatorname{Res}_{s=1}\zeta_F(s) = \mathfrak{C}_F/|\Delta_F|^{1/2}$, it follows that

$$\mathfrak{C}_L\mathfrak{C}_k^2|\Delta_L\Delta_k^2|^{-1/2} = \mathfrak{C}_{k_1}\mathfrak{C}_{k_2}\mathfrak{C}_{k_3}|\Delta_{k_1}\Delta_{k_2}\Delta_{k_3}|^{-1/2}\,.$$

Recall that we have a functional equation

$$\zeta_F(1-s) = (2^{-2r_2(F)}\pi^{-[F:\boldsymbol{Q}]}|\Delta_F|)^{s-1/2}\frac{\Gamma(s/2)^{r_1(F)}\Gamma(s)^{r_2(F)}}{\Gamma((1-s)/2)^{r_1(F)}\Gamma(1-s)^{r_2(F)}}\zeta_F(s)\,,$$

where $r_1(F)$ denotes the number of real places of F and $r_2(F)$ the number of complex places of F. It is easy to check that $[L:\boldsymbol{Q}] + 2[k:\boldsymbol{Q}] = \sum_{j=1}^{3}[k_j:\boldsymbol{Q}]$ and $r_i(L) + 2r_i(k) =$

$\sum_{j=1}^{3} r_i(k_j)$ for $i = 1, 2$. Comparing the factors in the functional equation on both sides of (7.2) now shows that $|\Delta_L \Delta_k^2| = |\Delta_{k_1}\Delta_{k_2}\Delta_{k_3}|$ and the identity follows. $\square$

For notational compactness we shall set $\varepsilon_v(x) = b_{x,v}^{-1}|P(x)|_v^2$ for all $v \in \mathfrak{M}$ and $x \in V_{k_v}^{\mathrm{ss}}$. These constants are related to the quantities calculated in later parts by the following result.

LEMMA 7.3. *Let $v \in \mathfrak{M}_{\mathrm{f}}$ and $x \in V_{k_v}^{\mathrm{ss}}$. Then*

$$\varepsilon_v(x) = \mathrm{vol}(K_v \cap G_{x\,k_v}^{\circ})\mathrm{vol}(K_v x)\,,$$

where the first volume is evaluated with respect to the canonical measure $dg''_{x,v}$ on $G_{x\,k_v}^{\circ}$ and the second with respect to the measure on V_{k_v} under which $V_{\mathcal{O}_v}$ has volume 1.

PROOF. We have

$$\begin{aligned} 1 &= \int_{K_v} dg_v \\ &= b_{x,v}\int_{K_v G_{x\,k_v}^{\circ}/G_{x\,k_v}^{\circ}} dg'_{x,v}\cdot\int_{K_v\cap G_{x\,k_v}^{\circ}} dg''_{x,v} \quad \text{by Definition 5.21} \\ &= b_{x,v}\mathrm{vol}(K_v\cap G_{x\,k_v}^{\circ})\int_{K_v x}|P(y)|_v^{-2}\,dy \end{aligned}$$

by (5.19) with $s = 0$ and Φ the characteristic function of $K_v x$. But $|P(y)|_v = |P(x)|_v$ for all $y \in K_v x$, and so $1 = b_{x,v}\mathrm{vol}(K_v \cap G_{x\,k_v}^{\circ})|P(x)|_v^{-2}\mathrm{vol}(K_v x)$. $\square$

Using this formula for $\varepsilon_v(x)$ and the results of Sections 3 and 4 in [17], we may determine the values of $\varepsilon_v(x)$ for all $v \notin \mathfrak{M}_{\mathrm{dy}} \cup \mathfrak{M}_\infty$ and all standard orbital representatives $x \in V_{k_v}^{\mathrm{ss}}$. We record the results in Table 1.

The first column displays the index of the orbit and the second, $\varepsilon_v(x)$, where x is the standard representative for the orbit. The values of $\mathrm{vol}(K_v \cap G_{x\,k_v}^{\circ})$ which we use here are contained in Propositions 3.2, 3.3, 3.5 and 3.6 in [17] and the values of $\mathrm{vol}(K_v x)$ in Propositions 4.14, 4.15 and 4.26 in [17].

The infinite and dyadic places of k both require special treatment. We shall begin with the infinite places as the easier of the two. We extend a classical notation (r_1 for the number of real places and r_2 for the number of complex places) by letting r_{11} be the number of real places of k which split in $\tilde{k}$ and r_{12} the number of real places of k which ramify in $\tilde{k}$.

PROPOSITION 7.4. *For any S-tuple ω_S we have*

$$\prod_{v\in\mathfrak{M}_\infty} \varepsilon_v(\omega_v) = 2^{2r_2 - r_{11}}\pi^{3r_{11}+2r_{12}+3r_2}\,.$$

In particular, the product does not depend on ω_S.

PROOF. For the standard orbital representatives, x, at the infinite places we have required that $|P(x)|_v = 1$, and so $\varepsilon_v(x) = b_{x,v}^{-1}$. If v is a real place of k which splits in $\tilde{k}$, then $V_{k_v}^{\mathrm{ss}}$ is the union of two orbits with indices (sp) and (sp rm), respectively. From Propositions 5.2 and 5.6 [17] we see that $\varepsilon_v(\omega_v) = \pi^3/2$ for both these orbits. In the product, the total contribution from these places is thus $2^{-r_{11}}\pi^{3r_{11}}$. If v is a real place of k which ramifies in $\tilde{k}$, then

TABLE 1. $\varepsilon_v(x)$ for v finite and non-dyadic.

Index	$\varepsilon_v(x)$
(sp)	$(1/2)(1+q_v^{-1})(1-q_v^{-2})^2$
(in)	$(1/2)(1-q_v^{-1})(1-q_v^{-4})$
(rm)	$(1/2)(1-q_v^{-2})^2$
(sp ur)	$(1/2)(1-q_v^{-1})^3(1-q_v^{-2})$
(sp rm)	$(1/2)q_v^{-1}(1-q_v^{-1})(1-q_v^{-2})^3$
(in ur)	$(1/2)(1-q_v^{-1})(1-q_v^{-4})$
(in rm)	$(1/2)q_v^{-1}(1-q_v^{-1})(1-q_v^{-2})(1-q_v^{-4})$
(rm ur)	$(1/2)(1-q_v^{-1})^2(1-q_v^{-2})$
(rm rm)*	$(1/2)q_v^{-2}(1-q_v^{-2})^2$
(rm rm ur)	$(1/2)q_v^{-2}(1-q_v^{-1})^2(1-q_v^{-2})$

$V_{k_v}^{\mathrm{ss}}$ is the union of two orbits with indices (rm) and (rm rm)*, respectively. From Propositions 5.4 and 5.7 in [17] we see that $\varepsilon_v(\omega_v) = \pi^2$ for both these orbits. In the product, the total contribution from these places is thus $\pi^{2r_{12}}$. Finally, if v is a complex place of k, then $V_{k_v}^{\mathrm{ss}}$ consists of a single orbit with index (sp) and, from Proposition 5.2 in [17], $\varepsilon_v(\omega_v) = 4\pi^3$ for this orbit. The total contribution to the product from the complex places of k is thus $2^{2r_2}\pi^{3r_2}$ and the formula follows. □

When $v \in \mathfrak{M}_{\mathrm{dy}}$ we shall not calculate the constants $\varepsilon_v(x)$ individually in all cases. Rather we shall sometimes calculate the sum of the $\varepsilon_v(x)$ over a set of orbits with similar arithmetical properties. This is because if $v \in \mathfrak{M}_{\mathrm{dy}}$, then it is difficult to deal with the ramified quadratic extensions of k_v individually. This leads to a final version of Theorem 6.22 which contains no unevaluated constants, but which employs an equivalence relation, denoted by $\asymp$, coarser than the relation $\approx$. Our next task is to define this relation.

Recall that, for $x, y \in V_{k_v}^{\mathrm{ss}}$, we write $x \approx y$ if $k_v(x) = k_v(y)$ (we have previously used this notation only when y was a standard orbital representative, but the extension is convenient here). If $v \notin \mathfrak{M}_{\mathrm{dy}}$ or if $v \in \mathfrak{M}_{\mathrm{dy}}$ but $k_v(x)/k_v$ is unramified (including the case $k_v(x) = k_v$), then $x \asymp y$ will have the same meaning as $x \approx y$. Suppose now that $v \in \mathfrak{M}_{\mathrm{dy}}$ and that $k_v(x)/k_v$ is ramified. If the type of x is (sp rm) or (in rm), then we shall write $x \asymp y$ if $\Delta_{k_v(x)/k_v} = \Delta_{k_v(y)/k_v}$. If the type of x is (rm rm)* or (rm rm ur), then we write $x \asymp y$ if y has the same type as x. Finally, if x has type (rm rm rm), then we write $x \asymp y$ if $\Delta_{k_v(x)/k_v} = \Delta_{k_v(y)/k_v}$ and $\Delta_{\tilde{k}_v(x)/\tilde{k}_v} = \Delta_{\tilde{k}_v(y)/\tilde{k}_v}$. This defines an equivalence relation on $V_{k_v}^{\mathrm{ss}}$

for all places v of k. If ω_S is an S-tuple of standard orbital representatives and $x \in L_0$, then we write $x \asymp \omega_S$ to mean that $x \asymp \omega_v$ for all $v \in S$.

The grouping of dyadic orbits is differently expressed in [18] and we must explain the connection between the two formulations. For any $v \in \mathfrak{M}_{\rm dy}$ we shall put $2\mathcal{O}_v = \mathfrak{p}_v^{m_v}$. If $x \in V_{k_v}^{\rm ss}$, then let $\Delta_{k_v(x)/k_v} = \mathfrak{p}_v^{\delta_{x,v}}$ and, if $v \notin \mathfrak{M}_{\rm sp}$, also let $\Delta_{\tilde{k}_v/k_v} = \mathfrak{p}_v^{\tilde{\delta}_v}$ and $\Delta_{\tilde{k}_v(x)/\tilde{k}_v} = \tilde{\mathfrak{p}}_v^{\tilde{\delta}_{x,v}}$. It is well-known that if $k_v(x)/k_v$ is ramified and v is dyadic, then $\delta_{x,v}$ takes one of the values $2, 4, \ldots, 2m_v, 2m_v + 1$. In [18] we introduce a natural number $\mathrm{lev}(k_1, k_2)$, the *level* of k_1 and k_2, which is defined whenever k_1 and k_2 are ramified quadratic extensions of a local field. Let us write $\lambda_{x,v} = \mathrm{lev}(k_v(x), \tilde{k}_v)$ when $v \in \mathfrak{M}_{\rm rm} \cap \mathfrak{M}_{\rm dy}$ and $k_v(x)/k_v$ is ramified. If $\delta_{x,v} \neq \tilde{\delta}_v$, then

$$\lambda_{x,v} = \min\left\{\left\lfloor \frac{1}{2}(\delta_{x,v}+1)\right\rfloor, \left\lfloor \frac{1}{2}(\tilde{\delta}_v+1)\right\rfloor\right\}, \tag{7.5}$$

but if $\delta_{x,v} = \tilde{\delta}_v$, then $\lambda_{x,v}$ may take any value from this minimum up to $\delta_{x,v}$. We have the relation

$$\tilde{\delta}_{x,v} = 2(\delta_{x,v} - \lambda_{x,v}), \tag{7.6}$$

and so, with $\Delta_{k_v(x)/k_v}$ fixed, $\Delta_{\tilde{k}_v(x)/\tilde{k}_v}$ and $\mathrm{lev}(k_v(x), \tilde{k}_v)$ determine one another. Thus the grouping of dyadic orbits with index (rm rm rm) in [18], by discriminant and level with $\tilde{k}_v$, coincides with the grouping defined here.

If x is a standard orbital representative in $V_{k_v}^{\rm ss}$ for any $v \in \mathfrak{M}$, then let us write

$$\bar{\varepsilon}_v(x) = \sum_{y \asymp x} \varepsilon_v(y),$$

where the sum is over standard orbital representatives that satisfy $y \asymp x$. Thus $\bar{\varepsilon}_v(x) = \varepsilon_v(x)$ unless $v \in \mathfrak{M}_{\rm dy}$ and $k_v(x)/k_v$ is ramified. Also $y \asymp x$ implies that x and y have the same type and since there is only one orbit corresponding to each of the indices (rm rm)* and (rm rm ur), $\bar{\varepsilon}_v(x) = \varepsilon_v(x)$ if x is the standard representative for either of these orbits. In Table 2 we collect the values of the constants $\varepsilon_v(x)$ for those dyadic orbits having $\bar{\varepsilon}_v(x) = \varepsilon_v(x)$ and in Table 3 we collect the values of the constants $\bar{\varepsilon}_v(x)$ for the remaining dyadic orbits.

The values of $\mathrm{vol}(K_v x)$ and $\mathrm{vol}(K_v \cap G^\circ_{x\,k_v})$ used to determine the entries in the two tables were drawn from Propositions 3.2, 3.3, 3.5, 4.14, 4.25 of [17] and Propositions 4.2, 5.11, 5.14 and Corollary 5.15 of [18]. In Table 3, the second column records the conditions on $\delta_{x,v}$, $\tilde{\delta}_v$ and $\lambda_{x,v}$ under which the entry is valid. From (7.5) and the observations made in the previous paragraph it is easy to see that the available conditions are exhaustive.

It will be convenient to extend the notation of Section 6 by writing

$$E_v = \sum_x \varepsilon_v(x)$$

for all $v \in \mathfrak{M}_{\rm f}$, where the sum is taken over all standard representatives, x, of orbits in $G_{k_v}\backslash V_{k_v}^{\rm ss}$. We call E_v the *local density* at the place v.

TABLE 2. $\varepsilon_v(x)$ for ungrouped dyadic orbits.

Index	$\varepsilon_v(x)$
(sp)	$(1/2)(1+q_v^{-1})(1-q_v^{-2})^2$
(in)	$(1/2)(1-q_v^{-1})(1-q_v^{-4})$
(rm)	$(1/2)(1-q_v^{-2})^2$
(sp ur)	$(1/2)(1-q_v^{-1})^3(1-q_v^{-2})$
(in ur)	$(1/2)(1-q_v^{-1})(1-q_v^{-4})$
(rm ur)	$(1/2)(1-q_v^{-1})^2(1-q_v^{-2})$
(rm rm)*	$(1/2)q_v^{-2\tilde{\delta}_v-2\lfloor\tilde{\delta}_v/2\rfloor}(1-q_v^{-2})^2$
(rm rm ur)	$q_v^{-2\tilde{\delta}_v}(1-(1/2)q_v^{-2\lfloor\tilde{\delta}_v/2\rfloor})(1-q_v^{-1})^2(1-q_v^{-2})$

TABLE 3. $\bar{\varepsilon}_v(x)$ for grouped dyadic orbits.

Index	Conditions	$\bar{\varepsilon}_v(x)$
(sp rm)	$\delta_{x,v} \le 2m_v$	$q_v^{-\delta_{x,v}/2}(1-q_v^{-1})^2(1-q_v^{-2})^3$
(sp rm)	$\delta_{x,v} = 2m_v+1$	$q_v^{-(m_v+1)}(1-q_v^{-1})(1-q_v^{-2})^3$
(in rm)	$\delta_{x,v} \le 2m_v$	$q_v^{-\delta_{x,v}/2}(1-q_v^{-1})^2(1-q_v^{-2})(1-q_v^{-4})$
(in rm)	$\delta_{x,v} = 2m_v+1$	$q_v^{-(m_v+1)}(1-q_v^{-1})(1-q_v^{-2})(1-q_v^{-4})$
(rm rm rm)	$\delta_{x,v} \ne \tilde{\delta}_v$, $\delta_{x,v} \le 2m_v$	$q_v^{-(\delta_{x,v}/2+\lambda_{x,v})}(1-q_v^{-1})^2(1-q_v^{-2})^2$
(rm rm rm)	$\delta_{x,v} \ne \tilde{\delta}_v$, $\delta_{x,v} = 2m_v+1$	$q_v^{-(m_v+\lambda_{x,v}+1)}(1-q_v^{-1})(1-q_v^{-2})^2$
(rm rm rm)	$\delta_{x,v} = \tilde{\delta}_v \le 2m_v$, $\lambda_{x,v} = (1/2)\tilde{\delta}_v$	$q_v^{-2\lambda_{x,v}}(1-q_v^{-1})(1-2q_v^{-1})(1-q_v^{-2})^2$
(rm rm rm)	$\delta_{x,v} = \tilde{\delta}_v \le 2m_v$, $\lambda_{x,v} > (1/2)\tilde{\delta}_v$	$q_v^{-2\lambda_{x,v}}(1-q_v^{-1})^2(1-q_v^{-2})^2$
(rm rm rm)	$\delta_{x,v} = \tilde{\delta}_v = 2m_v+1$	$q_v^{-2\lambda_{x,v}}(1-q_v^{-1})^2(1-q_v^{-2})^2$

PROPOSITION 7.7. *Let $v \in \mathfrak{M}_{\mathrm{f}}$. Then $E_v = (1-q_v^{-2})E'_v$, where*

$$E'_v = \begin{cases} 1-3q_v^{-3}+2q_v^{-4}+q_v^{-5}-q_v^{-6} & \text{if } v \in \mathfrak{M}_{\mathrm{sp}}\,, \\ (1+q_v^{-2})(1-q_v^{-2}-q_v^{-3}+q_v^{-4}) & \text{if } v \in \mathfrak{M}_{\mathrm{in}}\,, \\ (1-q_v^{-1})(1+q_v^{-2}-q_v^{-3}+q_v^{-2\tilde{\delta}_v-2\lfloor\tilde{\delta}_v/2\rfloor-1}) & \text{if } v \in \mathfrak{M}_{\mathrm{rm}}\,. \end{cases} \tag{7.8}$$

PROOF. First suppose that $v \notin \mathfrak{M}_{\mathrm{dy}}$. Then every index corresponds to a single orbit, with the exception of (sp rm) and (in rm), which correspond to two orbits each. Using this and the values of $\varepsilon_v(x)$ given in Table 1 it is routine to check the given expressions.

Now suppose that $v \in \mathfrak{M}_{\mathrm{dy}}$. We have $E_v = \sum_x \bar{\varepsilon}_v(x)$, where the sum now runs over a complete set of representatives for the $\asymp$ equivalence classes. The values of $\bar{\varepsilon}_v(x)$ are given in Tables 2 and 3 and using them one can easily establish the claim when $v \notin \mathfrak{M}_{\mathrm{rm}}$. We carry out the case $v \in \mathfrak{M}_{\mathrm{rm}}$ explicitly, since it is rather more elaborate.

First suppose that $\tilde{\delta}_v = 2\tilde{l}$ with $1 \le \tilde{l} \le m_v$. The indices which are possible with our assumptions are (rm), (rm ur), (rm rm)* and (rm rm ur), corresponding to one orbit each, and (rm rm rm), which corresponds to many orbits. By Table 2, the contribution to E_v from the first four of these indices is

$$\begin{aligned}
&\frac{1}{2}(1-q_v^{-2})^2 + \frac{1}{2}(1-q_v^{-1})^2(1-q_v^{-2}) \\
&+ \frac{1}{2}q_v^{-6\tilde{l}}(1-q_v^{-2})^2 + q_v^{-4\tilde{l}}\left(1-\frac{1}{2}q_v^{-2\tilde{l}}\right)(1-q_v^{-1})^2(1-q_v^{-2})\,.
\end{aligned} \tag{7.9}$$

Recall that the orbits with index (rm rm rm) have been grouped under $\asymp$ by $\delta_{x,v}$ if $\delta_{x,v} \ne \tilde{\delta}_v$ and by level if $\delta_{x,v} = \tilde{\delta}_v$. If $\delta_{x,v} \ne \tilde{\delta}_v$, then either $\delta_{x,v} = 2l$ with $l \ne \tilde{l}$ or $\delta_{x,v} = 2m_v + 1$. Using Table 3 and (7.5), we see that the contribution from these equivalence classes is

$$\begin{aligned}
&\sum_{l=1}^{\tilde{l}-1} q_v^{-2l}(1-q_v^{-1})^2(1-q_v^{-2})^2 + \sum_{l=\tilde{l}+1}^{m_v} q_v^{-(l+\tilde{l})}(1-q_v^{-1})^2(1-q_v^{-2})^2 \\
&\quad + q_v^{-(m_v+\tilde{l}+1)}(1-q_v^{-1})(1-q_v^{-2})^2 \\
&= (q_v^{-2} - q_v^{-2\tilde{l}})(1-q_v^{-1})^2(1-q_v^{-2}) \\
&\quad + (q_v^{-(2\tilde{l}+1)} - q_v^{-(m_v+\tilde{l}+1)})(1-q_v^{-1})(1-q_v^{-2})^2 \\
&\quad + q_v^{-(m_v+\tilde{l}+1)}(1-q_v^{-1})(1-q_v^{-2})^2 \\
&= q_v^{-2}(1-q_v^{-1})^2(1-q_v^{-2}) - q_v^{-2\tilde{l}}(1-q_v^{-1})^2(1-q_v^{-2}) \\
&\quad + q_v^{-(2\tilde{l}+1)}(1-q_v^{-1})(1-q_v^{-2})^2\,.
\end{aligned} \tag{7.10}$$

If $\delta_{x,v} = \tilde{\delta}_v$, then the level, $\lambda_{x,v}$, runs from $\tilde{l}$ up to $\tilde{\delta}_v - 1$. By (7.6), the value $\lambda_{x,v} = \tilde{\delta}_v = \delta_{x,v}$, although possible, corresponds to the orbit with index (rm rm ur), and so is excluded here. The contribution from the equivalence classes with $\delta_{x,v} = \tilde{\delta}_v$ is thus

$$
\begin{aligned}
&q_v^{-2\tilde{l}}(1-q_v^{-1})(1-2q_v^{-1})(1-q_v^{-2})^2\\
&\quad+\sum_{i=\tilde{l}+1}^{2\tilde{l}-1} q_v^{-2i}(1-q_v^{-1})^2(1-q_v^{-2})^2\\
&=q_v^{-2\tilde{l}}(1-q_v^{-1})(1-2q_v^{-1})(1-q_v^{-2})^2\\
&\quad+(q_v^{-(2\tilde{l}+2)}-q_v^{-4\tilde{l}})(1-q_v^{-1})^2(1-q_v^{-2})\,.
\end{aligned}
\tag{7.11}
$$

Let us now collect all the terms from (7.10) and (7.11) which have $q_v^{-2\tilde{l}}$ as a visible factor. The result is

$$
\begin{aligned}
&q_v^{-2\tilde{l}}[-(1-q_v^{-1})^2(1-q_v^{-2})+q_v^{-1}(1-q_v^{-1})(1-q_v^{-2})^2\\
&\qquad+(1-q_v^{-1})(1-2q_v^{-1})(1-q_v^{-2})^2+q_v^{-2}(1-q_v^{-1})^2(1-q_v^{-2})]\\
&=q_v^{-2\tilde{l}}(1-q_v^{-1})(1-q_v^{-2})[-(1-q_v^{-1})+q_v^{-1}(1-q_v^{-2})\\
&\qquad\qquad+(1-2q_v^{-1})(1-q_v^{-2})+q_v^{-2}(1-q_v^{-1})]\\
&=0
\end{aligned}
$$

on expanding the factor in the square brackets. It remains to add (7.9), the first term of (7.10) and the term $-q_v^{-4\tilde{l}}(1-q_v^{-1})^2(1-q_v^{-2})$ from (7.11) to obtain E_v. This is easily done. The situation where $\tilde{\delta}_v = 2m_v+1$ is similar, but simpler, and we leave it to the reader. □

In particular, this proposition verifies Condition 6.21 subject to the results of Sections 3 and 4 of [17].

If F/k is a quadratic extension distinct from $\tilde{k}/k$, then $F = k(x)$ for some $x \in L_0$ and we shall write $F \approx \omega_S$ if $x \approx \omega_S$ and $F \asymp \omega_S$ if $x \asymp \omega_S$.

THEOREM 7.12. *Let $S \supseteq \mathfrak{M}_\infty$ be a finite set of places of k and ω_S an S-tuple of standard orbital representatives. Then*

$$
\lim_{X\to\infty} X^{-2} \sum_{\substack{[F:k]=2,\, F\asymp\omega_S\\ \mathcal{N}(\Delta_{F/k})\le X}} \mathfrak{C}_F\mathfrak{C}_{F^*}
$$

exists and has the value

$$
2^{-(r_1+r_2+1)}|\Delta_{\tilde{k}}/\Delta_k|^{1/2}\mathfrak{C}_k^3\zeta_{\tilde{k}}(2)\prod_{v\in S\setminus\mathfrak{M}_\infty}(1-q_v^{-2})^{-1}\bar{\varepsilon}_v(\omega_v)\cdot\prod_{v\notin S}E'_v\,,
$$

where $\bar{\varepsilon}_v(x)$ is given by Tables 1, 2 *and* 3 *and E'_v by* (7.8).

PROOF. By Proposition 7.1 we have $\mathfrak{C}_{\tilde{k}(x)} = \mathfrak{C}_k^{-2}\mathfrak{C}_{\tilde{k}}\mathfrak{C}_F\mathfrak{C}_{F^*}$ if $F = k(x)$. Recall, from Proposition 6.19 and (24), that

$$
\begin{aligned}
\mathcal{R}_2 &= |\Delta_k|^{-1/2}\mathfrak{C}_k|\Delta_{\tilde{k}}|^{-1/2}\mathfrak{C}_{\tilde{k}}\cdot Z_k(2)Z_{\tilde{k}}(2)/|\Delta_k|\\
&= \mathfrak{C}_k\mathfrak{C}_{\tilde{k}}|\Delta_k|^{-3/2}|\Delta_{\tilde{k}}|^{-1/2}Z_k(2)Z_{\tilde{k}}(2)
\end{aligned}
$$

and, from (2.3), that

$$Z_k(2) = 2^{-r_2}\pi^{-(r_1+r_2)}|\Delta_k|\zeta_k(2)\,,$$
$$Z_{\tilde{k}}(2) = 2^{-\tilde{r}_2}\pi^{-(\tilde{r}_1+\tilde{r}_2)}|\Delta_{\tilde{k}}|\zeta_{\tilde{k}}(2)\,,$$

where $\tilde{r}_1$ is the number of real places of $\tilde{k}$ and $\tilde{r}_2$ the number of complex places of this field. Thus

$$\mathcal{R}_2 = 2^{-(r_2+\tilde{r}_2)}\pi^{-(r_1+\tilde{r}_1+r_2+\tilde{r}_2)}|\Delta_{\tilde{k}}/\Delta_k|^{1/2}\mathfrak{C}_k\mathfrak{C}_{\tilde{k}}\zeta_k(2)\zeta_{\tilde{k}}(2)\,. \tag{7.13}$$

Let $T = S \cup S_0$ and choose a T-tuple, $\omega'_T = (\omega'_v)$. According to Theorem 6.22,

$$\lim_{X\to\infty} \sum_{\substack{[F:k]=2,\, F\approx\omega'_T \\ \mathcal{N}(\Delta_{F/k})\le X}} \mathfrak{C}_F\mathfrak{C}_{F^*} \tag{7.14}$$

exists and equals

$$\frac{1}{2}\mathfrak{C}_k^2\mathfrak{C}_{\tilde{k}}^{-1}\mathcal{R}_2 \prod_{v\in T}\varepsilon_v(\omega'_v)\cdot\prod_{v\notin T}E_v\,.$$

By (7.13) and Proposition 7.4 this quantity equals

$$2^{r_2-r_{11}-\tilde{r}_2-1}\pi^{3r_{11}+2r_{12}+2r_2-r_1-\tilde{r}_1-\tilde{r}_2}|\Delta_{\tilde{k}}/\Delta_k|^{1/2}\mathfrak{C}_k^3\zeta_k(2)\zeta_{\tilde{k}}(2)\prod_{v\in T\setminus\mathfrak{M}_\infty}\varepsilon_v(\omega'_v)\cdot\prod_{v\notin T}E_v\,.$$

But $\tilde{r}_1 = 2r_{11}$, $\tilde{r}_2 = r_{12} + 2r_2$ and $r_1 = r_{11} + r_{12}$. Thus

$$\begin{aligned} r_2 - r_{11} - \tilde{r}_2 - 1 &= r_2 - r_{11} - r_{12} - 2r_2 - 1 \\ &= -(r_2 + r_{11} + r_{12} + 1) = -(r_1 + r_2 + 1) \end{aligned}$$

and

$$\begin{aligned} 3r_{11} + 2r_{12} + 2r_2 - r_1 - \tilde{r}_1 - \tilde{r}_2 &= 3r_{11} + 2r_{12} + 2r_2 - r_{11} - r_{12} - 2r_{11} - r_{12} - 2r_2 \\ &= 0 \end{aligned}$$

and we have evaluated (7.14) as

$$2^{-(r_1+r_2+1)}|\Delta_{\tilde{k}}/\Delta_k|^{1/2}\mathfrak{C}_k^3\zeta_k(2)\zeta_{\tilde{k}}(2)\prod_{v\in T\setminus\mathfrak{M}_\infty}\varepsilon_v(\omega'_v)\cdot\prod_{v\notin T}E_v\,. \tag{7.15}$$

Now

$$\begin{aligned} \prod_{v\notin T}E_v &= \prod_{v\notin T}(1-q_v^{-2})\cdot\prod_{v\notin T}E'_v \\ &= \zeta_k(2)^{-1}\prod_{v\in T\setminus\mathfrak{M}_\infty}(1-q_v^{-2})^{-1}\cdot\prod_{v\notin T}E'_v\,, \end{aligned}$$

and so (7.15) equals

$$2^{-(r_1+r_2+1)}|\Delta_{\tilde{k}}/\Delta_k|^{1/2}\mathfrak{C}_k^3\zeta_{\tilde{k}}(2)\prod_{v\in T\setminus\mathfrak{M}_\infty}(1-q_v^{-2})^{-1}\varepsilon_v(\omega'_v)\cdot\prod_{v\notin T}E'_v\,. \tag{7.16}$$

Now we sum (7.14) and (7.16) over all T-tuples $\omega'_T = (\omega'_v)$ which satisfy $\omega'_v \asymp \omega_v$ for all $v \in S$ to obtain the statement of the theorem. $\square$

Note that in Theorem 7.12, S does not have to contain S_0.

Given an S-tuple, ω_S, with $S \supseteq \mathfrak{M}_\infty$ let us define

$$\begin{aligned} n_{++} &= \#\{v \in \mathfrak{M}_{\boldsymbol{R}} \mid v \in \mathfrak{M}_{\mathrm{sp}} \text{ and } k_v(\omega_v) = k_v\}\,, \\ n_{+-} &= \#\{v \in \mathfrak{M}_{\boldsymbol{R}} \mid v \in \mathfrak{M}_{\mathrm{sp}} \text{ and } k_v(\omega_v) \neq k_v\}\,, \\ n_{-+} &= \#\{v \in \mathfrak{M}_{\boldsymbol{R}} \mid v \in \mathfrak{M}_{\mathrm{rm}} \text{ and } k_v(\omega_v) = k_v\}\,, \\ n_{--} &= \#\{v \in \mathfrak{M}_{\boldsymbol{R}} \mid v \in \mathfrak{M}_{\mathrm{rm}} \text{ and } k_v(\omega_v) \neq k_v\}\,. \end{aligned}$$

If F is a quadratic extension of k and $F \asymp \omega_S$, then we denote the composite of F and $\tilde{k}$ by $\tilde{F}$ (which corresponds to L in Proposition 7.1). Then it is easy to see that

$$\begin{aligned} r_1(F) &= 2(n_{++} + n_{-+})\,, & r_2(F) &= n_{--} + n_{+-} + 2r_2\,, \\ r_1(F^*) &= 2(n_{++} + n_{--})\,, & r_2(F^*) &= n_{+-} + n_{-+} + 2r_2\,, \\ r_1(\tilde{F}) &= 4n_{++}\,, & r_2(\tilde{F}) &= 2(n_{+-} + n_{-+} + n_{--}) + 4r_2\,, \end{aligned}$$

and so $r_1(F)$, $r_1(F^*)$, $r_1(\tilde{F})$, $r_2(F)$, $r_2(F^*)$, and $r_2(\tilde{F})$ depend only upon ω_S. This allows us to define

$$\begin{aligned} \mathfrak{c}(\omega_S) &= 2^{r_1(F)+r_1(F^*)}(2\pi)^{r_2(F)+r_2(F^*)}\,, \\ \tilde{\mathfrak{c}}(\omega_S) &= 2^{r_1(\tilde{F})}(2\pi)^{r_2(\tilde{F})}\,, \end{aligned}$$

where $F \neq \tilde{k}$ is any quadratic extension of k satisfying $F \asymp \omega_S$.

COROLLARY 7.17. *Let $S \supseteq \mathfrak{M}_\infty$ be a finite set of places of k and ω_S an S-tuple of standard orbital representatives. Then*

$$\lim_{X\to\infty} X^{-2} \sum_{\substack{[F:k]=2,\, F\asymp\omega_S \\ \mathcal{N}(\Delta_{F/k})\le X}} h_F R_F h_{F^*} R_{F^*}$$

exists and equals

$$2^{-(r_1+r_2+1)}\mathfrak{c}(\omega_S)^{-1} e_k^2 |\Delta_{\tilde{k}}/\Delta_k|^{1/2} \mathfrak{C}_k^3 \zeta_{\tilde{k}}(2) \prod_{v\in S\setminus\mathfrak{M}_\infty} (1-q_v^{-2})^{-1}\bar{\varepsilon}_v(\omega_v) \cdot \prod_{v\notin S} E'_v\,.$$

PROOF. Let F/k be a quadratic extension and suppose that F contains a primitive n^{th} root of unity, ζ_n, for some n. Since $[\boldsymbol{Q}(\zeta_n) : \boldsymbol{Q}] = \varphi(n)$, it follows that $\varphi(n) \le [F : \boldsymbol{Q}] = 2[k : \boldsymbol{Q}]$. But it is well-known that $\varphi(n) \to \infty$ as $n \to \infty$, and so there is some constant N, independent of F, such that $n \le N$. We conclude that $e_F = e_{F^*} = e_k$ for all but finitely-many quadratic extensions F of k. This finite list of exceptions may be ignored in the limit. Since

$$\mathfrak{C}_F = 2^{r_1(F)}(2\pi)^{r_2(F)} h_F R_F e_F^{-1}\,,$$

the corollary is now an immediate consequence of the theorem and the definition of $\mathfrak{c}(\omega_S)$. $\square$

COROLLARY 7.18. *With the same assumptions as in Corollary* 7.17,

$$\lim_{X\to\infty} X^{-2} \sum_{\substack{[F:k]=2,\, F\asymp\omega_S \\ \mathcal{N}(\Delta_{F/k})\le X}} h_{\tilde{F}} R_{\tilde{F}}$$

exists and equals

$$2^{-(r_1+r_2+1)} 2^{r_1(\tilde{k})} (2\pi)^{r_2(\tilde{k})} \tilde{\mathfrak{c}}(\omega_S)^{-1} |\Delta_{\tilde{k}}/\Delta_k|^{1/2} \mathfrak{C}_k h_{\tilde{k}} R_{\tilde{k}} \zeta_{\tilde{k}}(2) \\ \times \prod_{v\in S\setminus\mathfrak{M}_\infty} (1-q_v^{-2})^{-1} \bar{\varepsilon}_v(\omega_v) \cdot \prod_{v\notin S} E'_v \,.$$

PROOF. By Proposition 7.1, $\mathfrak{C}_{\tilde{F}} = \mathfrak{C}_k^{-2}\mathfrak{C}_F\mathfrak{C}_F\mathfrak{C}_{\tilde{k}}$. So

$$\begin{aligned} h_{\tilde{F}} R_{\tilde{F}} &= 2^{-r_1(\tilde{F})}(2\pi)^{-r_2(\tilde{F})} e_{\tilde{F}} \mathfrak{C}_{\tilde{F}} \\ &= \tilde{\mathfrak{c}}(\omega_S)^{-1} e_{\tilde{F}} \mathfrak{C}_k^{-2} \mathfrak{C}_F \mathfrak{C}_{F^*} \mathfrak{C}_{\tilde{k}} \\ &= 2^{r_1(\tilde{k})} (2\pi)^{r_2(\tilde{k})} \tilde{\mathfrak{c}}(\omega_S)^{-1} \mathfrak{c}(\omega_S) e_{\tilde{F}} e_{\tilde{k}}^{-1} e_F^{-1} e_{F^*}^{-1} h_{\tilde{k}} R_{\tilde{k}} \mathfrak{C}_k^{-2} h_F R_F h_{F^*} R_{F^*} \,. \end{aligned}$$

As in the proof of Corollary 7.17, $e_F = e_{F^*} = e_k$ and $e_{\tilde{F}} = e_{\tilde{k}}$ except for a finite number of quadratic extensions F. Therefore Corollary 7.18 follows from Corollary 7.17. □

We now specialize to the case $k = \boldsymbol{Q}$ and $S = \mathfrak{M}_\infty$. Suppose $\tilde{k} = \boldsymbol{Q}(\sqrt{d_0})$ where $d_0 \neq 1$ is a square-free integer. Then $r_1 = 1, r_2 = 0, h_k = 1, e_k = 2$ and $\mathfrak{C}_k = 1$. It is easy to verify that $2^{-(r_1+r_2+1)}\mathfrak{c}(\omega_S)^{-1}e_k^2 = \mathfrak{c}(\omega_S)^{-1}$ and $2^{-(r_1+r_2+1)}2^{r_1(\tilde{k})}(2\pi)^{r_2(\tilde{k})}\tilde{\mathfrak{c}}(\omega_S)^{-1}$ both coincide with $c_\pm(d_0)^{-1}$ as defined in the introduction. Therefore Theorems 1.1 and 1.2 are special cases of Corollaries 7.17 and 7.18.

8. The omega sets and their properties. The main purpose of this section is to verify Condition 6.12. Let $v \in \mathfrak{M}_{\mathrm{f}}$ and $x \in V_{k_v}^{\mathrm{ss}}$. The function $\varXi_{x,v}(s)$ is defined as an integral over $G_{k_v}/G^\circ_{x\,k_v}$ and our strategy is to replace this by an integral over a carefully chosen set $\varOmega_{x,v} \subseteq G_{k_v}$ called the omega set. We impose on the omega set, $\varOmega_{x,v}$, several conditions derived from an analysis of Datskovsky's calculations of standard local zeta functions in [3]. Once we show that these conditions can be satisfied, Condition 6.12 is an almost immediate consequence. Thus the bulk of the work in this section is devoted to finding the omega sets and verifying their properties.

For the sake of Condition 6.12, it is enough to assume that $v \in \mathfrak{M}\setminus S_0$. However, verifying Condition 6.12 will not be our only application of the existence of omega sets. We shall also require them in certain proofs in Section 4 of [17] and, for this, greater generality will be needed. Thus we shall allow v to be any finite place of k and consider orbits of types other than three types (rm rm)*, (rm rm ur), and (rm rm rm) at dyadic places $v \in \mathfrak{M}_{\mathrm{dy}}$.

Before we begin, we shall record as a lemma a simple observation which will be useful both later in this section and in the next.

LEMMA 8.1. *Suppose that* $v \in \mathfrak{M}$, $x \in V_{k_v}^{\mathrm{ss}}$ *and* $y \in G_{k_v}x$. *If* $|P(x)|_v = |P(y)|_v$, *then* $Z_{x,v}(\varPhi, s) = Z_{y,v}(\varPhi, s)$ *for all* $\varPhi \in \mathcal{S}(V_{k_v})$.

PROOF. Examining the second equation in Definition 5.22 we see, in light of Proposition 5.23 and the hypotheses, that every factor in the definition of the local orbital zeta function remains unchanged when we replace x by y. $\square$

For each $x \in V_{k_v}^{\mathrm{ss}}$ we choose an element $g_x \in G_{k_v(x)}$ such that $g_x w = x$ and g_x satisfies Condition 5.8 if $k_v(x) \neq k_v$. From this choice we obtain an isomorphism $\theta_{g_x} : G^\circ_{x\, k_v} \to H_{x\, k_v}$, where $H_{x\, k_v}$ is defined by (5.1).

DEFINITION 8.2. A set $\Omega_{x,v} \subseteq G_{k_v}$ is called an *omega set* for x if it has the following properties:

(1) $\Omega_{x,v} x = (G_{k_v} x) \cap V_{\mathcal{O}_v}$.
(2) $K_v \Omega_{x,v} \theta_{g_x}^{-1}(H_{x\, \mathcal{O}_v}) = \Omega_{x,v}$.
(3) If $g_1, g_2 \in \Omega_{x,v}$, $h \in G^\circ_{x\, k_v}$ and $g_1 = g_2 h$, then $h \in \theta_{g_x}^{-1}(H_{x\, \mathcal{O}_v})$.
(4) If $g \in \Omega_{x,v}$, then $|\chi(g)|_v \leq 1$ with equality only if $g \in K_v$.

Below we give omega sets for representatives of each of the orbit types that we require. These include the six orbit types possible under the restriction that $v \notin S_0$, as well as the orbits of type (rm) and (rm ur). For the orbits of type (sp), (in) and (rm) it will be convenient to use $x = w$ as the orbital representative instead of the standard w_p. This is permissible for the purpose at hand by Lemma 8.1. For the orbits of types (sp ur), (sp rm), (in ur), (in rm) and (rm ur) we shall use the standard representatives.

If $p(z) = z^2 + a_1 z + a_2 \in k_v[z]$, then we shall let $\alpha = \{\alpha_1, \alpha_2\}$ be the set of roots of p and write $e(\alpha) = {}^t(1 \;\; -\alpha_1)$ (a column vector in $k_v(w_p)^2$). If $l = {}^t(l_1 \;\; l_2)$ is any such column vector, then we set $\|l\| = \max\{|l_1|_{k_v(w_p)}, |l_2|_{k_v(w_p)}\}$. Let t be as in (3.19) for the field k_v and $n(u) = (n(u_1), n(u_2), n(u_3))$ for $u = (u_1, u_2, u_3) \in k_v^3$ or $n(u) = (n(u_1), n(u_2))$ for $u = (u_1, u_2) \in \tilde{k}_v \times k_v$. Let $g = \kappa t n(u)$ be the Iwasawa decomposition of $g \in G_{k_v}$. In Section 6 we described the form of the polynomial $p(z)$ for each of the standard orbital representatives. It will be convenient here to add the assumption that $a_1 = 0$ whenever v is not dyadic, as we may.

For the index (sp) with orbital representative $x = w$ we define

$$\Omega_{x,v} = \{g = \kappa t n(u) \mid t_{ij} = 1 \text{ for } i, j = 1, 2 \text{ and } gx \in V_{\mathcal{O}_v}\}\,. \tag{8.3}$$

For the indices (in) and (rm) with orbital representative $x = w$ we define

$$\Omega_{x,v} = \{g = \kappa t n(u) \mid t_{11} = t_{12} = 1 \text{ and } gx \in V_{\mathcal{O}_v}\}\,. \tag{8.4}$$

For the index (sp ur) with orbital representative $x = w_p$ we define

$$\begin{aligned} \Omega_{x,v} = \{g = (g_1, g_2, g_3) \mid & \,|\det(g_1)|_v = 1 \text{ or } q_v^{-1}, \\ & |\det(g_2)| = 1 \text{ or } q_v, \;\; gx \in V_{\mathcal{O}_v}\}\,. \end{aligned} \tag{8.5}$$

For the index (sp rm) with orbital representative $x = w_p$ we define

$$\Omega_{x,v} = \{g = (g_1, g_2, g_3) \mid |\det(g_i)|_v = 1 \text{ for } i = 1, 2, \;\; gx \in V_{\mathcal{O}_v}\}\,. \tag{8.6}$$

For the index (in ur) with orbital representative $x = w_p$ we define

$$\Omega_{x,v} = \{g = (g_1, g_2) \mid |\det(g_1)|_{\tilde{k}_v} = 1, \|g_1 e(\alpha)\| = 1,\ gx \in V_{\mathcal{O}_v}\}. \tag{8.7}$$

For the index (in rm) with orbital representative $x = w_p$ we define

$$\Omega_{x,v} = \{g = (g_1, g_2) \mid |\det(g_1)|_{\tilde{k}_v} = 1,\ gx \in V_{\mathcal{O}_v}\}. \tag{8.8}$$

Finally, for the index (rm ur) with orbital representative $x = w_p$ we define

$$\Omega_{x,v} = \{g = (g_1, g_2) \mid |\det(g_1)|_{\tilde{k}_v} = 1 \text{ or } q_v^{-1},\ gx \in V_{\mathcal{O}_v}\}. \tag{8.9}$$

In every case we shall write

$$\Omega^1_{x,v} = \{g \in \Omega_{x,v} \mid |\chi(g)|_v = 1\}. \tag{8.10}$$

PROPOSITION 8.11. *The sets defined by* (8.3)–(8.9) *have properties* (1), (2) *and* (3) *of Definition* 8.2.

PROOF. If $\kappa \in K_v$, then $\kappa V_{\mathcal{O}_v} = V_{\mathcal{O}_v}$, $|\det(\kappa)|_v = 1$ and $\|\kappa e\| = \|e\|$ for any vector e. This makes it clear that $K_v\Omega_{x,v} = \Omega_{x,v}$ in all cases. The rest of the argument will be case by case, but we make two observations which will be used repeatedly. First, it follows at once from the definition in every case that $\Omega_{x,v}x \subseteq G_{k_v}x \cap V_{\mathcal{O}_v}$, and so to establish (1) we need only prove the reverse inclusion. This will be done if we can show that given $g \in G_{k_v}$ with $gx \in V_{\mathcal{O}_v}$ we can find $h \in G^\circ_{x\,k_v}$ such that $gh \in \Omega_{x,v}$. Secondly, any $h \in G^\circ_{x\,k_v}$ may be expressed as $h = g_x s_x(t_x) g_x^{-1}$, in the notation of (5.2)–(5.6), and $h \in \theta^{-1}_{g_x}(H_{x\,\mathcal{O}_v})$ if and only if all the components of t_x are units.

Consider the cases (sp), (in) and (rm). We may assume, for simplicity, that g_x has been chosen to be the identity. Take $g \in G_{k_v}$ with $gx \in V_{\mathcal{O}_v}$ and let $g = \kappa(g)t(g)n(u(g))$ be its Iwasawa decomposition. Let $s_x(t_x)$ be as in (5.2) or (5.4). By choosing $t_x = (t_{11}(g)^{-1}, t_{12}(g)^{-1}, t_{21}(g)^{-1}, t_{22}(g)^{-1})$ in the first case and $t_x = (t_{11}(g)^{-1}, t_{12}(g)^{-1})$ in the second, we may arrange that $gs_x(t_x) \in \Omega_{x,v}$. This proves Property (1). Moreover, if $g \in \Omega_{x,v}$ and all the components of t_x are units, then commuting $s_x(t_x)$ past the T_{k_v} and N_{k_v} factors in the Iwasawa decomposition and absorbing it into the K_v factor shows that $gs_x(t_x) \in \Omega_{x,v}$ also, which proves Property (2). For Property (3), observe that in the Iwasawa decomposition, the T_{k_v} factor is unique up to multiplication of its diagonal elements by units. Thus if $g_1, g_2 \in \Omega_{x,v}$ and $g_1 = g_2 h$ with $h = s_x(t_x)$, then $s_x(t_x) \in H_{x\,\mathcal{O}_v}$. This proves Property (3).

We next turn to case (sp ur). Let $s_x(t_x)$ be as in (5.3) and $g \in G_{k_v}$ with $gx \in V_{\mathcal{O}_v}$. Note that

$$|\det s_{x1}(t_x)|_v = |\mathrm{N}_{k_v(x)/k_v}(t_{11})|_v$$

and since $k_v(x)/k_v$ is unramified, this may be any even power of q_v. The same holds for $|\det s_{x2}(t_x)|_v$ and the determinants of the components of $g_x s_x(t_x) g_x^{-1}$ are the same as those of the corresponding components of $s_x(t_x)$. It follows that we can arrange $g\left(g_x s_x(t_x) g_x^{-1}\right) \in \Omega_{x,v}$ for a suitable choice of t_x and this proves (1). If $s_x(t_x) \in H_{x\,\mathcal{O}_v}$, then the determinants of each of its components are units and this makes (2) obvious. Also, this argument shows that if $g_1, g_2 \in \Omega_{x,v}$, $h = g_x s_x(t_x) g_x^{-1}$ and $g_1 = g_2 h$, then t_{11} and t_{21} are units, which implies that $s_x(t_x) \in H_{x\,\mathcal{O}_v}$; hence (3).

The case (sp rm) is very similar, with the one difference that since $k_v(x)/k_v$ is ramified, $|\det s_{xj}(t_x)|_v$ can be any integer power of q_v.

Next we treat (in ur). Let $g = (g_1, g_2) \in G_{k_v}$ with $gx \in V_{\mathcal{O}_v}$ and $s_x(t_x)$ be as in (5.5). Note that $e(\alpha)$ is an eigenvector for the first component of $h = g_x s_x(t_x) g_x^{-1}$ with eigenvalue t_{11}. So if $h = (h_1, h_2)$, then $\|g_1 h_1 e(\alpha)\| = |t_{11}|_{\tilde{k}_v} \|g_1 e(\alpha)\|$. Also, $|\det(g_1 h_1)|_{\tilde{k}_v} = |\det(g_1)|_{\tilde{k}_v} |t_{11} t_{12}|_{\tilde{k}_v}$. We are free to choose the pair $(t_{11}, t_{11}t_{12}) \in \tilde{k}_v^2$ arbitrarily, and so there exists $h \in G^\circ_{x\,k_v}$ with $gh \in \Omega_{x,v}$, proving (1). If $g \in \Omega_{x,v}$ and $h \in \theta_{g_x}^{-1}(H_{x\,\mathcal{O}_v})$, then t_{11} and t_{12} are units, and so $\|ghe(\alpha)\| = \|ge(\alpha)\|$ and $|\det(gh)|_{\tilde{k}_v} = |\det(g)|_{\tilde{k}_v}$, which proves (2). Also, if $g_1, g_2 \in \Omega_{x,v}$, $h = g_x s_x(t_x) g_x^{-1}$ and $g_1 = g_2 h$, then $|t_{11}|_{\tilde{k}_v} = |t_{11}t_{12}|_{\tilde{k}_v} = 1$, which implies that $h \in \theta_{g_x}^{-1}(H_{x\,\mathcal{O}_v})$ and (3) follows.

Finally, Cases (in rm) and (rm ur) are very similar to Cases (sp rm) and (sp ur). Note that if $s_x(t_x)$ is as in (5.6), then $|\det s_{x1}(t_x)|_{\tilde{k}_v} = |\mathrm{N}_{\tilde{k}_v(x)/\tilde{k}_v}(t_{11})|_{\tilde{k}_v}$. In Case (in rm), $\tilde{k}_v(x)/\tilde{k}_v$ is ramified, and so this takes every value in $|\tilde{k}_v^\times|_{\tilde{k}_v}$. In Case (rm ur), $\tilde{k}_v(x)/\tilde{k}_v$ is unramified, and so $|\det s_{x1}(t_x)|_{\tilde{k}_v}$ takes every value in $|(\tilde{k}_v^\times)^2|_{\tilde{k}_v}$. The rest of the argument is identical to that in the cases already mentioned. □

Using only Parts (1), (2) and (3) of Definition 8.2 we can prove the following.

PROPOSITION 8.12. *Let $\Psi_{x,v}$ be the characteristic function of $\Omega_{x,v}$. Then*

$$Z_{x,v}(\Phi_{v,0}, s) = \int_{G_{k_v}} |\chi(g)|_v^s \Psi_{x,v}(g) dg_v\,.$$

PROOF. Since

$$dg_v = d\tilde{g}_v d^\times \tilde{t}_v\,, \quad dg''_{x,v} = d\tilde{g}''_{x,v} d^\times \tilde{t}_v\,, \quad dg_v = b_{x,v} dg'_{x,v} dg''_{x,v}\,,$$

$d\tilde{g}_v = b_{x,v} dg'_{x,v} d\tilde{g}''_{x,v}$. So the right hand side of the above identity is

$$b_{x,v} \int_{G_{k_v}/G^\circ_{x\,k_v}} |\chi(g'_{x,v})|_v^s \left(\int_{G^\circ_{k_v}} \Psi_{x,v}(g'_{x,v} g''_{x,v})\, dg''_{x,v} \right) dg'_{x,v}\,. \tag{8.13}$$

By (2) and (3) of Definition 8.2, $\Psi_{x,v}(g'_{x,v} g''_{x,v})$ is non-zero if and only if $g'_{x,v} \in \Omega_{x,v}$ and $g''_{x,v} \in \theta_{g_x}^{-1}(H_{x\,\mathcal{O}_v})$. Since we chose the measure $dg''_{x,v}$ so that the volume of this set is one,

$$\int_{G^\circ_{k_v}} \Psi_{x,v}(g'_{x,v} g''_{x,v}) dg''_{x,v}$$

is the characteristic function of $\Omega_{x,v} G^\circ_{x\,k_v}/G^\circ_{x\,k_v} \cong G_{k_v} x \cap V_{\mathcal{O}_v}$. Therefore, (8.13) is

$$b_{x,v} \int_{G_{k_v}/G^\circ_{x\,k_v}} |\chi(g'_{x,v})|_v^s \Phi_{v,0}(g'_{x,v} x) dg'_{x,v}\,,$$

which is the definition of $Z_{x,v}(\Phi_{v,0}, s)$. □

Before we verify Part (4) of Definition 8.2 it will be convenient to prove three lemmas. First note that we may let $\mathrm{GL}(2)_{k_v}$ act on the space of quadratic polynomials in $k_v[z]$ by regarding such polynomials as the inhomogeneous forms of binary quadratic forms. With this convention, if $p(z) = z^2 + a_1 z + a_2 \in k_v[z]$ and $g = a(t_1, t_2)n(u)$, then

$$g\,p(z) = t_1^2 z^2 + t_1 t_2 (2u + a_1) z + t_2^2 (u^2 + a_1 u + a_2)\,.$$

LEMMA 8.14. *Suppose that $p(z)$ is an Eisenstein polynomial. Let $t \in k_v^\times$, $u \in k_v$, $i = 0$ or 1 and suppose that $\pi_v^i a(t, t^{-1}\pi_v^{-i}) n(u) p(z) \in \mathcal{O}_v[z]$. Then $t \in \mathcal{O}_v^\times$ and $u \in \mathcal{O}_v$. Moreover, if $i = 1$, then $u \in \mathfrak{p}_v$.*

PROOF. We have $\pi_v^i t^2 \in \mathcal{O}_v$, which implies that $t \in \mathcal{O}_v$ since $i = 0$ or 1. Since $t^{-2}\pi_v^{-i}(u^2 + a_1 u + a_2) \in \mathcal{O}_v$, $(u^2 + a_1 u + a_2) \in t^2 \pi_v^i \mathcal{O}_v$. In particular, $u^2 + a_1 u + a_2 \in \mathcal{O}_v$, and so $u(u + a_1) \in \mathcal{O}_v$. If $u \notin \mathcal{O}_v$, then $\mathrm{ord}(u + a_1) = \mathrm{ord}(u)$ and we reach a contradiction. Hence $u \in \mathcal{O}_v$. The order of $u^2 + a_1 u + a_2$ is either 0 (if $u \in \mathcal{O}_v^\times$) or 1 (if $u \in \mathfrak{p}_v$). If $i = 0$, this forces $t \in \mathcal{O}_v^\times$ and if $i = 1$ it forces first $t \in \mathcal{O}_v^\times$ and then $u \in \mathfrak{p}_v$.

LEMMA 8.15. *Suppose that $p(z) = z^2 + a_2$ with $-a_2 \in \mathcal{O}_v^\times \setminus (\mathcal{O}_v^\times)^2$, if $v \notin \mathfrak{M}_{\mathrm{dy}}$, or that $p(z)$ is an Artin-Schreier polynomial, if $v \in \mathfrak{M}_{\mathrm{dy}}$. Let $t \in k_v^\times$, $u \in k_v$, $i = -1$, 0 or 1 and suppose that $\pi_v^i a(t, t^{-1}\pi_v^{-i}) n(u) p(z) \in \mathcal{O}_v[z]$. Then $i = 0$, $t \in \mathcal{O}_v^\times$ and $u \in \mathcal{O}_v$.*

PROOF. The conditions imply that $\pi_v^i t^2$ and $t^{-2}\pi_v^{-i} p(u)$ are integral. Since $-1 \le i \le 1$, $t \in \mathcal{O}_v$. Thus $p(u) \in \pi_v^{-1}\mathcal{O}_v$, which implies that $u(u + a_1) \in \pi_v^{-1}\mathcal{O}_v$. If $u \notin \mathcal{O}_v$, then $\mathrm{ord}(u) = \mathrm{ord}(u + a_1)$, and so $\mathrm{ord}(u(u + a_1))$ is a negative, even integer. This is a contradiction, and so $u \in \mathcal{O}_v$. The reduction of the polynomial $p(z)$ has no roots in $\mathcal{O}_v/\mathfrak{p}_v$ and thus $p(u) \in \mathcal{O}_v^\times$ for all $u \in \mathcal{O}_v$. It follows that $t^2 \pi_v^i \in \mathcal{O}_v^\times$. This gives $i = 0$ and $t \in \mathcal{O}_v^\times$, as required. □

LEMMA 8.16. *Let x be a standard orbital representative and suppose that $y \in V_{\mathcal{O}_v}$ lies in the orbit of x under G_{k_v}. Then $|P(y)|_v \le |P(x)|_v$.*

PROOF. If $k_v(x) = k_v$, then $|P(x)|_v = 1$ and $P(y) \in \mathcal{O}_v$ since $y \in V_{\mathcal{O}_v}$. The statement follows in this case. We now assume that $k_v(x) \ne k_v$. Let $F_y(v_1, v_2) = b_0 v_1^2 + b_1 v_1 v_2 + b_2 v_2^2$ and consider the polynomial $r(z) = z^2 + b_1 z + b_0 b_2$. Since $y \in V_{\mathcal{O}_v}$, $b_0, b_1, b_2 \in \mathcal{O}_v$, and so $r(z) \in \mathcal{O}_v[z]$. The discriminant of $r(z)$ is equal to the discriminant of F_y, and so if β is a root of $r(z)$, then $\beta \in k_v(y) = k_v(x)$. It follows that $\mathcal{O}_v[\beta] \subseteq \mathcal{O}_{k_v(x)}$ and hence that $P(y)\mathcal{O}_v \subseteq \Delta_{k_v(x)/k_v}$. But the standard orbital representative was chosen so that $\Delta_{k_v(x)/k_v} = P(x)\mathcal{O}_v$ and the statement follows in this case also. □

PROPOSITION 8.17. *The sets defined by* (8.3)–(8.9) *have property* (4) *of Definition* 8.2. *Consequently, they are omega sets.*

PROOF. If $g \in \Omega_{x,v}$, then $gx \in V_{\mathcal{O}_v}$, and so $|P(gx)|_v \le |P(x)|_v$ by Lemma 8.16. But $|P(gx)|_v = |\chi(g)|_v |P(x)|_v$ and it follows that $|\chi(g)|_v \le 1$. This establishes the first part of (4) in Definition 8.2.

We now have to show that if $g \in \Omega^1_{x,v}$, then $g \in K_v$. The orbital representatives have already been fixed in (8.3)–(8.9) and the notation introduced there will be used without comment below.

We begin with the Cases (sp), (in) and (rm). Let $g \in \Omega^1_{x,v}$; we have to show that $g \in K_v$. By (2) of Definition 8.2, $\Omega^1_{x,v}$ is left K_v-invariant, and so we may assume that $g = tn(u)$. Since $g \in \Omega_{x,v}$ we have $t_{11} = t_{12} = t_{21} = t_{22} = 1$ in Case (sp) and $t_{11} = t_{12} = 1$ in Cases (in) and (rm). The assumption that $|\chi(g)|_v = 1$ implies that $|t_{31}t_{32}|_v = 1$ in Case (sp) and that $|t_{21}t_{22}|_v = 1$ in Cases (in) and (rm). In Case (sp) we have

$$gw = \left(t_{31}\begin{pmatrix}1 & u_2\\ u_1 & u_1u_2\end{pmatrix}, t_{32}\begin{pmatrix}u_3 & u_2u_3\\ u_1u_3 & 1+u_1u_2u_3\end{pmatrix}\right), \tag{8.18}$$

and in Cases (in) and (rm) we have

$$gw = \left(t_{21}\begin{pmatrix}1 & u_1^\sigma\\ u_1 & \mathrm{N}_{\tilde{k}_v/k_v}(u_1)\end{pmatrix}, t_{22}\begin{pmatrix}u_3 & u_1^\sigma u_3\\ u_1u_3 & 1+\mathrm{N}_{\tilde{k}_v/k_v}(u_1)u_3\end{pmatrix}\right). \tag{8.19}$$

Let $a = \mathrm{ord}_{k_v}(t_{31})$ or $\mathrm{ord}_{k_v}(t_{21})$. Then, by assumption, $\mathrm{ord}_{k_v}(t_{32}) = -a$ or $\mathrm{ord}_{k_v}(t_{22}) = -a$. Consider Case (sp). Let $\bar{u}_i = \pi_v^a u_i$ for $i = 1, 2$, and $\bar{u}_3 = \pi_v^{-a}u_3$. Then $gw \in V_{\mathcal{O}_v}$ if and only if

$$\pi_v^a, \quad \bar{u}_1, \bar{u}_2, \bar{u}_3, \quad \pi^{-a}\bar{u}_1\bar{u}_2, \quad \pi^{-a}\bar{u}_1\bar{u}_3, \quad \pi^{-a}\bar{u}_2\bar{u}_3, \quad \pi_v^{-a}(1+\pi_v^{-a}\bar{u}_1\bar{u}_2\bar{u}_3)$$

are integral. So $a \geq 0$. We assume $a > 0$ and deduce a contradiction. Suppose $\bar{u}_1$ is not a unit. Then

$$\pi_v^{-a}\bar{u}_1\bar{u}_2\bar{u}_3 = (\pi_v^{-a}\bar{u}_2\bar{u}_3)\bar{u}_1 \equiv 0 \ (\mathfrak{p}_v).$$

Then $1+\pi_v^{-a}\bar{u}_1\bar{u}_2\bar{u}_3$ is a unit. This implies $\pi_v^{-a}(1+\pi_v^{-a}\bar{u}_1\bar{u}_2\bar{u}_3) \notin \mathcal{O}_v$, which is a contradiction. So $\bar{u}_1$ is a unit and similarly $\bar{u}_2, \bar{u}_3$ are units also. Then the order of $\pi_v^{-a}(1+\pi_v^{-a}\bar{u}_1\bar{u}_2\bar{u}_3)$ is $-2a$, which is a contradiction. This implies $a = 0$. Then $u_i \in \mathcal{O}_v$ for $i = 1, 2, 3$. Cases (in) and (rm) are similar using u_1, u_1^σ, u_2 in the places of u_1, u_2, u_3 above. The only difference is that we consider elements in $\tilde{\mathcal{O}}_v$.

Next we treat the Case (sp rm). Suppose $g = (g_1, g_2, g_3) \in \Omega^1_{x,v}$. Then $|\det g_i|_v = 1$ for $i = 1, 2, 3$. We may assume that g_1, g_2, g_3 are lower triangular. Note that $F_{w_p}(z, 1) = p(z)$. So $F_{gw_p}(z, 1) = (\det g_1 \det g_2)g_3 p(z)$ is integral. Since $\det g_1, \det g_2 \in \mathcal{O}_v^\times$, we have $g_3 \in \mathrm{GL}(2)_{\mathcal{O}_v}$ by Lemma 8.14.

In this case, we can regard V as $\mathrm{Aff}^2 \otimes \mathrm{Aff}^2 \otimes \mathrm{Aff}^2$. Instead of the third factor, we can use the first and the second factors to make equivariant maps similar to F_x. Then because of the symmetry of our element w_p, we have $g_1, g_2 \in \mathrm{GL}(2)_{\mathcal{O}_v}$ by Lemma 8.14 again. This concludes the verification in this case.

Now we consider the Case (sp ur). Let $g = (g_1, g_2, g_3) \in \Omega_{x,v}$ and $|\chi(g)|_v = 1$. In this case there are four possibilities as follows:

(A) $|\det g_1|_v = |\det g_2|_v = 1$,
(B) $|\det g_1|_v = 1$, $|\det g_2|_v = q_v^{-1}$,
(C) $|\det g_1|_v = q_v^{-1}$, $|\det g_2|_v = 1$,
(D) $|\det g_1|_v = q_v^{-1}$, $|\det g_2|_v = q_v$.

In these cases, $|\det g_3|_v = 1, q_v, q_v, 1$, respectively. The argument in Case (A) is similar to that used in Case (sp rm). In Case (B), $F_{gw_p}(z,1) = \pi_v g_3 p(z)$, and $\det g_3 = \pi_v^{-1}$. Since $F_{gw_p}(z,1)$ is integral, this corresponds to the case $i = 1$ in Lemma 8.15. Therefore this cannot happen. Cases (C), (D) are similar to Case (B) because of the symmetry (considering an equivariant map using the second Aff^2 factor in Case (D)).

Now we consider the case (in ur). Suppose that $g = (g_1, g_2) \in \Omega^1_{x,v}$. This implies that $|\det(g_1)|_{\tilde{k}_v} = |\det(g_2)|_v = 1$. We have

$$F_{gx}(z,1) = \mathrm{N}_{\tilde{k}_v/k_v}(\det g_1) g_2 p(z)$$

and, since $\mathrm{N}_{\tilde{k}_v/k_v}(\det g_1)$ is a unit by assumption, $g_2 \in \mathrm{GL}(2)_{\mathcal{O}_v}$ by Lemma 8.15. Since $\Omega_{x,v}$ is left K_v-invariant we may assume that $g_2 = 1$ and that g_1 is lower triangular, say $g_1 = a(t_{11}, t_{12})n(u_1)$. Note that

$$g_1 e(\alpha) = \begin{pmatrix} t_{11} \\ t_{12}(u_1 - \alpha_1) \end{pmatrix} \tag{8.20}$$

and this is a primitive integral vector. Computation gives $(g_1, 1)w_p = (M_1, M_2)$, where

$$\begin{aligned} M_1 &= \begin{pmatrix} 0 & t_{11}t_{12}^{\sigma} \\ t_{11}^{\sigma}t_{12} & \mathrm{N}_{\tilde{k}_v/k_v}(t_{12})[\mathrm{Tr}_{\tilde{k}_v/k_v}(u_1) + a_1] \end{pmatrix}, \\ M_2 &= \begin{pmatrix} \mathrm{N}_{\tilde{k}_v/k_v}(t_{11}) & t_{11}t_{12}^{\sigma}(u_1^{\sigma} + a_1) \\ t_{11}^{\sigma}t_{12}(u_1 + a_1) & \mathrm{N}_{\tilde{k}_v/k_v}(t_{12})m(u_1, p) \end{pmatrix} \end{aligned} \tag{8.21}$$

with

$$m(u_1, p) = a_1^2 - a_2 + a_1\mathrm{Tr}_{\tilde{k}_v/k_v}(u_1) + \mathrm{N}_{\tilde{k}_v/k_v}(u_1)$$

and both these matrices must be integral. Let $\bar{u}_1 = u_1 - \alpha_1$. Then $\mathrm{Tr}_{\tilde{k}_v/k_v}(u_1) + a_1 = \mathrm{Tr}_{\tilde{k}_v/k_v}(\bar{u}_1)$ and

$$m(u_1, p) = \mathrm{N}_{\tilde{k}_v/k_v}(\bar{u}_1) - \mathrm{Tr}_{\tilde{k}_v/k_v}(\alpha_1\bar{u}_1),$$

and so M_1 and M_2 are integral if and only if

$$t_{11}, \quad \bar{u}_1, \quad \mathrm{N}_{\tilde{k}_v/k_v}(t_{12})\mathrm{Tr}_{\tilde{k}_v/k_v}(\bar{u}_1), \quad \mathrm{N}_{\tilde{k}_v/k_v}(t_{12})[\mathrm{N}_{\tilde{k}_v/k_v}(\bar{u}_1) - \mathrm{Tr}_{\tilde{k}_v/k_v}(\alpha_1\bar{u}_1)]$$

are integral. Since $\alpha_1 \in \tilde{\mathcal{O}}_v$, it follows that $u_1 \in \tilde{\mathcal{O}}_v$. Also $t_{11} \in \tilde{\mathcal{O}}_v$ and it remains to show that t_{11} and t_{12} are units. From the definition of $\Omega_{x,v}$ we know that $|t_{11}t_{12}|_{\tilde{k}_v} = 1$. Let $\mathrm{ord}_{\tilde{k}_v}(t_{11}) = i$; we assume that $i > 0$ and deduce a contradiction. We have $\mathrm{ord}_{\tilde{k}_v}(t_{12}) = -i$ and, from (8.20), we conclude that $\mathrm{ord}_{\tilde{k}_v}(\bar{u}_1) = i$. Thus we may write $\bar{u}_1 = \pi_v^i(\bar{u}_{11} + \bar{u}_{12}\alpha_1)$, where $\bar{u}_{11}, \bar{u}_{12} \in \mathcal{O}_v$ and $\bar{u}_{11} + \bar{u}_{12}\alpha_1 \in \tilde{\mathcal{O}}_v^{\times}$. Then

$$\begin{aligned} \mathrm{N}_{\tilde{k}_v/k_v}(\bar{u}_1) &= \pi_v^{2i}[\bar{u}_{11}^2 - a_1\bar{u}_{11}\bar{u}_{12} + a_2\bar{u}_{12}^2], \\ \mathrm{Tr}_{\tilde{k}_v/k_v}(\bar{u}_1) &= \pi_v^i[2\bar{u}_{11} - a_1\bar{u}_{12}], \\ \mathrm{Tr}_{\tilde{k}_v/k_v}(\alpha_1\bar{u}_1) &= \pi_v^i[-a_1\bar{u}_{11} + (a_1^2 - 2a_2)\bar{u}_{12}] \end{aligned}$$

and, since $\mathrm{ord}_{k_v}(\mathrm{N}_{\tilde{k}_v/k_v}(t_{12})) = -2i$, it follows that

$$(8.22)\qquad \begin{aligned} -a_1\bar{u}_{11} + (a_1^2 - 2a_2)\bar{u}_{12} &\equiv 0\ (\mathfrak{p}_v^i)\,,\\ 2\bar{u}_{11} - a_1\bar{u}_{12} &\equiv 0\ (\mathfrak{p}_v^i)\,. \end{aligned}$$

Regarding this as a linear system for $(\bar{u}_{11}, \bar{u}_{12})$, the determinant of the coefficient matrix is $-a_1^2+4a_2 = -P(x)$. This is a unit by the choice of x, and so (8.22) implies that $(\bar{u}_{11}, \bar{u}_{12}) \equiv (0, 0)\ (\mathfrak{p}_v)$. This contradicts $\bar{u}_{11}+\bar{u}_{12}\alpha_1 \in \tilde{\mathcal{O}}_v^\times$, and so $i = 0$. This completes the case (in ur).

Next we must deal with the case (in rm). Suppose that $g = (g_1, g_2) \in \Omega^1_{x,v}$. By arguments similar to those in the previous case, using Lemma 8.14 in place of Lemma 8.15, we see that $g_2 \in \mathrm{GL}(2)_{\mathcal{O}_v}$. Hence we may assume that $g_2 = 1$ and that $g_1 = a(t_{11}, t_{12})n(u_1)$ is lower triangular. Then $(g_1, 1)w_p = (M_1, M_2)$, where M_1 and M_2 are given by (8.21). Since $t_{11}t_{12}^\sigma \in \tilde{\mathcal{O}}_v^\times$, M_1 and M_2 are integral if and only if

$$(8.23)\qquad t_{11}\,,\quad u_1\,,\quad \mathrm{N}_{\tilde{k}_v/k_v}(t_{12})[\mathrm{Tr}_{\tilde{k}_v/k_v}(u_1) + a_1]\,,\quad \mathrm{N}_{\tilde{k}_v/k_v}(t_{12})m(u_1, p)$$

are integral. Let $\mathrm{ord}_{\tilde{k}_v}(t_{11}) = i$; we shall again assume that $i > 0$ and derive a contradiction. We have $\mathrm{ord}_{\tilde{k}_v}(t_{12}) = -i$, so that $\mathrm{ord}_{k_v}(\mathrm{N}_{\tilde{k}_v/k_v}(t_{12})) = -2i$. Thus $\mathrm{Tr}_{\tilde{k}_v/k_v}(u_1) \equiv -a_1\ (\mathfrak{p}_v^{2i})$ and, since $p(z)$ is an Eisenstein polynomial, it follows that $\mathrm{Tr}_{\tilde{k}_v/k_v}(u_1) \equiv 0\ (\mathfrak{p}_v)$. Also, $m(u_1, p) \equiv 0\ (\mathfrak{p}_v^{2i})$ and, using our conclusion about $\mathrm{Tr}_{\tilde{k}_v/k_v}(u_1)$ together with the fact that $p(z)$ is an Eisenstein polynomial, we deduce that $\mathrm{N}_{\tilde{k}_v/k_v}(u_1) \equiv a_2\ (\mathfrak{p}_v^2)$. But $\mathrm{ord}_{k_v}(a_2) = 1$ and $\mathrm{ord}_{k_v}(\mathrm{N}_{\tilde{k}_v/k_v}(u_1)) = 2\mathrm{ord}_{\tilde{k}_v}(u_1)$ is always even, so this last congruence is impossible. This contradiction completes the case (in rm).

Finally we must deal with the case (rm ur). Suppose that $g = (g_1, g_2) \in \Omega^1_{x,v}$. There are apparently two possibilities: either $|\det(g_1)|_{\tilde{k}_v} = |\det(g_2)|_v = 1$ or $|\det(g_1)|_{\tilde{k}_v} = q_v^{-1}$ and $|\det(g_2)|_v = q_v$. However, Lemma 8.15 shows that the second possibility cannot occur and, moreover, that $g_2 \in \mathrm{GL}(2)_{\mathcal{O}_v}$. Thus we may assume, as usual, that $g_1 = a(t_{11}, t_{12})n(u_1)$ and $g_2 = 1$. The matrices M_1 and M_2 given by (8.21) must be integral and, since $t_{11}t_{12}^\sigma$ is a unit, this happens if and only if the quantities enumerated in (8.23) are all integral. Again assume that $\mathrm{ord}_{\tilde{k}_v}(t_{11}) = i$ and that $i > 0$. Then $\mathrm{Tr}_{\tilde{k}_v/k_v}(u_1) + a_1 \equiv 0\ (\mathfrak{p}_v^i)$. If v is dyadic, then $a_1 = -1$, and so this congruence forces $\mathrm{Tr}_{\tilde{k}_v/k_v}(u_1)$ to be a unit. However, since $\tilde{k}_v/k_v$ is ramified, $u_1^\sigma \equiv u_1\ (\tilde{\mathfrak{p}}_v)$, and so $\mathrm{Tr}_{\tilde{k}_v/k_v}(u_1) \equiv 2u_1 \equiv 0\ (\tilde{\mathfrak{p}}_v)$, which implies that $\mathrm{Tr}_{\tilde{k}_v/k_v}(u_1)$ is not a unit. This contradiction completes that proof in the dyadic case. Now assume that v is not dyadic. Then $a_1 = 0$, and so $\mathrm{Tr}_{\tilde{k}_v/k_v}(u_1)$ is not a unit. We can write $u_1 = u_{11} + u_{12}\sqrt{\pi_v}$ with $u_{11}, u_{12} \in \mathcal{O}_v$ and a suitable choice of uniformizer π_v. Since $\mathrm{Tr}_{\tilde{k}_v/k_v}(u_1) = 2u_{11}$, we conclude that u_{11} is not a unit and hence that u_1 is not a unit. However, $m(u_1, p) = -a_2 + \mathrm{N}_{\tilde{k}_v/k_v}(u_1) \equiv 0\ (\mathfrak{p}_v^i)$ and a_2 is a unit. This contradiction completes the proof in the non-dyadic case. □

Having completed the verification that $\Omega_{x,v}$ is an omega set in every case, we can now quickly achieve the aim of this section.

COROLLARY 8.24. *Condition* 6.12 *holds. Moreover*, $a_{x,v,n} = 0$ *if* n *is odd.*

PROOF. Let $v \in \mathfrak{M} \setminus S_0$ and $y \in V^{\mathrm{ss}}_{k_v}$. From Lemma 8.1, Proposition 8.12 and the choices made above we have

$$\Xi_{y,v}(s) = Z_{x,v}(\Phi_{v,0}, s) = \int_{\Omega_{x,v}} |\chi(g)|_v^s \, dg_v \,, \tag{8.25}$$

where x is the representative we have chosen here to represent the orbit of y. Let $V_j = \{g \in \Omega_{x,v} \mid |\chi(g)|_v = q_v^{-j}\}$. From (8.25) we obtain

$$\Xi_{y,v}(s) = \sum_{j=-\infty}^{\infty} \mathrm{vol}(V_j) q_v^{-js} \,.$$

However, we have $V_j = \emptyset$ if $j < 0$ from (4) in the definition of an omega set. Thus the sum really only extends from 0 to ∞ and $a_{y,v,n} = \mathrm{vol}(V_n)$ for $n \geq 0$. This makes it clear that $a_{y,v,n} \geq 0$ for all n. Since χ is the square of a rational character, we have $V_n = \emptyset$ if n is odd, and this gives the last statement. Finally, again by (4) of the definition, $V_0 = \Omega^1_{x,v} = K_v$, and so $a_{y,v,0} = \mathrm{vol}(K_v) = 1$. □

9. The estimate of the local zeta functions. The purpose of this section is to verify Condition 6.13. So we assume that $v \in \mathfrak{M} \setminus S_0$ and $x \in V^{\mathrm{ss}}_{k_v}$. Our method will be to estimate $\Xi_{x,v}(s)$ by expressing it as an integral over a domain, Γ_v, adapted to the purposes of this section as the omega sets were to those of Section 8. Throughout this section, if T_{x1} and T_{x2} are distributions depending on x and $T_{x1} = C_x T_{x2}$ for some constant $C_x \neq 0$, then we shall write $T_{x1} \propto T_{x2}$. After working with such proportionality statements, we shall appeal to the results of Section 8 to strengthen them to inequalities. Thus the results of this section depend logically on those of the last.

We introduce the following objects ($j \geq 0$ in the last equation).

$$\begin{aligned}
\gamma &= \begin{cases} (a(1,t_1)n(u_1), a(1,t_2)n(u_2), n(u_3)a(t_3,t_4)) & v \in \mathfrak{M}_{\mathrm{sp}}\,, \\ (a(1,t_1)n(u_1), n(u_2)a(t_2,t_3)) & v \notin \mathfrak{M}_{\mathrm{sp}}\,, \end{cases} \\
d\gamma &= \begin{cases} d^\times t_1 d^\times t_2 d^\times t_3 d^\times t_4 du_1 du_2 du_3 & v \in \mathfrak{M}_{\mathrm{sp}}\,, \\ d^\times t_1 d^\times t_2 d^\times t_3 du_1 du_2 & v \notin \mathfrak{M}_{\mathrm{sp}}\,, \end{cases} \\
\Gamma_v &= \begin{cases} \{\gamma \mid t_1, t_2, t_3, t_4 \in k_v^\times,\ u_1, u_2, u_3 \in k_v\} & v \in \mathfrak{M}_{\mathrm{sp}}\,, \\ \{\gamma \mid t_1 \in \tilde{k}_v^\times, t_2, t_3 \in k_v^\times,\ u_1 \in \tilde{k}_v, u_2 \in k_v\} & v \notin \mathfrak{M}_{\mathrm{sp}}\,, \end{cases} \\
\Gamma_v^j &= \begin{cases} \{\gamma \in \Gamma_v \mid |t_1t_2t_3t_4|_v = q_v^{-j}\} & v \in \mathfrak{M}_{\mathrm{sp}}\,, \\ \{\gamma \in \Gamma_v \mid |\mathrm{N}_{\tilde{k}_v/k_v}(t_1)t_2t_3|_v = q_v^{-j}\} & v \notin \mathfrak{M}_{\mathrm{sp}}\,. \end{cases}
\end{aligned} \tag{9.1}$$

In the above definition, $d^\times t_1, du_1$, etc., are the standard measures on $k_v^\times, \tilde{k}_v^\times, k_v$, or $\tilde{k}_v$, and $d\gamma$ is thus a measure on Γ_v right invariant with respect to the last entry and left invariant with respect to the other entries.

LEMMA 9.2. *If $x \in V_{k_v}^{\mathrm{ss}}$ is a standard orbital representative, then*

$$\int_{G_{k_v}/G^\circ_{x\,k_v}} f(g'_{x,v}x)\,dg'_{x,v} \propto \int_{\Gamma_v} f(\gamma x)\,d\gamma$$

for every $f \in L^1(G_{k_v}x)$ which is invariant on the left by the action of elements of the form $(1,1,\kappa)$ or $(1,\kappa)$ with $\kappa \in \mathrm{GL}(2)_{\mathcal{O}_v}$.

PROOF. We begin with the case $v \in \mathfrak{M}_{\mathrm{sp}}$. Define

$$\bar{\Gamma}_v = \left\{ \bar{\gamma} \in G_{k_v} \;\middle|\; \begin{array}{c} \bar{\gamma} = (a(1,t_1)n(u_1), a(1,t_2)n(u_2), g_3) \\ t_1, t_2 \in k_v^\times,\ u_1, u_2 \in k_v \end{array} \right\}. \tag{9.3}$$

Suppose that $x = w_p$, where $p(z) = z^2 + a_1 z + a_2$ (recall that all the standard orbital representatives have this form). We claim that $\bar{\Gamma}_v \cap G^\circ_{x\,k_v} = \{1\}$ and that

$$\bar{\Gamma}_v G^\circ_{x\,k_v} = \{(g_1, g_2, g_3) \mid g_{i11}^2 + a_1 g_{i11} g_{i12} + a_2 g_{i12}^2 \neq 0,\ i = 1,2\}.$$

The elements of the group $G^\circ_{x\,k_v}$ have the form described in Lemma 3.27. If an element $a(1,t)n(u)$ is of the form $A_p(c,d)$ in (3.26) then

$$\begin{pmatrix} 1 & 0 \\ tu & t \end{pmatrix} = \begin{pmatrix} c & -d \\ a_2 d & c - a_1 d \end{pmatrix}.$$

Therefore, $c = 1$ and $d = 0$. This implies that $\bar{\Gamma}_v \cap G^\circ_{x\,k_v} = \{1\}$. Since the last entry in elements of $\bar{\Gamma}_v$ is unrestricted, we need only to show that the equation

$$\begin{pmatrix} 1 & 0 \\ u' & t \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} c & -d \\ a_2 d & c - a_1 d \end{pmatrix} \tag{9.4}$$

is always solvable for $t \neq 0$, u' and c and d satisfying $c^2 - a_1 cd + a_2 d^2 \neq 0$ provided that $m_{11}^2 + a_1 m_{11} m_{12} + a_2 m_{12}^2 \neq 0$ and the matrix (m_{ij}) is non-singular.

If (9.4) holds, we must take $c = m_{11}$ and $d = -m_{12}$ and then the equation is equivalent to

$$\begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix} \begin{pmatrix} u' \\ t \end{pmatrix} = \begin{pmatrix} -a_2 m_{12} \\ m_{11} + a_1 m_{12} \end{pmatrix},$$

which is solvable for t and u' since the coefficient matrix is non-singular by hypothesis. If $t = 0$, then we have $u' m_{11} = -a_2 m_{12}$ and $u' m_{12} = m_{11} + a_1 m_{12}$. Multiplying the first equation by m_{12}, the second by m_{11} and subtracting, we obtain $m_{11}^2 + a_1 m_{11} m_{12} + a_2 m_{12}^2 = 0$, contrary to hypothesis. This proves the second claim.

Let $d_l \bar{\gamma} = d^\times t_1\, d^\times t_2\, du_1\, du_2\, dg_3$. Then $d_l \bar{\gamma}$ is a left Haar measure on the (non-unimodular) group $\bar{\Gamma}_v$. From what we have just shown, it follows that $G_{k_v} \setminus \bar{\Gamma}_v \cdot G^\circ_{x\,k_v}$ always has measure zero. Thus we have

$$\begin{aligned} \int_{G_{k_v}/G^\circ_{x\,k_v}} f(g'_{x,v}x)\,dg'_{x,v} &= \int_{\bar{\Gamma}_v \cdot G^\circ_{x\,k_v}/G^\circ_{x\,k_v}} f(g'_{x,v}x)\,dg'_{x,v} \\ &\propto \int_{\bar{\Gamma}_v} f(\gamma x)\,d_l \bar{\gamma} \end{aligned} \tag{9.5}$$

for all $f \in L^1(G_{k_v}x)$. Now if $\varphi \in L^1(\mathrm{GL}(2)_{k_v})$ is left invariant under $\mathrm{GL}(2)_{\mathcal{O}_v}$, then the Iwasawa decomposition implies that

$$\int_{\mathrm{GL}(2)_{k_v}} \varphi(h)dh \propto \int_B \varphi(b)d_r b\,,$$

where $B = \{n(u_3)a(t_3, t_4) \mid t_3, t_4 \in k^\times, u_3 \in k\}$ and $d_r b$ denotes the right Haar measure on the group B. It is easy to check that $d_r b = d^\times t_3\, d^\times t_4\, du_3$, and applying this in (9.5) we obtain the conclusion.

Finally, almost identical arguments apply in the case where (G_{k_v}, V_{k_v}) is not split and we shall not repeat them. □

PROPOSITION 9.6. *If $p(z) = z^2 - z$, then we have*

$$\Xi_{w_p,v}(s) = (1 - q_v^{-(2s-1)})^{-1}(1 - q_v^{-(2s-2)})^{-1}\,.$$

PROOF. Our work will be simplified if we compute with the element $x = n_0 w_p$ with $n_0 = (1, 1, {}^t n(1))$ or $(1, {}^t n(1))$ instead of with the element w_p. By Lemma 8.1, $Z_{w_p,v}(\Phi_{v,0}, s) = Z_{x,v}(\Phi_{v,0}, s)$, and so this is permissible.

Suppose that $v \in \mathfrak{M}_{\mathrm{sp}}$. Then, by Lemma 3.27, elements of $G^\circ_{x\,k_v}$ have the form

$$\left(\begin{pmatrix} c_{11} & c_{11} - c_{12} \\ 0 & c_{12} \end{pmatrix}, \begin{pmatrix} c_{21} & c_{21} - c_{22} \\ 0 & c_{22} \end{pmatrix}, *\right) \tag{9.7}$$

where $*$ is determined by the other two entries. Note that the conjugation by n_0 does not change the first two components. Let

$$\begin{aligned} \mu &= ({}^t n(u_1), {}^t n(u_2), a(t_1, t_2)n(u_3))\,, \\ d\mu &= |t_1^{-1}t_2|_v d^\times t_1 d^\times t_2 du_1 du_2 du_3\,, \\ S &= \{\mu \mid t_1, t_2 \in k_v^\times,\ u_1, u_2, u_3 \in k_v\}\,. \end{aligned} \tag{9.8}$$

From (9.7) and the Iwasawa decomposition it follows that $K_v S G^\circ_{x\,k_v} = G_{k_v}$ and $dg \propto d\kappa\, d\mu\, dg''_{x,v}$. Since $\Phi_{v,0}$ is K_v-invariant,

$$\begin{aligned} \Xi_{x,v}(s) &= b_{x,v} \int_{G_{k_v}/G^\circ_{x\,k_v}} |\chi(g'_{x,v})|_v^s \Phi_{v,0}(g'_{x,v}x)dg'_{x,v} \\ &\propto \int_S |\chi(\mu)|_v^s \Phi_{v,0}(\mu x)d\mu\,. \end{aligned}$$

Computation gives

$$\mu x = \left(\begin{pmatrix} t_1 & 0 \\ 0 & 0 \end{pmatrix}, t_2 \begin{pmatrix} u_3 - u_1 - u_2 + u_1u_2 + 1 & u_1 - 1 \\ u_2 - 1 & 1 \end{pmatrix}\right).$$

Introducing the variables

$$\bar{u}_1 = t_2(u_1 - 1)\,, \quad \bar{u}_2 = t_2(u_2 - 1)\,, \quad \bar{u}_3 = t_2(u_3 - u_1 - u_2 + u_1u_2 + 1)$$

we have $d\bar{u}_1\, d\bar{u}_2\, d\bar{u}_3 = |\, t_2|_v^3 du_1 du_2 du_3$. So

$$\begin{aligned}\Xi_{x,v}(s) &\propto \int |\, t_1|_v^{2s-1} |\, t_2|_v^{2s-2} \Phi_{v,0}\left(\begin{pmatrix} t_1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \bar{u}_3 & \bar{u}_1 \\ \bar{u}_2 & t_2 \end{pmatrix} \right) d^\times t_1 d^\times t_2 d\bar{u}_1 d\bar{u}_2 d\bar{u}_3 \\ &= \int_{|t_1|_v, |t_2|_v \le 1} |\, t_1|_v^{2s-1} |\, t_2|_v^{2s-2}\, d^\times t_1 d^\times t_2 \\ &= (1 - q_v^{-(2s-1)})^{-1} (1 - q_v^{-(2s-2)})^{-1}\,.\end{aligned}$$

But we know from Condition 6.12 that the constant term in $\Xi_{x,v}(s)$ is 1, and so $\Xi_{x,v}(s)$ has the stated value. When $v \in \mathfrak{M}_{\mathrm{in}}$ the calculation is a simple variation on the above and we shall not reproduce it here. □

PROPOSITION 9.9. *Let $v \in \mathfrak{M}_{\mathrm{sp}}$ and suppose that x is the standard orbital representative for an orbit with $k_v(x) \neq k_v$. If*

$$L_v(s) = 1 + 8(1 - q_v^{-2(s-1)})^{-3} q_v^{-2(s-1)} (4 - 3q_v^{-2(s-1)} + q_v^{-4(s-1)})\,,$$

then $\Xi_{x,v}(s) \preccurlyeq L_v(s)$.

PROOF. The standard orbital representative is $x = w_p$ for some quadratic polynomial $p(z) = z^2 + a_2$ which is irreducible over k_v (we may assume that $a_1 = 0$ since $v \notin \mathfrak{M}_{\mathrm{dy}}$). Let γ, $d\gamma$, Γ_v and Γ_v^j be as in (9.1). By Definition 5.22 and Lemma 9.2,

$$\Xi_{x,v}(s) = Z_{x,v}(\Phi_{v,0}, s) = C_x \int_{\Gamma_v} |\chi(\gamma)|_v^s \Phi_{v,0}(\gamma x) d\gamma$$

for some constant $C_x \neq 0$. Since $\Gamma_v = \coprod_j \Gamma_v^j$,

$$\Xi_{x,v}(s) = C_x \sum_{j=0}^{\infty} q_v^{-2js} \int_{\Gamma_v^j} \Phi_{v,0}(\gamma x) d\gamma\,,$$

which implies that

$$a_{x,v,2j} = C_x \int_{\Gamma_v^j} \Phi_{v,0}(\gamma x) d\gamma \tag{9.10}$$

for all $j \ge 0$. (Recall that $a_{x,v,n} = 0$ if n is odd by Corollary 8.24.)

Computing, we find that $\gamma x = (M_1, M_2)$, where

$$\begin{aligned} M_1 &= \begin{pmatrix} 0 & t_2 t_3 \\ t_1 t_3 & t_1 t_2 t_3 (u_1 + u_2) \end{pmatrix}, \\ M_2 &= \begin{pmatrix} t_4 & t_2 (t_4 u_2 + t_3 u_3) \\ t_1 (t_4 u_1 + t_3 u_3) & m(t, u) \end{pmatrix}\end{aligned}$$

with

$$m(t, u) = t_1 t_2 t_3 (u_1 + u_2) u_3 + t_1 t_2 t_4 (u_1 u_2 - a_2)\,.$$

If we make $t_1, \ldots, t_4$ units and $u_1, \ldots, u_3$ integers, then $\gamma x \in V_{\mathcal{O}_v}$ and the volume of the set $\{\gamma \mid t_j \in \mathcal{O}_v^\times, u_j \in \mathcal{O}_v\}$ under $d\gamma$ is 1, and so it follows from this, Condition 6.12 and (9.10) that

$$1 = a_{x,v,0} = C_x \int_{\Gamma_v^0} \Phi_{v,0}(\gamma x)\, d\gamma \geq C_x\,.$$

Therefore, from (9.10) again,

$$a_{x,v,2j} \leq \int_{\Gamma_v^j} \Phi_{v,0}(\gamma x)\, d\gamma \tag{9.11}$$

for all $j \geq 0$.

We introduce new variables defined by

$$\bar{t}_1 = t_4\,, \quad \bar{t}_2 = t_2t_3\,, \quad \bar{t}_3 = t_1t_3\,, \quad \bar{t}_4 = t_1t_2t_3t_4\,.$$

Then

$$t_1 = \bar{t}_1^{-1}\bar{t}_2^{-1}\bar{t}_4\,, \quad t_2 = \bar{t}_1^{-1}\bar{t}_3^{-1}\bar{t}_4\,, \quad t_3 = \bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4^{-1}\,, \quad t_4 = \bar{t}_1.$$

Note that $\bar{t}_1, \ldots, \bar{t}_4$ are monomials of $t_1, \ldots, t_4$. So they correspond to a lattice in $\mathbf{Z}^4$. Since the correspondence between $(t_1, \ldots, t_4)$ and $(\bar{t}_1, \ldots, \bar{t}_4)$ is bijective, this lattice must be unimodular. This implies that

$$d^\times\bar{t}_1 d^\times\bar{t}_2 d^\times\bar{t}_3 d^\times\bar{t}_4 = d^\times t_1 d^\times t_2 d^\times t_3 d^\times t_4\,. \tag{9.12}$$

Suppose that $\gamma x \in V_{\mathcal{O}_v}$. Then $\bar{t}_1, \bar{t}_2, \bar{t}_3 \in \mathcal{O}_v$. Since $|P(x)|_v$ is the maximum of $|P(y)|_v$ for $y \in G_{k_v}x \cap V_{\mathcal{O}_v}$, $|P(x)|_v \geq |P(\gamma x)|_v = |\bar{t}_4|_v^2 |P(x)|_v$, which implies that $\bar{t}_4 \in \mathcal{O}_v$. The conditions that the $(2,2)$ entry in M_1 and the $(2,1)$ and $(1,2)$ entries in M_2 are integers may be expressed as $N\,{}^t(u_1, u_2, u_3) \in \mathcal{O}_v^3$ where

$$N = \begin{pmatrix} t_1t_2t_3 & t_1t_2t_3 & 0 \\ t_1t_4 & 0 & t_1t_3 \\ 0 & t_2t_4 & t_2t_3 \end{pmatrix} = \begin{pmatrix} \bar{t}_1^{-1}\bar{t}_4 & \bar{t}_1^{-1}\bar{t}_4 & 0 \\ \bar{t}_2^{-1}\bar{t}_4 & 0 & \bar{t}_3 \\ 0 & \bar{t}_3^{-1}\bar{t}_4 & \bar{t}_2 \end{pmatrix}.$$

This matrix factors as $N = D_1^{-1}CD_2$, where we have set $D_1 = \operatorname{diag}(\bar{t}_1, \bar{t}_2, \bar{t}_3)$, $D_2 = \operatorname{diag}(\bar{t}_4, \bar{t}_4, \bar{t}_2\bar{t}_3)$ and

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Let

$$\begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{pmatrix} = CD_2 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

Then the three conditions are equivalent to ${}^t(\bar{u}_1\ \ \bar{u}_2\ \ \bar{u}_3) \in D_1\mathcal{O}_v^3$, which in turn is equivalent to the conditions

$$\bar{u}_1 \in \bar{t}_1\mathcal{O}_v\,, \qquad \bar{u}_2 \in \bar{t}_2\mathcal{O}_v\,, \qquad \bar{u}_3 \in \bar{t}_3\mathcal{O}_v\,. \tag{9.13}$$

By computation,

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \bar{t}_4^{-1}(\bar{u}_1 + \bar{u}_2 - \bar{u}_3) \\ \bar{t}_4^{-1}(\bar{u}_1 - \bar{u}_2 + \bar{u}_3) \\ \bar{t}_2^{-1}\bar{t}_3^{-1}(-\bar{u}_1 + \bar{u}_2 + \bar{u}_3) \end{pmatrix}$$

and

$$du_1 du_2 du_3 = |\bar{t}_2\bar{t}_3\bar{t}_4^2|_v^{-1} d\bar{u}_1 d\bar{u}_2 d\bar{u}_3 \,. \tag{9.14}$$

The remaining condition for $\gamma x \in V_{\mathcal{O}_v}$ is that $m(t,u) \in \mathcal{O}_v$. Expressing $m(t,u)$ in terms of the coordinates $(\bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{t}_4, \bar{u}_1, \bar{u}_2, \bar{u}_3)$ we find that

$$m(t,u) = (1/4)\bar{t}_1^{-1}\bar{t}_2^{-1}\bar{t}_3^{-1}[-Q(\bar{u}_1, \bar{u}_2, \bar{u}_3) - 4\bar{t}_4^2 a_2]\,,$$

where $Q(\bar{u}_1, \bar{u}_2, \bar{u}_3) = \bar{u}_1^2 + \bar{u}_2^2 + \bar{u}_3^2 - 2(\bar{u}_1\bar{u}_2 + \bar{u}_1\bar{u}_3 + \bar{u}_2\bar{u}_3)$. Since $v \notin \mathfrak{M}_{\mathrm{dy}}$ and $P(x) = -4a_2$, we have $m(t,u) \in \mathcal{O}_v$ if and only if

$$Q(\bar{u}_1, \bar{u}_2, \bar{u}_3) - \bar{t}_4^2 P(x) \in \bar{t}_1\bar{t}_2\bar{t}_3\mathcal{O}_v \,. \tag{9.15}$$

We claim that at least one of $|\bar{t}_1|_v$, $|\bar{t}_3|_v$ and $|\bar{t}_2|_v$ must be greater than or equal to $|\bar{t}_4|_v$. Suppose to the contrary that $|\bar{t}_1|_v, |\bar{t}_2|_v, |\bar{t}_3|_v < |\bar{t}_4|_v$. Then $|\bar{u}_1|_v, |\bar{u}_2|_v, |\bar{u}_3|_v < |\bar{t}_4|_v$ also, by (9.13), and so $|Q(\bar{u}_1, \bar{u}_2, \bar{u}_3)|_v \le |\bar{t}_4|_v^2 q_v^{-2}$. Furthermore, since $\bar{t}_4 \in \mathcal{O}_v$,

$$|\bar{t}_1\bar{t}_2\bar{t}_3|_v \le |\bar{t}_4|_v^3 q_v^{-3} < |\bar{t}_4|_v^2 q_v^{-2}\,,$$

and it follows from (9.15) that $|\bar{t}_4|_v^2 |P(x)|_v \le |\bar{t}_4|_v^2 q_v^{-2}$, and so $|P(x)|_v \le q_v^{-2}$. However, by the choice of the standard orbital representatives, $|P(x)|_v \ge q_v^{-1}$ and we have a contradiction. This establishes our claim.

Next we claim that $|\bar{t}_1|_v, |\bar{t}_2|_v, |\bar{t}_3|_v \ge |\bar{t}_4|_v^2 q_v^{-1}$. Suppose to the contrary that one of these quantities is less than $|\bar{t}_4|_v^2 q_v^{-1}$. In light of the symmetry between the roles of the pairs $(\bar{t}_1, \bar{u}_1)$, $(\bar{t}_2, \bar{u}_2)$ and $(\bar{t}_3, \bar{u}_3)$ we may suppose without loss of generality that $|\bar{t}_3|_v$ is the greatest of $|\bar{t}_1|_v$, $|\bar{t}_2|_v$ and $|\bar{t}_3|_v$ and that $|\bar{t}_1|_v < |\bar{t}_4|_v^2 q_v^{-1}$. By the previous paragraph, $|\bar{t}_3|_v \ge |\bar{t}_4|_v$. Dividing (9.15) through by $\bar{t}_3^2$ we obtain

$$Q(\bar{t}_3^{-1}\bar{u}_1, \bar{t}_3^{-1}\bar{u}_2, \bar{t}_3^{-1}\bar{u}_3) - (\bar{t}_3^{-1}\bar{t}_4)^2 P(x) \in \bar{t}_3^{-1}\bar{t}_1\bar{t}_2\mathcal{O}_v \subseteq \bar{t}_3^{-1}\bar{t}_1\mathcal{O}_v \,.$$

We have $\bar{u}_1/\bar{t}_3 \in (\bar{t}_1/\bar{t}_3)\mathcal{O}_v$, and so we may drop the terms involving $\bar{u}_1/\bar{t}_3$ to obtain

$$(\bar{t}_3^{-1}(\bar{u}_2 - \bar{u}_3))^2 - (\bar{t}_3^{-1}\bar{t}_4)^2 P(x) \in \bar{t}_3^{-1}\bar{t}_1\mathcal{O}_v \,. \tag{9.16}$$

Now

$$|(\bar{t}_3^{-1}\bar{t}_4)^2 P(x)|_v \ge |\bar{t}_3|_v^{-2}|\bar{t}_4|_v^2 q_v^{-1} > |\bar{t}_3|_v^{-2}|\bar{t}_1|_v \ge |\bar{t}_3|_v^{-1}|\bar{t}_1|_v\,,$$

and hence $|\bar{t}_3^{-1}(\bar{u}_2 - \bar{u}_3)|_v^2 = |(\bar{t}_3^{-1}\bar{t}_4)^2 P(x)|_v$. This implies that $|\bar{t}_4^{-1}(\bar{u}_2 - \bar{u}_3)|_v^2 = |P(x)|_v \ge q_v^{-1}$, and so $\mathrm{ord}_{k_v}(\bar{t}_4^{-1}(\bar{u}_2 - \bar{u}_3)) \le 0$. By (9.16),

$$(\bar{t}_4^{-1}(\bar{u}_2 - \bar{u}_3))^2 - P(x) \in \bar{t}_4^{-2}\bar{t}_1\bar{t}_3\mathcal{O}_v \subseteq \bar{t}_4^{-2}\bar{t}_1\mathcal{O}_v \subseteq \mathfrak{p}_v^2 \,.$$

These last two facts allow us to apply Hensel's lemma to conclude that $P(x) \in (k_v^\times)^2$, which contradicts the assumption that $k_v(x) \ne k_v$. Thus $|\bar{t}_1|_v, |\bar{t}_2|_v, |\bar{t}_3|_v \ge |\bar{t}_4|_v^2 q_v^{-1}$, as claimed.

Changing variables to $(\bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{t}_4, \bar{u}_1, \bar{u}_2, \bar{u}_3)$ in (9.11) and using (9.12), (9.14), we obtain

$$\begin{aligned} a_{x,v,2j} &\le \int |\,\bar{t}_2\bar{t}_3\bar{t}_4^2|_v^{-1} d^\times\bar{t}_1 d^\times\bar{t}_2 d^\times\bar{t}_3 d^\times\bar{t}_4 d\bar{u}_1 d\bar{u}_2 d\bar{u}_3 \\ &= q_v^{2j}\int |\,\bar{t}_2\bar{t}_3|_v^{-1} d^\times\bar{t}_1 d^\times\bar{t}_2 d^\times\bar{t}_3 d\bar{u}_1 d\bar{u}_2 d\bar{u}_3 \end{aligned}$$

where, on the domain of integration, $|\bar{u}_i|_v \le |\,\bar{t}_i|_v$ and $1 \ge |\,\bar{t}_i|_v \ge |\,\bar{t}_4|_v^2 q_v^{-1} = q_v^{-2j-1}$ for $i = 1, 2, 3$. Note that $|\,\bar{t}_4|_v = q_v^{-j}$ on Γ_v^j. Carrying out the integration with respect to $\bar{u}_1$, $\bar{u}_2$ and $\bar{u}_3$ we get

$$\begin{aligned} a_{x,v,2j} &\le q_v^{2j}\int |\,\bar{t}_1|_v d^\times\bar{t}_1\, d^\times\bar{t}_2\, d^\times\bar{t}_3 \\ &\le q_v^{2j}(1-q_v^{-1})^{-1}\int_{1\ge|\bar{t}_3|_v,|\bar{t}_2|_v\ge q_v^{-2j-1}} d^\times\bar{t}_2 d^\times\bar{t}_3 \\ &\le 2q_v^{2j}(2j+2)^2 \\ &= 8q_v^{2j}(j+1)^2\,. \end{aligned}$$

Note that the volume of the set $\bigcup_{i=0}^{2j+1}\pi_v^i\mathcal{O}_v^\times$ is $2j+2$ and $(1-q_v^{-1})^{-1} \le 2$. Put $B_j(v) = 8q_v^{2j}(j+1)^2$. Using the formulas

$$\text{(9.17)}\qquad \begin{aligned} \sum_{j=1}^\infty q_v^{-js} &= q_v^{-s}(1-q_v^{-s})^{-1}\,, \\ \sum_{j=1}^\infty jq_v^{-js} &= q_v^{-s}(1-q_v^{-s})^{-2}\,, \\ \sum_{j=1}^\infty j^2 q_v^{-js} &= q_v^{-s}(1+q_v^{-s})(1-q_v^{-s})^{-3}\,, \end{aligned}$$

valid for $\mathrm{Re}(s) > 0$, we obtain

$$\sum_{j=1}^\infty B_j(v)q_v^{-2js} = L_v(s) - 1\,,$$

valid for $\mathrm{Re}(s) > 1$, where $L_v(s)$ is given in the statement of the proposition. This completes the proof. □

PROPOSITION 9.18. *Let $v \in \mathfrak{M}_{\mathrm{in}}$ and suppose that x is the standard orbital representative for an orbit with $k_v(x) \neq k_v$. If*

$$L_v(s) = 1 + 4(1-q_v^{-2(s-1)})^{-2}q_v^{-2(s-1)}(2-q_v^{-2(s-1)})\,,$$

then $\Xi_{x,v}(s) \preccurlyeq L_v(s)$.

PROOF. The structure of this proof will be very similar to that of the proof of Proposition 9.9, and so we shall abbreviate somewhat. We have $x = w_p$ for some irreducible quadratic polynomial $p(z) = z^2 + a_2 \in k_v[z]$. Let γ, $d\gamma$, Γ_v and Γ_v^j be as in (9.1). Arguing as in the previous proposition we obtain the inequality

$$a_{x,v,2j} \leq \int_{\Gamma_v^j} \Phi_{v,0}(\gamma x)d\gamma \tag{9.19}$$

for all $j \geq 0$.

Calculation gives $\gamma x = (M_1, M_2)$, where

$$M_1 = \begin{pmatrix} 0 & t_1^\sigma t_2 \\ t_1 t_2 & t_2 \mathrm{N}_{\tilde{k}_v/k_v}(t_1)\mathrm{Tr}_{\tilde{k}_v/k_v}(u_1) \end{pmatrix},$$

$$M_2 = \begin{pmatrix} t_3 & t_1^\sigma(t_3 u_1^\sigma + t_2 u_2) \\ t_1(t_3 u_1 + t_2 u_2) & m(t, u) \end{pmatrix}$$

with

$$m(t, u) = t_2 \mathrm{N}_{\tilde{k}_v/k_v}(t_1)\mathrm{Tr}_{\tilde{k}_v/k_v}(u_1)u_2 + t_3 \mathrm{N}_{\tilde{k}_v/k_v}(t_1)[\mathrm{N}_{\tilde{k}_v/k_v}(u_1) - a_2]\,.$$

We introduce new variables defined by

$$\bar{t}_1 = t_1 t_2\,, \quad \bar{t}_2 = t_3\,, \quad \bar{t}_3 = t_2 t_3 \mathrm{N}_{\tilde{k}_v/k_v}(t_1)\,.$$

Then

$$t_1 = \bar{t}_1^{-\sigma}\bar{t}_2^{-1}\bar{t}_3\,, \quad t_2 = \bar{t}_2\bar{t}_3^{-1}\mathrm{N}_{\tilde{k}_v/k_v}(\bar{t}_1)\,, \quad t_3 = \bar{t}_2\,.$$

Since we are dealing with coordinates in two different fields, k_v and $\tilde{k}_v$, a small digression is required to calculate the relationship between $d^\times\bar{t}_1 d^\times\bar{t}_2 d^\times\bar{t}_3$ and $d^\times t_1 d^\times t_2 d^\times t_3$. Let us fix an element $\beta \in \tilde{k}_v^\times$ which satisfies $\beta^\sigma = -\beta$. For $u \in \tilde{k}_v$, we define $u^+ = u + u^\sigma$ and $u^- = (u - u^\sigma)/\beta$. Both u^+ and u^- lie in k_v and since $u = (1/2)(u^+ + \beta u^-)$, u^+ and u^- serve as k_v coordinates for $\tilde{k}_v$. We use this notation replacing u by other letters. The measure corresponding to $dt_1^+ dt_1^-$ is invariant under addition and hence there is a constant C_v, depending only on k, $\tilde{k}$ and v, such that

$$d^\times t_1 = C_v \frac{dt_1^+ dt_1^-}{|\,\mathrm{N}_{\tilde{k}_v/k_v}(t_1)|_v}\,.$$

We also have $\mathrm{N}_{\tilde{k}_v/k_v}(t_1) = (1/4)[(t_1^+)^2 - \beta^2(t_1^-)^2]$ and a calculation gives

$$\left|\frac{\partial(\bar{t}_1^+, \bar{t}_1^-, \bar{t}_2, \bar{t}_3)}{\partial(t_1^+, t_1^-, t_2, t_3)}\right|_v = |\,t_3 \mathrm{N}_{\tilde{k}_v/k_v}(\bar{t}_1)|_v\,,$$

so that $d\bar{t}_1^+ d\bar{t}_1^- d^\times\bar{t}_2 d^\times\bar{t}_3/|\,\mathrm{N}_{\tilde{k}_v/k_v}(\bar{t}_1)|_v = dt_1^+ dt_1^- d^\times t_2 d^\times t_3\,/|\,\mathrm{N}_{\tilde{k}_v/k_v}(t_1)|_v$. Multiplying both sides by C_v we obtain

$$d^\times\bar{t}_1 d^\times\bar{t}_2 d^\times\bar{t}_3 = d^\times t_1 d^\times t_2 d^\times t_3\,. \tag{9.20}$$

Suppose that $\gamma x \in V_{\mathcal{O}_v}$. Then $\bar{t}_1 \in \tilde{\mathcal{O}}_v$ and $\bar{t}_2 \in \mathcal{O}_v$. Also $\bar{t}_3 \in \mathcal{O}_v$ by Lemma 8.16. If we set

$$\begin{aligned} \bar{u}_1 &= \mathrm{N}_{\tilde{k}_v/k_v}(\bar{t}_1)u_2 + \bar{t}_3 u_1 \\ \bar{u}_2 &= \bar{t}_3 \mathrm{Tr}_{\tilde{k}_v/k_v}(u_1)\,, \end{aligned}$$

then the $(2,2)$ entry in M_1 is $\bar{t}_2^{-1}\bar{u}_2$ and the $(2,1)$ entry in M_2 is $\bar{t}_1^{-\sigma}\bar{u}_1$, and it follows that

$$\bar{u}_1 \in \bar{t}_1^{\sigma}\tilde{\mathcal{O}}_v \quad \text{and} \quad \bar{u}_2 \in \bar{t}_2\mathcal{O}_v\,.$$

We have

$$\begin{aligned} u_1 &= (1/2)\bar{t}_3^{-1}(\bar{u}_1 - \bar{u}_1^{\sigma} + \bar{u}_2)\,, \\ u_2 &= (1/2)\mathrm{N}_{\tilde{k}_v/k_v}(\bar{t}_1)^{-1}(\bar{u}_1 + \bar{u}_1^{\sigma} - \bar{u}_2)\,, \end{aligned}$$

and so

$$\begin{aligned} u_1^+ &= \bar{t}_3^{-1}\bar{u}_2\,, \\ u_1^- &= \bar{t}_3^{-1}\bar{u}_1^-\,, \\ u_2 &= (1/2)\mathrm{N}_{\tilde{k}_v/k_v}(\bar{t}_1)^{-1}(\bar{u}_1^+ - \bar{u}_2)\,. \end{aligned}$$

Hence $du_1^+du_1^-du_2 = |\,\bar{t}_3|_v^{-2}|\,\mathrm{N}_{\tilde{k}_v/k_v}(\bar{t}_1)|_v^{-1}d\bar{u}_1^+d\bar{u}_1^-d\bar{u}_2$, which implies that

$$du_1du_2 = |\,\bar{t}_3|_v^{-2}|\,\mathrm{N}_{\tilde{k}_v/k_v}(\bar{t}_1)|_v^{-1}d\bar{u}_1d\bar{u}_2\,. \tag{9.21}$$

The remaining condition for $\gamma x \in V_{\mathcal{O}_v}$ is that $m(t,u) \in \mathcal{O}_v$. In the coordinates $(\bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{u}_1, \bar{u}_2)$ we have

$$m(t,u) = (1/4)\bar{t}_2^{-1}\mathrm{N}_{\tilde{k}_v/k_v}(\bar{t}_1)^{-1}[-Q(\bar{u}_1,\bar{u}_2) + \bar{t}_3^2 P(x)]\,,$$

where

$$Q(\bar{u}_1,\bar{u}_2) = \bar{u}_1^2 + \bar{u}_2^2 + (\bar{u}_1^{\sigma})^2 - 2(\bar{u}_1\bar{u}_2 + \bar{u}_1^{\sigma}\bar{u}_2 + \bar{u}_1\bar{u}_1^{\sigma})\,.$$

Thus $m(t,u) \in \mathcal{O}_v$ if and only if

$$Q(\bar{u}_1,\bar{u}_2) - \bar{t}_3^2 P(x) \in \bar{t}_2\mathrm{N}_{\tilde{k}_v/k_v}(\bar{t}_1)\mathcal{O}_v\,. \tag{9.22}$$

Note that for any $a \in k_v$ we have $|\,a|_{\tilde{k}_v} = |\,a|_v^2$. We claim that either $|\,\bar{t}_2|_v \geq |\,\bar{t}_3|_v$ or $|\,\bar{t}_1|_{\tilde{k}_v} \geq |\,\bar{t}_3|_v^2 q_v^{-1}$. Suppose to the contrary that $|\,\bar{t}_2|_v \leq |\,\bar{t}_3|_v q_v^{-1}$ and $|\,\bar{t}_1|_{\tilde{k}_v} \leq |\,\bar{t}_3|_v^2 q_v^{-2}$, so that $|\,\bar{t}_2|_{\tilde{k}_v} \leq |\,\bar{t}_3|_v^2 q_v^{-2}$. Then $|\bar{u}_1|_{\tilde{k}_v}, |\bar{u}_2|_{\tilde{k}_v} \leq |\,\bar{t}_3|_{\tilde{k}_v}^2 q_v^{-2}$, and so $|Q(\bar{u}_1,\bar{u}_2)|_{\tilde{k}_v} \leq |\,\bar{t}_3|_v^4 q_v^{-4}$. Also

$$|\,\bar{t}_2\mathrm{N}_{\tilde{k}_v/k_v}(\bar{t}_1)|_{\tilde{k}_v} \leq |\,\bar{t}_3|_v^2 q_v^{-2}|\,\bar{t}_3|_v^4 q_v^{-4} < |\,\bar{t}_3|_v^4 q_v^{-4}.$$

So, from (9.22), $|\,\bar{t}_3^2 P(x)|_{\tilde{k}_v} \leq |\bar{t}_3|_v^4 q_v^{-4}$. Thus $|P(x)|_v \leq q_v^{-2}$, which is a contradiction. The claim follows.

Next we claim that $|\,\bar{t}_1|_{\tilde{k}_v} \geq |\,\bar{t}_3|_v^4 q_v^{-2}$. Suppose to the contrary that $|\,\bar{t}_1|_{\tilde{k}_v} \leq |\,\bar{t}_3|_v^4 q_v^{-3}$. Then, from the previous paragraph, $|\,\bar{t}_2|_{\tilde{k}_v} \geq |\,\bar{t}_3|_v^2$. Dividing (9.22) by $\bar{t}_2^2$ we obtain

$$Q(\bar{t}_2^{-1}\bar{u}_2, \bar{t}_2^{-1}\bar{u}_1) - \big(\bar{t}_2^{-1}\bar{t}_3\big)^2 P(x) \in \bar{t}_2^{-1}\mathrm{N}_{\tilde{k}_v/k_v}(\bar{t}_1)\mathcal{O}_v \subseteq \bar{t}_2^{-1}\bar{t}_1\tilde{\mathcal{O}}_v\,.$$

Since $\bar{u}_1/\bar{t}_2, \bar{u}_1^\sigma/\bar{t}_2 \in (\bar{t}_1/\bar{t}_2)\tilde{\mathcal{O}}_v$, this inclusion implies that

$$(\bar{t}_2^{-1}\bar{u}_2)^2 - (\bar{t}_2^{-1}\bar{t}_3)^2 P(x) \in \bar{t}_2^{-1}\bar{t}_1\tilde{\mathcal{O}}_v\,. \tag{9.23}$$

Now

$$|(\bar{t}_2^{-1}\bar{t}_3)^2 P(x)|_{\tilde{k}_v} \geq |\bar{t}_2|_{\tilde{k}_v}^{-2}|\bar{t}_3|_v^4 q_v^{-2} > |\bar{t}_2|_{\tilde{k}_v}^{-2}|\bar{t}_1|_{\tilde{k}_v} \geq |\bar{t}_2|_{\tilde{k}_v}^{-1}|\bar{t}_1|_{\tilde{k}_v}\,.$$

Hence

$$|(\bar{t}_2^{-1}\bar{u}_2)^2|_{\tilde{k}_v} = |(\bar{t}_2^{-1}\bar{t}_3)^2 P(x)|_{\tilde{k}_v}\,.$$

This implies that $|\bar{u}_2/\bar{t}_3|_v^2 = |P(x)|_v \geq q_v^{-1}$ and so $\mathrm{ord}_{k_v}(\bar{u}_2/\bar{t}_3) \leq 0$. By (9.23),

$$(\bar{t}_3^{-1}\bar{u}_2)^2 - P(x) \in \bar{t}_3^{-2}\bar{t}_1\bar{t}_2\tilde{\mathcal{O}}_v \subseteq \bar{t}_3^{-2}\bar{t}_1\tilde{\mathcal{O}}_v\,.$$

Thus $|(\bar{u}_2/\bar{t}_3)^2 - P(x)|_{\tilde{k}_v} \leq |\bar{t}_1/\bar{t}_3^2|_{\tilde{k}_v} \leq q_v^{-3}$, and so $|(\bar{u}_2/\bar{t}_3)^2 - P(x)|_v \leq q_v^{-2}$. We may now apply Hensel's lemma to conclude that $P(x) \in (k_v^\times)^2$, which contradicts the assumption that $k_v(x) \neq k_v$. Thus $|\bar{t}_1|_{\tilde{k}_v} \geq |\bar{t}_3|_v^4 q_v^{-2}$.

Changing variables to $(\bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{u}_1, \bar{u}_2)$ in (9.19) and using (9.20), (9.21), we obtain

$$\begin{aligned} a_{x,v,2j} &\leq \int |\bar{t}_3|_v^{-2}|\mathrm{N}_{\tilde{k}_v/k_v}(\bar{t}_1)|_v^{-1} d^\times\bar{t}_1 d^\times\bar{t}_2 d^\times\bar{t}_3 d\bar{u}_1 d\bar{u}_2 \\ &= q_v^{2j}\int |\mathrm{N}_{\tilde{k}_v/k_v}(\bar{t}_1)|_v^{-1} d^\times\bar{t}_1 d^\times\bar{t}_2 d\bar{u}_1 d\bar{u}_2\,, \end{aligned}$$

where, on the domain of integration, $|\bar{u}_2|_v \leq |\bar{t}_2|_v$, $|\bar{u}_1|_{\tilde{k}_v} \leq |\bar{t}_1|_{\tilde{k}_v} = |\mathrm{N}_{\tilde{k}_v/k_v}(\bar{t}_1)|_v$, $|\bar{t}_2| \leq 1$ and $|\bar{t}_3|_{k_v}^4 q_v^{-2} \leq |\bar{t}_1|_{\tilde{k}_v} \leq 1$. Carrying out the integration with respect to $\bar{u}_1$ and $\bar{u}_2$, we get

$$\begin{aligned} a_{x,v,2j} &\leq q_v^{2j}\int |\bar{t}_2| d^\times\bar{t}_1 d^\times\bar{t}_2 \\ &\leq q_v^{2j}(1-q_v^{-1})^{-1}\int_{1\geq|\bar{t}_1|_{\tilde{k}_v}\geq q_v^{-2}|\bar{t}_3|_v^4} d^\times\bar{t}_1 \\ &\leq 2q_v^{2j}(2j+2)\,, \end{aligned}$$

since $\tilde{k}_v/k_v$ is unramified. Set $B_j(v) = 4q_v^{2j}(j+1)$. Using (9.17), we obtain

$$\sum_{j=1}^{\infty} B_j(v) q_v^{-2js} = L_v(s) - 1\,,$$

valid for $\mathrm{Re}(s) > 1$, where $L_v(s)$ is given in the statement of the proposition. □

We define

$$L_v(s) = \begin{cases} \dfrac{1 + 29q_v^{-2(s-1)} - 21q_v^{-4(s-1)} + 7q_v^{-6(s-1)}}{(1-q_v^{-(2s-1)})(1-q_v^{-2(s-1)})^4} & v \in \mathfrak{M}_{\mathrm{sp}}\,, \\[2ex] \dfrac{1 + 6q_v^{-2(s-1)} - 3q_v^{-4(s-1)}}{(1-q_v^{-(2s-1)})(1-q_v^{-2(s-1)})^3} & v \in \mathfrak{M}_{\mathrm{in}}\,. \end{cases} \tag{9.24}$$

PROPOSITION 9.25. *Let $L_v(s)$ be as defined by* (9.24) *. Then $\Xi_{x,v}(s) \preccurlyeq L_v(s)$ for all $v \in \mathfrak{M} \setminus S_0$ and all $x \in V_{k_v}^{\mathrm{ss}}$. The product $\prod_{v \in \mathfrak{M} \setminus S_0} L_v(s)$ converges absolutely and locally uniformly in the region* $\mathrm{Re}(s) > 3/2$. *Moreover, if $L_v(s) = \sum_{n=0}^{\infty} l_{v,n} q_v^{-ns}$, then $l_{v,0} = 1$, $l_{v,n} \geq 0$ for all n and the series is convergent in the region* $\mathrm{Re}(s) > 1$. *Thus Condition* 6.13 *is satisfied.*

PROOF. Suppose we have two series

$$L_{i,v}(s) = 1 + \sum_{j=1}^{\infty} B_{i,j}(v) q_v^{-js}, \quad i = 1, 2,$$

with $B_{i,j}(v) \geq 0$ for all i and j. Then

$$L_{1,v}(s) L_{2,v}(s) = 1 + \sum_{j=1}^{\infty} C_j(v) q_v^{-js}$$

with

$$C_j(v) = B_{1,j}(v) + B_{2,j}(v) + \sum_{m=1}^{j-1} B_{1,m}(v) B_{2,j-m}(v),$$

and so if we set $L_v(s) = L_{1,v}(s) L_{2,v}(s)$, then $L_{1,v}(s) \preccurlyeq L_v(s)$, $L_{2,v}(s) \preccurlyeq L_v(s)$ and $C_j(v) \geq 0$ for all j.

We have shown that if $v \in \mathfrak{M}_{\mathrm{sp}}$, then

$$\Xi_{x,v}(\Phi_{v,0}, s) = (1 - q_v^{-(2s-1)})^{-1} (1 - q_v^{-(2s-2)})^{-1}$$

if $k_v(x) = k_v$, and

$$\Xi_{x,v}(\Phi_{v,0}, s) \preccurlyeq (1 - q_v^{-2(s-1)})^{-3} [1 + 29 q_v^{-2(s-1)} - 21 q_v^{-4(s-1)} + 7 q_v^{-6(s-1)}]$$

if $k_v(x) \neq k_v$ (the right hand side comes from writing $L_v(s)$ in Proposition 9.9 over a common denominator). Multiplying these two gives the value of $L_v(s)$ recorded in (9.24). The case $v \in \mathfrak{M}_{\mathrm{in}}$ is similar.

From their construction, the series for $L_v(s)$ in (9.24) have non-negative coefficients and constant term 1. It follows by inspection that these series converge when $\mathrm{Re}(s) > 1$. The discussion in the first paragraph shows that $\Xi_{x,v}(s) \preccurlyeq L_v(s)$ for all $v \in \mathfrak{M} \setminus S_0$ and $x \in V_{k_v}^{\mathrm{ss}}$. Finally, it is well-known that the series $\sum_{v \in \mathfrak{M} \setminus S_0} q_v^{-s}$ is absolutely and locally uniformly convergent in the region $\mathrm{Re}(s) > 1$. The usual convergence test for products now shows that $\prod_{v \in \mathfrak{M} \setminus S_0} L_v(s)$ has the stated convergence properties. □

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DEPARTMENT OF MATHEMATICS
OKLAHOMA STATE UNIVERSITY
STILLWATER OK 74078
U.S.A.

E-mail address: akable@mail.math.okstate.edu

MATHEMATICAL INSTITUTE
TOHOKU UNIVERSITY
SENDAI 980–8578
JAPAN

E-mail address: yukie@math.tohoku.ac.jp