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# ON AFFINE HYPERSURFACES WITH PARALLEL SECOND FUNDAMENTAL FORM

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**Abstract.** We investigate the classification problem of hypersurfaces with affine normal parallel second fundamental (cubic) form. A new method of approaching the solution to this problem is here presented; it consists in showing and using the equivalence of the mentioned problem with the classification of a certain class of solutions to the equation of Monge-Ampère type $\det(\partial_{ij} f) = \pm 1$.

**Introduction.** An interesting open problem in affine differential geometry is that of classifying hypersurfaces with affine normal parallel second fundamental (cubic) form, which are not hyperquadrics. The first instance of this classification was achieved by Nomizu and Pinkall, for dimension $n = 2$, i.e., for surfaces immersed in affine 3-space, in [4]. See also the book by Nomizu and Sasaki [5], where a different proof is presented. For dimensions $n \geq 3$, the only known result so far is the article by Vrancken ($n = 3$) [6]. It does not seem, or at least it is not apparent to the present author, that the methods employed in any of the mentioned articles are reasonably extendible to other cases of higher dimensions. Thus, it is the object of this paper to present a new method of approaching the solution to the problem which is, very seemingly, extendible to every case of higher dimensional hypersurfaces.

Among all of the geometrical properties of hypersurfaces satisfying the given condition of parallelism there is one which is very remarkable: the hypersurface can be represented in the form of Monge, i.e., as a graph immersion and, with respect to a suitable affine system of coordinates in the ambient space, such a graph function, say $f$, satisfies a partial differential equation of Monge-Ampère type: $\det(\partial_{ij} f) = \pm 1$. Thus, it is not merely a coincidence that the method exposed here is intimately related to the classification of certain kinds of solutions of this type of equation. For the same reason, we expect that the method shall be useful to solve other kinds of problems, mostly those which can be expressed, or are equivalent, to existence and properties of solutions of such a partial differential equation. Roughly speaking, the method consists of finding a special kind of coordinate system in which the given equation can be integrated fairly easily.

This article is organized as follows: in Section 1 we summarize notation, and main properties for dimensions greater than or equal to two, $n \geq 2$, related mostly with the topic under consideration here. In Section 2, we present the so-called *method of algorithmic sequence of coordinate changes*, for every case of dimension $n$ greater than or equal to two, and use it to furnish new proofs of the previously known, classificatory results by Nomizu, Pinkall, Sasaki

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$(n = 2)$, and Vrancken $(n = 3)$. Finally, in Section 3, we obtain the classification of the given family for the case of dimension $n = 4$.

**1. Affine hypersurface geometry: Notation and summary of known properties.** Let $X : M^n \to E^{n+1}$ be a differentiable, codimension-one immersion of the real, oriented, $n$-dimensional, abstract differentiable manifold $M$ into the $(n+1)$-dimensional real vector space $E$. (We could take, for example $E = R^{n+1}$, by considering only the real vector space structure of $R^{n+1}$.) Under suitable geometrical and analytical conditions one can develop from the above the so-called Affine Differential Geometry of Hypersurfaces ([1, 2, 3, 5]), where in the first three of these references it is used, for notation, the method of moving frames, while in the last it is developed by means of the "structural point of view", first considered for the topic by K. Nomizu, i.e., the language of Koszul for connections. We shall consider in this work the geometrical theory of invariants under the action of the unimodular affine group, $ASL(n + 1, \boldsymbol{R})$, as acting on the hypersurface $X(M)$. We keep notation as in our previously mentioned works to describe the main geometrical objects, i.e., we use the method of *moving frames*, with the following ranges for indices: Small Latin letters shall run from 1 to $n = \dim(M)$, i.e., $1 \le i, j, k, p, q, \ldots \le n$. Small Greek letters shall run from 1 to $n + 1 = \dim(E)$: $1 \le \alpha, \beta, \gamma, \ldots \le n + 1$. Thus, if $(f_1, f_2, \ldots, f_n)$ denotes a positively oriented frame field, locally defined on an open subset $U$ of $M$, and $(\sigma^1, \sigma^2, \ldots, \sigma^n)$ is the corresponding dual coframe, we can introduce a general affine frame field $(X, (e_1, e_2, \ldots, e_{n+1}))$ on the image hypersurface $X(U)$, by writing $e_i = dX(f_i)$ and prescribing that $e_{n+1}$ be a non-zero differentiable vector field, transversal to $X(U)$ at each point. For this purpose it is enough to require that $[e_1, e_2, \ldots, e_{n+1}] \neq 0$. Here we have denoted by square brackets $[\,, \ldots, ]$ the choice of a non-zero exterior $(n + 1)$-form in $E$, or determinant function.

Then, the *first fundamental form* $\mathrm{I}_{\mathrm{ua}}$ of unimodular affine geometry is denoted by

$$\mathrm{I}_{\mathrm{ua}} := \sum g_{ij}\sigma^i\sigma^j\,, \quad g_{ij} := |H|^{-1/(n+2)}h_{ij}\,, \tag{1.1}$$

where $H := \det(h_{ij}) \neq 0$.

The local expression of the unimodular affine normal $N_{\mathrm{ua}}$ is given by

$$N_{\mathrm{ua}} := |H|^{1/(n+2)}e_{n+1}\,. \tag{1.2}$$

While the *affine normal connection* $\nabla$ is defined by projecting the ambient space covariant derivative $D$, along the affine normal direction, onto the corresponding image tangent space, and then pulling back to $M$.

From $\nabla$ one constructs the *second fundamental form* of the geometry, whose corresponding local expression is given by

$$\mathrm{II}_{\mathrm{ua}} := \nabla(\mathrm{I}_{\mathrm{ua}}) = \sum g_{ijk}\sigma^i\sigma^j\sigma^k\,, \quad g_{ijk} := |H|^{-1/(n+2)}h_{ijk}\,, \tag{1.3}$$

where the scalar components $g_{ijk}$ are symmetric in all of their indices. Let us observe that the second fundamental form is also known as the "*cubic form*" in the terminology of other authors ([4, 5, 6]).

The *third fundamental form*, represented locally by the expression

$$\mathrm{III}_{\mathrm{ga}} = \sum L_{ij}\sigma^i\sigma^j\,, \tag{1.4}$$

where $L_{ij} = L_{ji}$, is invariant under the action of the *full general affine group*. Similarly, the (1, 1)-tensor whose local scalar components are defined by $L_i^j := \sum L^{jk} g_{ik}$ is obviously a unimodular affine invariant, which is also known as the *affine shape operator* ([5]).

In particular, if we assume ([2, 3]) that the immersion $X$ can be expressed in the form of Monge, i.e., as a graph immersion, then $X(M)$ is projectable onto (part of) a hyperplane, which can be represented, with respect to a suitable affine coordinate system $(t^1, t^2, \ldots, t^n, t^{n+1})$ of the vector space $E$, by the equation

$$X(t^1, \ldots, t^n) = (t^1, \ldots, t^n, f(t^1, \ldots, t^n))\,, \tag{1.5}$$

with the point $(t^1, \ldots, t^n)$ varying in an open, connected subset of $\boldsymbol{R}^n$. Also, if the map $f$ is assumed to be sufficiently differentiable, we have the following expressions for those objects:

$$\mathrm{I}_{\mathrm{ua}} = F^{-1/(n+2)}\left(\sum f_{ij}dt^i dt^j\right), \tag{1.6}$$

where $F := |\det(f_{ij})|$.

Next, if we choose the vector field $e_{n+1}$ to lie in the affine normal direction, in such a way that the frame $(e_1, e_2, \ldots, e_{n+1})$ be positively oriented, and write its components in the ambient space coordinate system as $e_{n+1} = (a^1, a^2, \ldots, a^{n+1})$, then we have that

$$a^p = -\frac{1}{n+2}F^{-(n+1)/(n+2)}\sum f^{kp}F_k\,. \tag{1.7}$$

Moreover, the Christoffel symbols of the affine normal connection are given by

$$\Gamma^i_{jk} = \frac{1}{n+2}f_{jk}\sum f^{pi}(\log F)_p\,, \tag{1.8}$$

and the scalar components $h_{ijk}$ of the unimodular affine second fundamental forms $\mathrm{II}_{\mathrm{ua}}$, by

$$h_{ijk} = f_{ijk} - \frac{1}{n+2}(f_{ij}(\log F)_k + f_{ik}(\log F)_j + f_{jk}(\log F)_i)\,. \tag{1.9}$$

Finally, for the components of the *third fundamental form* we obtain that

$$L_{ij} = -\frac{1}{n+2}\left((\log F)_{ij} + \frac{1}{n+2}(\log F)_i(\log F)_j - f_{ijp}f^{pk}(\log F)_k\right). \tag{1.10}$$

The properties described in the next proposition, pertaining to the class of hypersurfaces under consideration, are well-known: they can be deduced from the expressions above. See also [5], where, as pointed out before, different notation is used.

PROPOSITION 1.1. *Let $X : M^n \to E^{n+1}$ be a nondegenerate hypersurface with parallel second fundamental (cubic) form, $\nabla(\mathrm{II}_{\mathrm{ua}}) = 0$, which is not a hyperquadric, i.e., with $\mathrm{II}_{\mathrm{ua}} \neq 0$. Then the following properties hold:*

1) *$X(M)$ is an improper affine hypersphere.*

2) *$X(M)$ is expressible in the form of Monge, i.e., a graph immersion, and with respect to a suitable affine system of coordinates the graph function $f$ satisfies a Monge-Ampère type*

*equation* $\det(f_{ij}) = \pm 1$. *Moreover, it is representable as a polynomial function of degree exactly equal to three.*

3) *The following geometrical objects associated with* $X(M)$ *are all vanishing*: $\mathrm{III}_{\mathrm{ga}} = 0$, $\widetilde{\mathrm{Ric}} = 0$, $R = L = J = 0$.

4) *The first fundamental form* $\mathrm{I}_{\mathrm{ua}}$ *is indefinite.*

Finally, we want to stress the fact that the conditions expressed by property 2) in the above Proposition are characterizing. In fact, we have the following complementary result.

PROPOSITION 1.2. *Let* $X : M^n \to E^{n+1}$ *be a nondegenerate hypersurface which is expressible in the form of Monge, i.e., a graph immersion with respect to some affine system of coordinates in the ambient space, such that the graph function* $f$ *is a polynomial of degree exactly equal to three and satisfies the Monge-Ampère type equation* $\det(f_{ij}) = \pm 1$. *Then,* $X(M)$ *is an improper affine hypersphere with parallel second fundamental (cubic) form,* $\nabla(\mathrm{II}_{\mathrm{ua}}) = 0$, *which is not a hyperquadric, i.e., with* $\mathrm{II}_{\mathrm{ua}} \neq 0$.

PROOF. We have $F = |\det(f_{ij})| = 1$, so that by equation (1.7), $a^p = 0$, with $p = 1, \ldots, n$; and $X(M)$ is an improper affine hypersphere. By equation (1.8) all of the Christoffel symbols for the affine normal connection vanish so that, on one hand, the components of the second fundamental (cubic) form are equal to $h_{ijk} = f_{ijk}$ by equation (1.9), and since by hypothesis some of them is nonvanishing, $X(M)$ is not a hyperquadric; and, on the other hand, by the same token, the normal covariant derivatives of these components are equal to $h_{ijk;l} = f_{ijkl} = 0$, since $f$ is a polynomial function of degree three, i.e., $\nabla(\mathrm{II}_{\mathrm{ua}}) = 0$, and the second fundamental (cubic) form $\mathrm{II}_{\mathrm{ua}}$ is parallel with respect to the affine normal connection. The proposition is proved.

**2. An algorithmic sequence of coordinate changes.** Since we are interested in studying nondegenerate hypersurfaces $X(M)$ with parallel second fundamental (cubic) form, $\nabla(\mathrm{II}_{\mathrm{ua}}) = 0$, which are not hyperquadrics, i.e., with $\mathrm{II}_{\mathrm{ua}} \neq 0$, we may apply the characterizing properties described by Propositions 1.1 and 1.2. Thus, by means of a translation, if necessary, we may assume that a linear system of coordinates has been chosen in the ambient space in such a way that the origin of coordinates lies in the hypersurface $X(M)$, that the hyperplane on which $X(M)$ is projectable is precisely the tangent hyperplane $T_0(X(M))$ to $X(M)$ at that point, and that the last coordinate is chosen in the (constant) direction of the affine normal vector field $e_{n+1}$. We denote again, as in the previous section, by $(t^1, t^2, \ldots, t^n, t^{n+1})$ such an affine system of the vector space $E$, and represent the immersed hypersurface by the equation (1.5) with the point $(t^1, \ldots, t^n)$ varying in an open, connected subset $U \subset T_0(X(M))$, which is obviously identifiable with $\boldsymbol{R}^n$. By the choices made we have that

$$f(0, 0, \ldots, 0) = f_1(0, 0, \ldots, 0) = \cdots = f_n(0, 0, \ldots, 0) = 0 \,. \tag{2.1}$$

All of the remaining affine changes of coordinates shall occur in the tangent hyperplane $T_0(X(M))$ and shall be of a linear nature, i.e., given by a system of linear equations like

$$t^{*i} = \sum a^i_k t^k \,, \quad t^{*n+1} = t^{n+1} \,.$$

Most usually the change shall be unimodular, i.e., with det $(a^i_j) = 1$, although we may allow, occasionally, a rescaling in order to make the exposition less involved.

Once such a change is made, in the new coordinate system, conditions expressed by equations (2.1) remain unchanged, and the Hessian matrix $H(f) := (f_{ij})$ changes as indicated by

$$H^*(f) = PH(f)P^t\,, \tag{2.2}$$

where the matrix $P$ is nonsingular and $P^t$ denotes the transpose of $P$. Now, it is well-known that, since $P$ is expressible as a product of elementary matrices, the product to the left by $P$ is equivalent to performing the corresponding row elementary operations to $H(f)$, and the product to the right by $P^t$ is obtained by performing the equivalent kinds of column elementary operations, both in the same order of execution. Thus, to obtain $H^*(f)$ from $H(f)$ we may do so by means of the following row and column elementary operations, which we define next:

1) $R_{ij}$ interchanges rows $i$ and $j$. $C_{ij}$ interchanges columns $i$ and $j$.

2) $R_i + \sum a_{ij} R_j$, with $j \neq i$, substitutes the $i$-th row by the linear combination as indicated.

Similarly, the notation for columns shall be indicated by $C_i + \sum a_{ij} C_j$.

Obviously, these two kinds of elementary operation are unimodular. The third kind consists of multiplying a row, and the corresponding column, by a nonzero constant. This produces a rescaling.

LEMMA 2.1. *Let $X : M^n \to E^{n+1}$ be a nondegenerate hypersurface with parallel second fundamental (cubic) form, $\nabla(\mathrm{II}_{\mathrm{ua}}) = 0$, which is not a hyperquadric, i.e., $\mathrm{II}_{\mathrm{ua}} \neq 0$. Then there exists an affine coordinate system in the ambient space such that $X(M)$ is expressible in the form of Monge (i.e., by means of a graph function $f$) and such that the corresponding Hessian matrix is given by*

$$H(f) = (f_{ij}) = J_k + (x_{ij})\,,$$

*where $J_k$ is a matrix with $k$ $(\geq 1)$ blocks of the form*

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

*in diagonal position, occupying the first $2k$ diagonal entries; the (possible) remaining diagonal elements are equal to $1$, and with all of the rest of entries equal $0$; while all of the entries of the matrix $(x_{ij})$ are linear functions of the (domain) coordinates $t^1, t^2, \ldots, t^n$, i.e., $x_{ij} = \sum a_{ijk} t^k$. Moreover, the matrix of linear functions $(x_{ij})$ is everywhere singular, whose maximal rank $r$ is attained on an open, dense subset of the domain, and we have $1 \leq r \leq n-1$.*

PROOF. First, we choose the affine coordinate system as indicated at the beginning of this section so that equations (2.1) hold for the graph function $f$. Thus, since $f$ is a polynomial function of degree less than or equal to three, we can write

$$H(f) = (f_{ij}(0)) + (x_{ij}) ,$$

where $f_{ij}(0)$ is the value of $f_{ij}$ at the origin, and $x_{ij} = \sum a_{ijk} t^k$. Now, since by Proposition 2.1 the first fundamental form and hence the matrix $(f_{ij})$ is indefinite, one can find a nonsingular matrix $P$ such that $P(f_{ij}(0))P^t = (\varepsilon_i \delta_{ij})$, with $\varepsilon_i = \pm 1$. Also, we can assume, by changing the direction of the affine normal $e_{n+1}$ if necessary, that the number of positive entries in the diagonal is greater than or equal to the number of negative ones, i.e., the signature of the last matrix is greater than or equal to zero. Next, by the operations indicated previously we can transform the last matrix into the one having $k$ blocks of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

in diagonal position, occupying the first $2k$ diagonal entries; the (possible) remaining diagonal elements being equal to 1, and with all of the rest of entries equal 0. Then, by performing elementary row and column operations like $R_1 - R_2$ and $C_1 - C_2$, $R_3 - R_4$ and $C_3 - C_4$, and so on, we get $(f_{ij}(0))$ transformed into $J_k$, $k \geq 1$. On the other hand, the same procedure applied to the *complementary matrix* $(x_{ij})$ transforms this into a matrix with the same characteristic, i.e., with all of its entries being linear functions of the (domain) coordinates $t^1, t^2, \ldots, t^n$. In order to avoid unnecessary complications in notation we shall still denote the transformed complementary matrix by the same notation. Finally, developing the determinant of the Hessian matrix, we may write

$$\det(f_{ij}) = \det J_k + P^1 + P^2 + \cdots + P^{n-1} + P^n ,$$

where the $P^d$'s are all homogeneous polynomials of respective degree $d$ in the coordinates $t^1, t^2, \ldots, t^n$. It is obvious that all these polynomials must vanish, and that the last one is precisely the determinant of the complementary matrix. Thus $\det(x_{ij}) = P^n = 0$, and the matrix $(x_{ij})$ is everywhere singular . However, $(x_{ij})$ can not be equal to the null matrix at all points of the domain, because in this case it is easy to see that the graph function $f$ would be a polynomial function of degree 2, and then, by equation (1.9), the second fundamental form would vanish, $\mathrm{II}_{\mathrm{ua}} = 0$, contradicting the hypothesis. Thus, we have that $1 \leq r \leq n-1$. Therefore there exists some point $p_o \in U$, and a symmetric minor with $r$ rows and $r$ columns such that at that point, and hence on a neighborhood of it, it is nonsingular. Since the mentioned minor can be singular only on the intersection of $U$ with a finite union of vector subspaces, the last assertion of the lemma follows immediately.

The two positive integers $k$, with $1 \leq k \leq n/2$, and $r$, with $1 \leq r \leq n-1$, are characteristic of, and determined by, each hypersurface with the required geometrical properties of having parallel second fundamental form with respect to the affine normal connection, i.e., $\nabla(\mathrm{II}_{\mathrm{ua}}) = 0$, and not being a hyperquadric, i.e., with $\mathrm{II}_{\mathrm{ua}} \neq 0$. Thus, these two numbers shall play an essential role in the classification procedure that we begin next. First we present new proofs of two results previously obtained by other authors, with methods different to the one shown here.

THEOREM 2.2. *Let $X : M^2 \to E^3$ be a nondegenerate surface with parallel second fundamental* (*cubic*) *form*, $\nabla(\mathrm{II}_{\mathrm{ua}}) = 0$, *which is not a quadric*, $\mathrm{II}_{\mathrm{ua}} \neq 0$. *Then $X(M)$ is affinely congruent to the Cayley Surface*, *i.e.*, *expressible as the graph function $t^3 = t^1t^2 + (t^1)^3$.*

PROOF (compare to [4, 5]). By Lemma 2.1 we have, in the present case, only one possibility for the values of $k$ and $r$, namely, $k = 1$, $r = 1$. Thus, we assume, also by the lemma, that the surface is expressed in the form of Monge, and that the first step in the procedure for the algorithmic sequence of coordinate changes has already been taken, so that we can write the Hessian matrix in the form

$$H(f) = (f_{ij}) = J_1 + (x_{ij}) = \begin{bmatrix} x_{11} & 1 + x_{12} \\ 1 + x_{12} & x_{22} \end{bmatrix}.$$

Since one of the two symmetric minors must be different from zero, we may assume, by making the elementary operations $R_{12}$ and $C_{12}$ if necessary, that $x_{11} \neq 0$ on an open dense subset. It is then easy to see that $x_{12} = x_{22} = 0$ everywhere. Hence we have that $f_{12} = 1$, from which we obtain by integration that $f_1 = t^2 + A(t^1)$, where $A(t^1)$ is a function which depends only on $t^1$. From this we obtain $f_{11} = A'(t^1) = x_{11} = a_{111}t^1$, and it follows that

$$f_1 = t^2 + A(t^1) = t^2 + \frac{1}{2}a_{111}(t^1)^2,$$

by using (2.1). Finally, integrating once again, we get that

$$t^3 = f(t^1, t^2) = t^1t^2 + \frac{1}{6}a_{111}(t^1)^3,$$

where, of course, one can absorb the constant, by a suitable transformation, and write

$$t^3 = f(t^1, t^2) = t^1t^2 + (t^1)^3.$$

The theorem is proved.

THEOREM 2.3. *Let $X : M^3 \to E^4$ be a nondegenerate hypersurface with parallel second fundamental* (*cubic*) *form*, $\nabla(\mathrm{II}_{\mathrm{ua}}) = 0$, *which is not a hyperquadric*, $\mathrm{II}_{\mathrm{ua}} \neq 0$, *Then $X(M)$ is affinely congruent to one of the following graph immersions*:

a) $t^4 = t^1t^2 + (t^3)^2 + (t^1)^3$, *in this case $k = r = 1$.*

b) $t^4 = t^1t^2 + (t^3)^2 + (t^1)^2t^3$, *in this case $k = 1, r = 2$.*

PROOF (compare to [6]). Once again we use Lemma 2.1 and have only one possibility for the value of $k = 1$, and two possible values for $r = 1, 2$.

Let us first take $k = r = 1$. We assume, also by the Lemma, that the surface is expressed in the form of Monge, and that the first step in the procedure for the algorithmic sequence of coordinate changes has already been taken. Then the Hessian matrix can be written as

$$H(f) = (f_{ij}) = J_1 + (x_{ij}) = \begin{bmatrix} x_{11} & 1 + x_{12} & x_{13} \\ 1 + x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & 1 + x_{33} \end{bmatrix}. \tag{2.3}$$

We consider now possible subcases:

a$_1$) Assume the linear function $x_{11} \neq 0$. Then, considering the complementary matrix $(x_{ij})$, we can write the (row and column) vectors $X_2 := (x_{12}, x_{22}, x_{23})$, $X_3 := (x_{13}, x_{23}, x_{33})$, in terms of $X_1 := (x_{11}, x_{12}, x_{13})$, as $X_2 = a_{21}X_1$, $X_3 = a_{31}X_1$. Next, by applying to the Hessian matrix the elementary operations $R_2 - a_{21}R_1$, $R_3 - a_{31}R_1$; $C_2 - a_{21}C_1$, $C_3 - a_{31}C_1$; and then $R_2 + a_{31}R_3$, $C_2 + a_{31}C_3$, we obtain the original matrix transformed into

$$\begin{bmatrix} x_{11} & 1 & 0 \\ 1 & c_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with the constant value $c_{22} = 0$, since otherwise the determinant would take the value $-1 - c_{22}x_{11}$, and this represents a contradiction. Thus, the Hessian matrix becomes

$$\begin{bmatrix} x_{11} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and we proceed to integrate the latter; from $f_{12} = 1$ we obtain that $f_1 = t^2 + A(t^1)$, since $f_{13} = 0$. Hence, $f_{11} = A'(t^1) = x_{11} = a_{111}t^1$, from which it follows, by also using (2.1), that

$$f_1 = t^2 + A(t^1) = t^2 + \frac{1}{2}a_{111}(t^1)^2 .$$

Therefore, integrating once again and observing that $f_{22} = f_{23} = 0$, while $f_{33} = 1$, we see that

$$f = t^1t^2 + \frac{1}{6}a_{111}(t^1)^3 + \frac{1}{2}(t^3)^2 ,$$

and this is, except for the constants which can be absorbed, the solution indicated in the statement.

a$_2$) The case $x_{22} \neq 0$ can be reduced to a$_1$) by performing the elementary operations $R_{12}$ and $C_{12}$, so that we pass to the final possibility.

a$_3$) Assume $x_{33} \neq 0$. Then we can write the vectors $X_1$, $X_2$, in terms of $X_3$, as $X_1 = a_{13}X_3$, $X_2 = a_{23}X_3$. If $a_{13} = a_{23} = 0$, the matrix becomes

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 + x_{33} \end{bmatrix},$$

which, by evaluating the determinant, leads to the contradiction that $x_{33} = 0$. Thus we can assume that $a_{13} \neq 0$ or $a_{23} \neq 0$, and then make the elementary operations $R_1 - a_{13}R_3$, $R_2 - a_{23}R_3$; $C_1 - a_{13}C_3$, $C_2 - a_{23}C_3$, to obtain the Hessian matrix expressed by

$$\begin{bmatrix} a_{13}^2 & 1 + a_{13}a_{23} & -a_{13} \\ 1 + a_{13}a_{23} & a_{23}^2 & -a_{23} \\ -a_{13} & -a_{23} & 1 + x_{33} \end{bmatrix}.$$

Evaluation of the determinant gives $-1 - x_{33}(1 + 2a_{13}a_{23})$, from which we obtain the condition $1 + 2a_{13}a_{23} = 0$, since $x_{33} \neq 0$. Hence we perform in the last expression for the

Hessian the elementary operations $R_2 - 2a_{23}^2 R_1$, $R_3 - 2a_{23}R_1$; $C_2 - 2a_{23}^2 C_1$, $C_3 - 2a_{23}C_1$, to obtain

$$\begin{bmatrix} a_{13}^2 & 0 & 0 \\ 0 & 0 & -2a_{23} \\ 0 & -2a_{23} & x_{33} \end{bmatrix},$$

and the further operations $a_{13}R_2$, $2a_{23}R_1$; $a_{13}C_2$, $2a_{23}C_1$; $R_{13}$, $C_{13}$, to reduce this to the case a$_1$).

We now take the second case $k = 1$, $r = 2$. We assume again that the first step in the procedure has been performed and obtain the Hessian matrix expressed by equation (2.3). Since the role of the first two rows (and columns) is interchangeable by applying, if necessary, the elementary operations $R_{12}$ and $C_{12}$, we have two subcases: b$_1$) The vectors $X_1$ and $X_3$ are linearly independent (on an open, dense subset of the domain). b$_2$) The linearly independent vectors are the ones labeled as $X_1$ and $X_2$.

b$_1$) We can write the vector $X_2 = a_{21}X_1 + a_{23}X_3$, and apply to the Hessian matrix the elementary operations $R_2 - a_{21}R_1 - a_{23}R_3$ and $C_2 - a_{21}C_1 - a_{23}C_3$ to obtain

$$\begin{bmatrix} x_{11} & 1 & x_{13} \\ 1 & a_{23}^2 - 2a_{21} & -a_{23} \\ x_{13} & -a_{23} & 1 + x_{33} \end{bmatrix}. \tag{2.4}$$

If we evaluate the determinant of the latter, the factor $a_{23}^2 - 2a_{21}$ appears multiplied by the quadratic function represented by the second order determinant

$$\begin{vmatrix} x_{11} & x_{13} \\ x_{13} & x_{33} \end{vmatrix},$$

which is different from zero on an open, dense subset of the domain. Hence we must have $a_{23}^2 - 2a_{21} = 0$. Next, we apply to the matrix in (2.4) the elementary operations $R_3 + a_{23}R_1$, $C_3 + a_{23}C_1$ and get

$$\begin{bmatrix} x_{11} & 1 & x_{13} \\ 1 & 0 & 0 \\ x_{13} & 0 & 1 + x_{33} \end{bmatrix},$$

where we have labeled the linear functions $x_{13}$ and $x_{33}$ with the same notation for the sake of simplicity. Finally, evaluating the determinant of the last matrix, we conclude that $x_{33} = 0$, so that the Hessian becomes

$$\begin{bmatrix} x_{11} & 1 & x_{13} \\ 1 & 0 & 0 \\ x_{13} & 0 & 1 \end{bmatrix}. \tag{2.5}$$

We proceed to integrate this, having (2.1) in mind, as follows: $f_{21} = 1$, $f_{22} = f_{23} = 0$, imply $f_2 = t^1$, from which it follows that $f = t^1t^2 + A(t^1, t^3)$. From $A_{33} = f_{33} = 1$ it follows that $A_3 = f_3 = t^3 + B(t^1)$, so that $A_{31} = f_{31} = B'(t^1) = a_{311}t^1$ and $A_3 = f_3 = t^3 + (1/2)a_{311}(t^1)^2$. Thus $f = t^1t^2 + (1/2)(t^3)^2 + (1/2)a_{311}(t^1)^2t^3 + (1/6)a_{111}(t^1)^3$. Finally, by performing a couple of elementary operations to the resulting Hessian matrix, we can assume that $a_{111} = 0$, and on the other hand absorb the rest of constants to obtain b) in the statement of the theorem.

b$_2$) We write the vector $X_3 = a_{31}X_1 + a_{32}X_2$, and apply to the Hessian matrix the elementary operations $R_3 - a_{31}R_1 - a_{32}R_2$, $C_3 - a_{31}C_1 - a_{32}C_2$, to obtain

$$\begin{bmatrix} x_{11} & 1+x_{12} & -a_{32} \\ 1+x_{12} & x_{22} & -a_{31} \\ -a_{32} & -a_{31} & b \end{bmatrix}, \tag{2.6}$$

where, since the minor determinant $x_{11}x_{22} - (x_{12})^2 \neq 0$, we must have $b = 1 + 2a_{31}a_{32} = 0$. Then, by successive elementary operations we can transform the above matrix, first into

$$\begin{bmatrix} x_{11} & 1+x_{12} & -a_{32} \\ 1+x_{12} & c_{22}+x_{22} & 0 \\ -a_{32} & 0 & 0 \end{bmatrix},$$

from which we observe that we must have $x_{22} = 0$, then into

$$\begin{bmatrix} x_{11} & 1+x_{12} & 1 \\ 1+x_{12} & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

afterwards into

$$\begin{bmatrix} x_{11} & x_{12} & 1 \\ x_{12} & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

and finally, into the form indicated in (2.5). This shows that the present case is reduced to the previous one labeled as b$_1$). The theorem is now proved.

## 3. The classification of the four dimensional case.

THEOREM 3.1. *Let $X : M^4 \to E^5$ be a nondegenerate hypersurface with parallel second fundamental (cubic) form, $\nabla(\mathrm{II}_{\mathrm{ua}}) = 0$, which is not a hyperquadric, $\mathrm{II}_{\mathrm{ua}} \neq 0$. Then $X(M)$ is affinely congruent to one of the following graph immersions:*

a) *For $k = r = 1$:* $t^5 = t^1t^2 + (t^1)^3 + (t^3)^2 + (t^4)^2$.

b) *For $k = 1$, $r = 2$:* $t^5 = t^1t^2 + (t^3)^2 + (t^1)^2t^3 + (t^4)^2$.

c) *The case where $k = 1$, $r = 3$ is not possible, i.e., there does not exist any nondegenerate hypersurface immersion with the required geometrical properties in the case where $k = 1$ and $r = 3$.*

d) *For $k = 2$, $r = 1$:* $t^5 = t^1t^2 + (t^1)^3 + t^3t^4$.

e) *For $k = r = 2$, we have the following subcases:*

e$_{11}$) $t^5 = t^1t^2 + \frac{a}{6}(t^1)^3 + \frac{b}{2}(t^1)^2t^3 + \frac{c}{2}t^1(t^3)^2 + t^3t^4 + \frac{d}{6}(t^3)^3$,

*with the condition that the minor determinant of the complementary matrix be different from zero, i.e.,* $\det(x_{ij}) = (at^1 + bt^3)(ct^1 + dt^3) - (bt^1 + ct^3)^2 \neq 0$.

e$_{12}$) $t^5 = t^1t^2 + \frac{c}{2}t^1(t^3)^2 + \beta\frac{c}{2}t^2(t^3)^2 + t^3t^4 + \frac{d}{6}(t^3)^3$, *with the condition* $c \neq 0$.

f) *For* $k = 2$, $r = 3$:

$$t^5 = t^1t^2 + \frac{a}{2}t^2(t^3)^2 + t^3t^4 + \frac{b}{2}(t^1)^2t^3 + \frac{c}{2}t^1(t^3)^2 + \frac{d}{6}(t^1)^3 + \frac{e}{6}(t^3)^3 ,$$

*with the condition that* $(at^3)^2(bt^3 + dt^1) \neq 0$, *i.e.*, $a \neq 0$, *and* $b \neq 0$ *or* $d \neq 0$.

PROOF. From Lemma 2.1 we have, as possible values for $k$ and $r$, $k = 1, 2$ and $r = 1, 2, 3$, respectively.

a) Let us first take $k = 1$, $r = 1$. After the first step in the procedure, the Hessian matrix is reduced to

$$\begin{bmatrix} x_{11} & 1+x_{12} & x_{13} & x_{14} \\ 1+x_{12} & x_{22} & x_{23} & x_{24} \\ x_{13} & x_{23} & 1+x_{33} & x_{34} \\ x_{14} & x_{24} & x_{34} & 1+x_{44} \end{bmatrix} . \tag{3.1}$$

By making suitable elementary operations, this can be reduced to two subcases to be labeled as: a$_1$) $x_{11} \neq 0$; a$_2$) $x_{44} \neq 0$.

We observe that, in both subcases, the proof may follow an argument similar to that in the proof of Theorem 2.3, case a), to obtain the solution as stated above. Thus, we can shorten this part of the argument and proceed to the following case.

b) We now take $k = 1$, $r = 2$. Again, by means of suitable elementary operations, this is reduced to three subcases that we shall label as b$_1$), b$_2$) and b$_3$), considered below.

b$_1$) In the display (3.1), the vectors $X_1$, $X_3$ are linearly independent (on an open, dense subset of the domain) and we may write $X_2 = a_{21}X_1 + a_{23}X_3$, $X_4 = a_{41}X_1 + a_{43}X_3$. Then, by performing the elementary operations suggested by the latter equalities, i.e., $R_i - a_{i1}R_1 - a_{i3}R_3$ and $C_i - a_{i1}C_1 - a_{i3}C_3$, for $i = 2, 4$, the Hessian matrix is transformed into

$$\begin{bmatrix} x_{11} & 1 & x_{13} & 0 \\ 1 & b_{22} & -a_{23} & b_{24} \\ x_{13} & -a_{23} & 1+x_{33} & -a_{43} \\ 0 & b_{24} & -a_{43} & b_{44} \end{bmatrix} .$$

It is easy to see that the $2 \times 2$ submatrix $(b_{ij})$ must be singular, since by the Laplace development of the determinant, according to rows 2 and 4, its value appears multiplied by the determinant $x_{11}x_{33} - (x_{13})^2$, assumed to be $\neq 0$. Moreover, since $b_{44} = 1 + a_{43}^2$, we may write $(b_{22}, b_{24}) = c(b_{24}, b_{44})$, for some $c \in \boldsymbol{R}$, and perform the elementary operations $R_2 - cR_4$ and $C_2 - cC_4$, followed by $R_3 + a_{23}R_1$ and $C_3 + a_{23}C_1$, so that the Hessian matrix is transformed into

$$\begin{bmatrix} x_{11} & 1 & x_{13} & 0 \\ 1 & 0 & 0 & 0 \\ x_{13} & 0 & 1+x_{33} & -a_{43} \\ 0 & 0 & -a_{43} & b_{44} \end{bmatrix} .$$

By evaluating the determinant of the latter we conclude that $x_{33} = 0$; and, by making further elementary operations, we obtain that the Hessian matrix reduces to

$$\begin{bmatrix} x_{11} & 1 & x_{13} & 0 \\ 1 & 0 & 0 & 0 \\ x_{13} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{3.2}$$

Therefore, by Theorem 2.3, subcase b), we can integrate (and absorb the constants) to obtain

$$f = t^1 t^2 + (t^3)^2 + (t^1)^2 t^3 + (t^4)^2 . \tag{3.3}$$

b$_2$) Let us now assume that the vectors $X_1, X_2$ are linearly independent and write $X_i = a_{i1}X_1 + a_{i2}X_2$, with $i = 3, 4$. Then, we perform the elementary operations $R_i - a_{i1}R_1 - a_{i2}R_2$ and $C_i - a_{i1}C_1 - a_{i2}C_2$, for $i = 3, 4$, so that the Hessian matrix is transformed into

$$\begin{bmatrix} x_{11} & 1 + x_{12} & -a_{32} & -a_{42} \\ 1 + x_{12} & x_{22} & -a_{31} & -a_{41} \\ -a_{32} & -a_{31} & b_{33} & b_{34} \\ -a_{42} & -a_{41} & b_{34} & b_{44} \end{bmatrix} ,$$

where the submatrix $(b_{ij})$ must be singular, for the same reason as in the previous case. But, on the other hand, $(b_{ij}) \neq 0$, because otherwise the determinant of the Hessian matrix would equal to $(a_{32}a_{41} - a_{31}a_{42})^2 = -1$, a contradiction. Thus, we can assume that $b_{44} \neq 0$, and by making some further elementary operations, which are easy to determine, transform the latter into

$$\begin{bmatrix} b_{11} + x_{11} & b_{12} + x_{12} & b_{13} & 0 \\ b_{12} + x_{12} & b_{22} + x_{22} & b_{23} & 0 \\ b_{13} & b_{23} & 0 & 0 \\ 0 & 0 & 0 & b_{44} \end{bmatrix} ,$$

where we must have $b_{13} \neq 0$ or $b_{23} \neq 0$. We may assume the latter, and then further transform the Hessian matrix into

$$\begin{bmatrix} b_{11} + x_{11} & x_{12} & 0 & 0 \\ x_{12} & x_{22} & b_{23} & 0 \\ 0 & b_{23} & 0 & 0 \\ 0 & 0 & 0 & b_{44} \end{bmatrix} .$$

Then, by evaluating its determinant $(= -(b_{11} + x_{11})b_{23}^2 b_{44})$, we conclude that $x_{11} = 0$. Finally, by making some further elementary operations, we transform the Hessian into the form (3.2), so that this subcase is equivalent to the previous one, b$_1$).

b$_3$) Assume now that the vectors $X_3, X_4$ are linearly independent, write $X_i = a_{i3}X_3 + a_{i4}X_4$, for $i = 1, 2$, and perform the elementary operations $R_i - a_{i3}R_3 - a_{i4}R_4$ and $C_i - a_{i3}C_3 - a_{i4}C_4$, for $i = 3, 4$. Then, the Hessian matrix is transformed into

$$\begin{bmatrix} b_{11} & b_{12} & -a_{13} & -a_{14} \\ b_{12} & b_{22} & -a_{23} & -a_{24} \\ -a_{13} & -a_{23} & 1 + x_{33} & x_{34} \\ -a_{14} & -a_{24} & x_{34} & 1 + x_{44} \end{bmatrix} ,$$

where, again, the submatrix $(b_{ij})$ must be singular but $\neq 0$, so that we can assume $b_{11} \neq 0$ and, by means of elementary operations, easily determined, transform the latter, first into

$$\begin{bmatrix} b_{11} & 0 & 0 & 0 \\ 0 & 0 & b_{23} & b_{24} \\ 0 & b_{23} & b_{33}+x_{33} & b_{34}+x_{34} \\ 0 & b_{24} & b_{34}+x_{34} & b_{44}+x_{44} \end{bmatrix},$$

and then, since one can further assume that $b_{23} \neq 0$, into

$$\begin{bmatrix} b_{11} & 0 & 0 & 0 \\ 0 & 0 & b_{23} & 0 \\ 0 & b_{23} & x_{33} & x_{34} \\ 0 & 0 & x_{34} & b_{44}+x_{44} \end{bmatrix}.$$

Finally, it follows that $x_{44} = 0$ and, by further elementary operations, we can also reduce the latter to the form (3.2); i.e., this subcase is also equivalent to the first one $\mathrm{b}_1$).

c) The third possible case corresponds to the values $k = 1$, $r = 3$. It is easy to see, by means of suitable elementary operations, that this is reduced to two subcases: $\mathrm{c}_1$) and $\mathrm{c}_2$).

$\mathrm{c}_1$) First, let us assume that the vectors $X_1$, $X_2$, $X_3$ are linearly independent on an open, dense subset of the domain. Represent the remaining one by $X_4 = a_{41}X_1 + a_{42}X_2 + a_{43}X_3$ and perform on the Hessian matrix defined by equation (3.1) the elementary operations suggested by this equality, i.e., $R_4 - a_{41}R_1 - a_{42}R_2 - a_{43}R_3$ and $C_4 - a_{41}C_1 - a_{42}C_2 - a_{43}C_3$, to obtain

$$\begin{bmatrix} x_{11} & 1+x_{12} & x_{13} & -a_{42} \\ 1+x_{12} & x_{22} & x_{23} & -a_{41} \\ x_{13} & x_{23} & 1+x_{33} & -a_{43} \\ -a_{42} & -a_{41} & -a_{43} & b_{44} \end{bmatrix},$$

where, since the $3 \times 3$ principal minor of the complementary matrix $\det(x_{ij}) \neq 0$, we must have $b_{44} = 1 + 2a_{41}a_{42} + a_{43}^2 = 0$. From this we have that $a_{42} \neq 0$, so that, by applying further elementary operations, the latter expression of the Hessian matrix can be reduced to

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & 1 \\ x_{12} & b_{22}+x_{22} & x_{23} & 0 \\ x_{13} & x_{23} & 1+x_{33} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

and since the determinant of the latter must equal to $-1$, it is easy to see that we must also have $b_{22} = 1$ and $x_{22} = x_{23} = x_{33} = 0$. But then, the maximal rank of the complementary matrix is $r < 3$, which contradicts our hypothesis. Thus, this subcase is not possible.

$\mathrm{c}_2$) Second, let us assume that the linearly independent vectors are the ones labeled as $X_2$, $X_3$, $X_4$. Then, represent the first one by $X_1 = a_{12}X_2 + a_{13}X_3 + a_{14}X_4$ and perform on the Hessian matrix defined by equation (3.1) the elementary operations $R_1 - a_{12}R_2 - a_{13}R_3 -$

$a_{14}R_4$ and $C_1 - a_{12}C_2 - a_{13}C_3 - a_{14}C_4$. We then get the Hessian matrix transformed into

$$\begin{bmatrix} b_{11} & 1 & -a_{13} & -a_{14} \\ 1 & x_{22} & x_{23} & x_{24} \\ -a_{13} & x_{23} & 1+x_{33} & x_{34} \\ -a_{14} & x_{24} & x_{34} & 1+x_{44} \end{bmatrix},$$

where we must have $b_{11} = -2a_{12} + a_{13}^2 + a_{14}^2 = 0$. Further elementary operations transform the latter into

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & x_{22} & x_{23} & x_{24} \\ 0 & x_{23} & 1+x_{33} & x_{34} \\ 0 & x_{24} & x_{34} & 1+x_{44} \end{bmatrix},$$

from which it follows, by evaluating its determinant, that $x_{33} = x_{34} = x_{44} = 0$. Then we would have that the maximal rank of the complementary matrix is again $r < 3$, a contradiction. Therefore, this second subcase is not possible, and so is the whole case $k = 1, r = 3$.

d) We now consider the case $k = 2$, $r = 1$. Then, by making suitable elementary operations, it is easy to see that this can be reduced to a single case: $x_{11} \neq 0$. Hence, by the first step in the procedure, the Hessian matrix is reduced to

$$\begin{bmatrix} x_{11} & 1+x_{12} & x_{13} & x_{14} \\ 1+x_{12} & x_{22} & x_{23} & x_{24} \\ x_{13} & x_{23} & x_{33} & 1+x_{34} \\ x_{14} & x_{24} & 1+x_{34} & x_{44} \end{bmatrix}, \tag{3.4}$$

where we can write $X_i = a_{i1}X_1$, for $i = 2, 3, 4$. Then, after some elementary operations, easily determined, the Hessian matrix is transformed into

$$\begin{bmatrix} x_{11} & 1 & 0 & 0 \\ 1 & b_{22} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

with $b_{22} = -2(a_{21} + a_{31}a_{41}) = 0$, since the evaluation of the determinant gives $1 + b_{22}x_{11}$, and we assumed that $x_{11} \neq 0$.

Hence, also by using Theorem (2.2), we can integrate straightforwardly to obtain, in the last system of coordinates on which the Hessian matrix is represented by the above:

$$f(t^1, t^2, t^3, t^4) = t^1t^2 + (t^1)^3 + t^3t^4 .$$

e) Next, we consider the case $k = 2$, $r = 2$. It is easy to see, by means of elementary operations, that this contains two subcases: $e_1$) the vectors $X_1$ and $X_3$ are linearly independent (on an open, dense subset of the domain); $e_2$) $X_1$ and $X_2$ are linearly independent.

$e_1$) We may write $X_i = a_{i1}X_1 + a_{i3}X_3$, $i = 2, 4$, and perform on the Hessian matrix (3.4), the elementary operations $R_i - a_{i1}R_1 - a_{i3}R_3$ and $C_i - a_{i1}C_1 - a_{i3}C_3$, $i = 2, 4$, to obtain

$$\begin{bmatrix} x_{11} & 1 & x_{13} & 0 \\ 1 & b_{22} & 0 & b_{24} \\ x_{13} & 0 & x_{33} & 1 \\ 0 & b_{24} & 1 & b_{44} \end{bmatrix}, \tag{3.5}$$

Now, since the determinant of the $2 \times 2$ submatrix labeled $(x_{ij})$ is assumed to be different from zero we must have that the determinant of the complementary $2 \times 2$ submatrix $(b_{ij})$ must vanish, i.e., $\det(b_{ij}) = b_{22}b_{44} - b_{24}^2 = 0$. Hence, we have two possibilities: $(b_{ij}) = 0$ or $(b_{ij}) \neq 0$.

$e_{11}$) Let us first assume that $(b_{ij}) = 0$. Then the Hessian matrix becomes

$$\begin{bmatrix} x_{11} & 1 & x_{13} & 0 \\ 1 & 0 & 0 & 0 \\ x_{13} & 0 & x_{33} & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{3.6}$$

and straightforward integration allows to compute the solution as

$$f(t^1, t^2, t^3, t^4) = t^1t^2 + \frac{a}{6}(t^1)^3 + \frac{b}{2}(t^1)^2t^3 + \frac{c}{2}t^1(t^3)^2 + t^3t^4 + \frac{d}{6}(t^3)^3 . \tag{3.7}$$

Now, since the Hessian matrix for this is

$$\begin{bmatrix} at^1 + bt^3 & 1 & bt^1 + ct^3 & 0 \\ 1 & 0 & 0 & 0 \\ bt^1 + ct^3 & 0 & ct^1 + dt^3 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

the constants appearing in the solution must satisfy the natural condition that the value of the minor determinant of the complementary matrix be $\det(x_{ij}) = (at^1 + bt^3)(ct^1 + dt^3) - (bt^1 + ct^3)^2 \neq 0$.

$e_{12}$) The second possibility is to have $\det(b_{ij}) = 0$, and $(b_{ij}) \neq 0$. In this case we may assume that $b_{22} \neq 0$. Hence there exists $\alpha \in \boldsymbol{R}$ such that $(b_{24}, b_{44}) = \alpha(b_{22}, b_{24})$, and we may apply to the matrix in (3.5) the elementary operations $R_4 - \alpha R_2$ and $C_4 - \alpha C_2$, followed by $R_1 + \alpha R_3$ and $C_1 + \alpha C_3$ in order to obtain the Hessian matrix represented by

$$\begin{bmatrix} x_{11} & 1 & x_{13} & 0 \\ 1 & b_{22} & 0 & 0 \\ x_{13} & 0 & x_{33} & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

By evaluating the determinant of the latter it follows that $x_{11} = 0$. Then, by making the further elementary operations $R_2 + \beta R_1$ and $C_2 + \beta C_1$, $\beta = -(b_{22}/2)$, the Hessian matrix can be written as

$$\begin{bmatrix} 0 & 1 & x_{13} & 0 \\ 1 & 0 & \beta x_{13} & 0 \\ x_{13} & \beta x_{13} & x_{33} & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

By integrating, we obtain the solution represented by

$$f(t^1,t^2,t^3,t^4) = t^1t^2 + \frac{c}{2}t^1(t^3)^2 + \beta\frac{c}{2}t^2(t^3)^2 + t^3t^4 + \frac{d}{6}(t^3)^3\,. \tag{3.8}$$

We observe that this solution is, in a sense, comparable to the one obtained previously. In fact, if we put $a = b = 0$ in (3.7) and $\beta = 0$ in (3.8), then they do coincide . Besides, in (3.8) we may always make $d = 0$, by means of elementary operations.

$\mathrm{e}_2$) In case that $X_1$ and $X_2$ are linearly independent, we may write $X_i = a_{i1}X_1 + a_{i2}X_2$, $i = 3, 4$, and apply the elementary operations $R_i - a_{i1}R_1 - a_{i2}R_2$ and $C_i - a_{i1}C_1 - a_{i2}C_2$, $i = 3, 4$, to transform the Hessian matrix (3.4) into the form

$$\begin{bmatrix} x_{11} & 1+x_{12} & -a_{32} & -a_{42} \\ 1+x_{12} & x_{22} & -a_{31} & -a_{41} \\ -a_{32} & -a_{31} & b_{33} & b_{34} \\ -a_{42} & -a_{41} & b_{34} & b_{44} \end{bmatrix},$$

where we must have $\det(b_{ij}) = 0$. Then, we consider two subcases: $\mathrm{e}_{21}$) $(b_{ij}) = 0$; $\mathrm{e}_{22}$) $(b_{ij}) \neq 0$.

$\mathrm{e}_{21}$) If $(b_{ij}) = 0$, we conclude that $a_{32}a_{41} \neq 0$ or $a_{31}a_{42} \neq 0$, since the evaluation of the determinant of the latter matrix gives the quantity $(a_{32}a_{41} - a_{31}a_{42})^2 = 1$. Thus, in any case, it is easy to see that we can make a finite number of elementary operations to transform the above, first into

$$\begin{bmatrix} x_{11} & x_{12} & 1 & 0 \\ x_{12} & x_{22} & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

and, afterwards, into the form described by (3.6), case $\mathrm{e}_{11}$).

$\mathrm{e}_{22}$) If $(b_{ij}) \neq 0$, we may assume that $b_{44} \neq 0$, and transform the Hessian matrix, by means of successive elementary operations into

$$\begin{bmatrix} x_{11} & 1 & x_{13} & 0 \\ 1 & 0 & 0 & 0 \\ x_{13} & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

which is a special case of (3.6).

f) Finally, let us consider the case $k = 2$, $r = 3$. Then, we may assume that $X_1, X_2, X_3$ are linearly independent (on an open, dense subset of the domain), and write $X_4 = a_{41}X_1 + a_{42}X_2 + a_{43}X_3$. Then, perform on the Hessian matrix represented by equation (3.4) the elementary operations $R_4 - a_{41}R_1 - a_{42}R_2 - a_{43}R_3$ and $C_4 - a_{41}C_1 - a_{42}C_2 - a_{43}C_3$ to obtain

$$\begin{bmatrix} x_{11} & 1+x_{12} & x_{13} & -a_{42} \\ 1+x_{12} & x_{22} & x_{23} & -a_{41} \\ x_{13} & x_{23} & x_{33} & 1 \\ -a_{42} & -a_{41} & 1 & b \end{bmatrix},$$

where, in order to avoid a contradiction, we must have $b = 2(a_{41}a_{42} - a_{43}) = 0$. Next, by applying further elementary operations, we may transform the above into the form

$$\begin{bmatrix} x_{11} & 1+x_{12} & x_{13} & 0 \\ 1+x_{12} & x_{22} & x_{23} & 0 \\ x_{13} & x_{23} & x_{33} & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

and by evaluating the determinant of the latter $(= 1 + 2x_{12} + x_{12}^2 - x_{11}x_{22})$, we conclude that the Hessian is written as

$$\begin{bmatrix} x_{11} & 1 & x_{13} & 0 \\ 1 & 0 & x_{23} & 0 \\ x_{13} & x_{23} & x_{33} & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Now, a straightforward process of integration allows to write the solution as

$$f(t^1, t^2, t^3, t^4) = t^1t^2 + \frac{a}{2}t^2(t^3)^2 + t^3t^4 + \frac{b}{2}(t^1)^2t^3 + \frac{c}{2}t^1(t^3)^2 + \frac{d}{6}(t^1)^3 + \frac{e}{6}(t^3)^3, \tag{3.9}$$

with the condition that $(at^3)^2(bt^3+dt^1) \neq 0$, i.e., $a \neq 0$, and $b \neq 0$ or $d \neq 0$. This completes the proof of the theorem.

Since some of the solutions listed in the last theorem contain constant parameters, it is quite appropriate to make the following comments and observations:

1) The solution obtained as subcase $\mathrm{e}_{11}$), equation (3.7), is the limit of that corresponding to the case f), equation (3.9), when the parameter $a$ tends to zero. However, when $a \neq 0$ in the latter, we can perform further elementary operations to make the coefficient represented by parameter $e$ equal to zero.

2) As we have observed, the solutions given by (3.7) and (3.8) are comparable. They do coincide if we take $a = b = \beta = 0$; and in the latter, one can always make $d = 0$.

3) In solution $\mathrm{e}_{11}$), equation (3.7), the natural condition to be satisfied, i.e., $\det(x_{ij}) = (at^1+bt^3)(ct^1+dt^3)-(bt^1+ct^3)^2 \neq 0$, can give rise to the consideration of various, further subcases. For example, if we take $b = c = 0$, one can absorb the remaining constant parameters, by means of rescaling, and obtain an expression without parameters at all. However, we have left the solution expressed as it is, because this represents the most general form.

Apart from these considerations, and the further possibility of performing rescalings, the solutions obtained are inequivalent for different values of the parameters, i.e., they do belong to different classes under the action of the unimodular affine group $ASL(n+1, \boldsymbol{R})$.

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