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ON CONNECTIONS BETWEEN HANKEL, LAGUERRE AND JACOBI TRANSPLANTATIONS

KRZYSZTOF STEMPAK

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Abstract. Proved are two results showing connections between the Hankel transplantation and a transplantation for a certain kind of Laguerre and Jacobi expansions. An asymptotic formula of Hilb's type for Laguerre and Jacobi polynomials is used. As an application of this link we obtain an extension of Guy's transplantation theorem for the Hankel transform to the case $\alpha, \gamma > -1$ also with more weights allowed. This is done by transferring a corresponding transplantation result for Jacobi expansions which was proved by Muckenhoupt. In the case when $\alpha, \gamma \geq -1/2$ the same is obtained by using Schindler's explicit kernel formula for the transplantation operator.

1. Introduction and statement of results. Given $\alpha > -1$ and a suitable function f on $(0, \infty)$, its (non-modified) Hankel transform is defined by

$$\mathcal{H}_\alpha f(x) = \int_0^\infty (xy)^{1/2} J_\alpha(xy) f(y)dy\,, \quad x > 0\,.$$

Here $J_\alpha(x)$ denotes the Bessel function of the first kind of order α, [Sz, (1.71.1)]. The kernels $\varphi_x^\alpha(y) = (xy)^{1/2} J_\alpha(xy)$, $x > 0$, appearing in this integral transformation satisfy the differential equation

$$\left(\frac{d^2}{dy^2} + \frac{1/4 - \alpha^2}{y^2}\right)\varphi_x^\alpha(y) = -x^2 \varphi_x^\alpha(y)\,, \quad y > 0\,.$$

Guy [Guy] showed that the size of the Hankel transform of any suitable function, when measured in the (weighted) L^p-norm, remains the same whatever the order of the Hankel transform is. More precisely, given $\alpha, \gamma \geq -1/2$, $1 < p < \infty$ and $-1 < a < p - 1$, there is a constant $C = C(\alpha, \gamma, p, a)$ such that for every appropriate function f

$$C^{-1}\|\mathcal{H}_\gamma f\|_{p,a} \leq \|\mathcal{H}_\alpha f\|_{p,a} \leq C\|\mathcal{H}_\gamma f\|_{p,a}\,.$$

In another way, this can be expressed as

$$\|(\mathcal{H}_\alpha \circ \mathcal{H}_\gamma) f\|_{p,a} \leq C\|f\|_{p,a}\,, \quad f \in C_c^\infty(0, \infty)\,,$$

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$C_c^\infty(0,\infty)$ being the space of all compactly supported C^∞ functions on $(0,\infty)$. Here, for any p, $1 \le p < \infty$, and any real number a,

$$\|g\|_{p,a} = \left(\int_0^\infty |g(x)|^p x^a dx \right)^{1/p}$$

and $L^{p,a}(dx)$ denotes the weighted Lebesgue space of all measurable functions on $(0,\infty)$ for which the above quantity is finite. For $a = 0$ we simplify the notation by writing $\|g\|_p$ and $L^p(dx)$.

Another proof of Guy's transplantation theorem (in the last formulation) was furnished by Schindler [Sch]. She found an explicit expression of the kernel of the transplantation operator $\mathcal{H}_\alpha \circ \mathcal{H}_\gamma$.

Guy's result initiated a series of transplantation theorems for both continuous and discrete orthogonal expansions. Recently, Kanjin [Ka2] proved a transplantation theorem for Laguerre expansions. Given $\alpha > -1$, the Laguerre functions $\mathcal{L}_n^\alpha(x)$, $n = 0, 1, 2, \ldots$, are defined by

$$\mathcal{L}_n^\alpha(x) = \left(\frac{n!}{\Gamma(n+\alpha+1)} \right)^{1/2} e^{-x/2} x^{\alpha/2} L_n^\alpha(x) \,,$$

where $L_n^\alpha(x)$ denotes the Laguerre polynomial of order α, [Sz, p. 101]. This set of functions is a complete orthonormal system in $L^2((0,\infty), dx)$. Kanjin's result says that if $\alpha, \gamma \ge 0$ and $1 < p < \infty$, then there is a constant $C = C(\alpha, \gamma, p)$ such that for every f in $C_c^\infty(0,\infty)$

$$\left\| \sum_{n=0}^\infty \langle f, \mathcal{L}_n^\gamma \rangle \mathcal{L}_n^\alpha \right\|_p \le C \|f\|_p \,. \tag{1.1}$$

In the case when $-1 < \tau = \min(\alpha, \gamma) < 0$, the above inequality holds in the restricted range $(1+\tau/2)^{-1} < p < -2/\tau$. Here and later on we write $\langle f, g \rangle = \int_0^\infty f(x) g(x) dx$ whenever it makes sense.

Thangavelu [Th] gave a modification of Kanjin's result by replacing the Lebesgue measure dx by $x^{p/4-1/2}dx$ (under the assumption $\tau \ge -1/2$). This means a transplantation for another system of Laguerre functions

$$\psi_n^\alpha(x) = \mathcal{L}_n^\alpha(x^2)\sqrt{2x} = \left(\frac{2n!}{\Gamma(n+\alpha+1)} \right)^{1/2} e^{-x^2/2} x^{\alpha+1/2} L_n^\alpha(x^2) \,,$$

which is also a complete orthonormal system in $L^2((0,\infty), dx)$. In several cases the system $\{\psi_n^\alpha\}$ is better suited for considerations than the system $\{\mathcal{L}_n^\alpha\}$, since the functions ψ_n^α satisfy the Sturm-Liouville type differential equation

$$\left(\frac{d^2}{dy^2} + \frac{1/4 - \alpha^2}{y^2} - y^2 \right) \psi_n^\alpha(y) = -(4n + 2\alpha + 2) \psi_n^\alpha(y) \,, \quad n \ge 0 \,.$$

We remark that Thangavelu's result may be regarded as a special case of a more general weighted transplantation result proved in [ST]: (1.1) holds with $\|\cdot\|_{p,a}$ replacing $\|\cdot\|_p$, $-1 < a < p - 1$.

In [St1] we proved a theorem relating L^p-multipliers for Laguerre expansions with those for the Hankel transform. Then, in [St2], we supplemented this result by showing how to relate L^p-norm maximal inequalities for Laguerre multipliers with those for the Hankel transform. To be precise, in [St1–2] we considered the modified Hankel transform and a slightly different system of Laguerre functions than that considered here. A close examination of the argument we used reveals, however, that the aforementioned results have their weighted analogues in the setting of $\{\psi_n^\alpha\}$-expansions and the (non-modified) Hankel transform $\mathcal{H}_\alpha$ (see remarks in [St2, §3]).

In short, both results say the following: let $\alpha > -1$, $1 < p < \infty$ and $a \in \boldsymbol{R}$, and assume that m is a bounded function on $(0, \infty)$, which is continuous except on a set of Lebesgue measure zero. Then, if $\liminf_{\varepsilon\to 0^+} \|m(\varepsilon n^{1/2})\|_{p,a}$ is finite, then $|m|_{p,a}$ is also finite and

$$|m|_{p,a} \le \liminf_{\varepsilon\to 0^+} \|m(\varepsilon n^{1/2})\|_{p,a} .$$

Here $|m|_{p,a}$ and $\|m(\varepsilon n^{1/2})\|_{p,a}$ denote the operator norms of multipliers (for the Hankel transform $\mathcal{H}_\alpha$ or for the $\{\psi_n^\alpha\}$-expansion) given by the function $m(x)$ or the sequence $m(\varepsilon n^{1/2})$, considered on the weighted Lebesgue space $L^{p,a}(dx)$ (cf. [St2] for the precise definition of these multipliers).

Next, let $\tilde{M}_m^* f(x) = \sup_{\varepsilon>0} |\tilde{T}_\varepsilon f(x)|$ be the maximal operator, where $\tilde{T}_\varepsilon$ is the Laguerre multiplier operator associated with the sequence $\{m(\varepsilon n^{1/2})\}$ (for the $\{\psi_n^\alpha\}$-expansion), and let $M_m^* f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|$ be the maximal operator, where T_ε is the Hankel multiplier operator associated with the function $m(\varepsilon y)$ (for the Hankel transform $\mathcal{H}_\alpha$). Then, if $\tilde{M}_m^*$ is bounded on $L^{p,a}(dx)$, then M_m^* is also bounded on $L^{p,a}(dx)$.

The main goal of this paper is to exhibit another connection between Laguerre (or Jacobi) expansions and the Hankel transform, on the level of transplantation.

THEOREM 1.1. *Let $1 < p < \infty$, $a \in \boldsymbol{R}$ and $\alpha, \gamma > -1$. If the Laguerre transplantation inequality*

$$\left\| \sum_0^\infty \langle f, \psi_n^\gamma \rangle \psi_n^\alpha \right\|_{p,a} \le C\|f\|_{p,a} , \quad f \in C_c^\infty(0,\infty) ,$$

holds, then the Hankel transplantation inequality

$$\|(\mathcal{H}_\alpha \circ \mathcal{H}_\gamma) f\|_{p,a} \le C\|f\|_{p,a} , \quad f \in C_c^\infty(0,\infty) ,$$

is also satisfied (*with the same constant $C > 0$*).

As already mentioned, we will also analyse a connection between Jacobi expansions and the Hankel transform. Given α and β, $\alpha > -1$, $\beta > -1$, consider the orthonormalized Jacobi functions

$$\phi_n^{(\alpha,\beta)}(x) = t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos x)\left(\sin\frac{x}{2}\right)^{\alpha+1/2}\left(\cos\frac{x}{2}\right)^{\beta+1/2} ,$$

where $P_n^{(\alpha,\beta)}$, $n = 0, 1, \dots$, are Jacobi polynomials, [Sz, (4.22.1)], and

$$t_n^{(\alpha,\beta)} = \left[\frac{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \right]^{1/2}.$$

Note that $t_n^{(\alpha,\beta)} = (2n)^{1/2} + O(n^{-1/2})$. This system of functions is a complete orthonormal system in $L^2((0,\pi), dx)$. For any p, $1 \le p < \infty$, and any real numbers a and b, we will consider the weighted Lebesgue space $L^{p,a,b}(dx)$ of those measurable functions on $(0,\pi)$ for which the norm

$$\|g\|_{p,a,b} = \left(\int_0^\pi |g(x)|^p \left(\sin\frac{x}{2} \right)^a \left(\cos\frac{x}{2} \right)^b dx \right)^{1/p}$$

is finite.

Askey [A] proved the following transplantation theorem for Jacobi expansions. Assume $1 < p < \infty$, $\alpha, \beta, \gamma, \delta \ge -1/2$, $-1 < a < p-1$, and $-1 < b < p-1$. Then there is a constant $C > 0$ such that for every f in $C_c^\infty(0,\pi)$

$$\left\| \sum_{n=0}^\infty \langle f, \phi_n^{(\gamma,\delta)} \rangle \phi_n^{(\alpha,\beta)} \right\|_{p,a,b} \le C \|f\|_{p,a,b}.$$

This result was then generalized by Muckenhoupt [M1] by admitting, among others, $\alpha, \beta, \gamma, \delta$ to be greater than -1 and considerably extending the range of a's and b's. Our second result is

THEOREM 1.2. *Let $1 < p < \infty$, $a, b \in \boldsymbol{R}$ and $\alpha, \beta, \gamma, \delta > -1$. If the Jacobi transplantation inequality*

$$\left\| \sum_0^\infty \langle f, \phi_n^{(\gamma,\delta)} \rangle \phi_n^{(\alpha,\beta)} \right\|_{p,a,b} \le C \|f\|_{p,a,b}, \quad f \in C_c^\infty(0,\pi),$$

holds, then the Hankel transplantation inequality

$$\|(\mathcal{H}_\alpha \circ \mathcal{H}_\gamma) f\|_{p,a} \le C \|f\|_{p,a}, \quad f \in C_c^\infty(0,\infty),$$

is also satisfied (*with the same constant $C > 0$*).

This connection between Jacobi expansions and the Hankel transform, exhibited again on the level of transplantation, has its ancessors on the levels of multipliers and maximal multiplier operators. Historically, it was the case of Jacobi expansions where the first connection with the Hankel transform was found. Igari [I] proved de Leeuw's type theorem linking Jacobi and Hankel multipliers, and then Kanjin [Ka1] proved a theorem that transferred L^p-norm maximal multiplier inequalities from Jacobi to Hankel side. Actually, both of our earlier papers, [St1] and [St2], were motivated by the results of Igari and Kanjin (needless to say, Igari's paper motivates the present paper, too). To be precise, the results of Igari and Kanjin were proved in the setting of modified Hankel transform and Jacobi polynomial expansions, but they have their (weighted) analogues in the setting we prefer: the (non-modified) Hankel transform $\mathcal{H}_\alpha$ and $\{\phi_n^{(\alpha,\beta)}\}$ Jacobi function expansions.

The following is a simplified version of a much more general Muckenhoupt's transplantation result [M1, Theorem (1.6)].

PROPOSITION 1.3. *If* $1 < p < \infty$, $0 < r < 1$, $\alpha > -1$, $\gamma > -1$, $-p(\alpha+1/2)-1 < a < p(\gamma+3/2)-1$ *and* $f \in L^{p,a,0}(dx)$, *then*

$$T_r f(x) = \sum_{n=0}^{\infty} r^n \langle f, \phi_n^{(\gamma,-1/2)} \rangle \phi_n^{(\alpha,-1/2)}(x)$$

converges for every $x \in (0,\pi)$,

$$\|T_r f\|_{p,a,0} \le C \|f\|_{p,a,0}\,,$$

with C *independent of* f *and* r, *and there is a function* $Tf(x)$ *in* $L^{p,a,0}$ *such that* $T_r f$ *converges to* Tf *in* $L^{p,a,0}$ *as* $r \to 1^-$ (*in consequence*, $\|Tf\|_{p,a,0} \le C\|f\|_{p,a,0}$).

Using Theorem 1.2, we generalize Guy's result by transferring Muckenhoupt's transplantation theorem for the Jacobi expansions to the Hankel transform setting. Besides the fact that the range of parameters α and γ is now enlarged to $(-1,\infty)$, the range of weights is also considerably extended.

COROLLARY 1.4. *Let* $1 < p < \infty$, $\alpha, \gamma > -1$ *and* $-p(\alpha+1/2)-1 < a < p(\gamma+3/2)-1$. *Then*

$$\|(\mathcal{H}_\alpha \circ \mathcal{H}_\gamma) f\|_{p,a} \le C \|f\|_{p,a}\,, \quad f \in C_c^\infty(0,\infty)\,.$$

PROOF. Let $f \in C_c^\infty(0,\pi)$ and $T_r f(x)$, $Tf(x)$ be as in Proposition 1.3. Then $\|Tf\|_{p,a,0} \le C\|f\|_{p,a,0}$ with C independent of f. The series

$$\sum_{n=0}^{\infty} \langle f, \phi_n^{(\gamma,-1/2)} \rangle \phi_n^{(\alpha,-1/2)}(x)$$

converges for every $x \in (0,\pi)$ by Lemma 2.2. Therefore

$$\lim_{r\to 1^-} T_r f(x) = \sum_{n=0}^{\infty} \langle f, \phi_n^{(\gamma,-1/2)} \rangle \phi_n^{(\alpha,-1/2)}(x)$$

for every $x \in (0,\pi)$. By choosing a sequence $r_1 < r_2 < \ldots, r_j \to 1^-$, such that $Tf(x) = \lim_{j\to\infty} T_{r_j} f(x)$ for almost every $x \in (0,\pi)$, we then get

$$Tf(x) = \sum_{n=0}^{\infty} \langle f, \phi_n^{(\gamma,-1/2)} \rangle \phi_n^{(\alpha,-1/2)}(x)$$

for almost every $x \in (0,\pi)$ and

$$\left\| \sum_{0}^{\infty} \langle f, \phi_n^{(\gamma,-1/2)} \rangle \phi_n^{(\alpha,-1/2)} \right\|_{p,a,0} \le C \|f\|_{p,a,0}\,.$$

Corollary 1.4 then follows by using Theorem 1.2.

It should be noted that in the case when the two parameters α and β differ by an even positive integer, only one side of the inequality restricting a in Corollary 1.4 is needed for the Hankel transplantation inequality to hold. Indeed, if $\alpha = \gamma + 2k$, $k = 1, 2, \dots$, then

$$\|(\mathcal{H}_{\gamma+2k} \circ \mathcal{H}_\gamma) f\|_{p,a} \le C\|f\|_{p,a}, \quad f \in L^{p,a}(dx),$$

easily follows for $1 \le p < \infty$ and $a < p(\gamma + 3/2) - 1$, by using a simple explicit integral kernel form of the operator $\mathcal{H}_{\gamma+2k} \circ \mathcal{H}_\gamma$. Considering only the case $k = 1$, we have

$$\mathcal{H}_{\gamma+2} f(x) = \frac{2(\gamma+1)}{x} \int_0^x \left(\frac{y}{x}\right)^{\gamma+1/2} \mathcal{H}_\gamma f(y) dy - \mathcal{H}_\gamma f(x).$$

This follows from recursion formulas for the Bessel functions. Therefore, we only need to show the boundedness of the operator

$$Tg(x) = \frac{1}{x} \int_0^x \left(\frac{y}{x}\right)^{\gamma+1/2} g(y) dy$$

on $L^{p,a}(dx)$, $1 \le p < \infty$, $a < p(\gamma + 3/2) - 1$. This leads to the inequality

$$\int_0^\infty \left| \int_0^x h(y) dy \right|^p x^{a-p(\gamma+3/2)} dx \le C_{p,a} \int_0^\infty |h(x)|^p x^{a-p(\gamma+1/2)} dx,$$

that follows from Hardy's inequality (1.7), since $a - p(\gamma + 3/2) < -1$. Also, in the case when $\gamma = \alpha + 2k$, $k = 1, 2, \dots$, the Hankel transplantation inequality

$$\|(\mathcal{H}_\alpha \circ \mathcal{H}_{\alpha+2k}) f\|_{p,a} \le C\|f\|_{p,a}$$

easily follows for $1 \le p < \infty$ and $-p(\alpha + 1/2) - 1 < a$, by using the formula

$$\mathcal{H}_\alpha f(x) = \int_x^\infty \left(\sum_{j=0}^{k-1} c_j \frac{1}{y} \left(\frac{x}{y}\right)^{2j+\alpha+1/2} \right) \mathcal{H}_{\alpha+2k} f(y) dy + (-1)^k \mathcal{H}_{\alpha+2k} f(x).$$

Indeed, what we need is to check the boundedness of the integral operators

$$T_j g(x) = \int_x^\infty \frac{1}{y} \left(\frac{x}{y}\right)^{2j+\alpha+1/2} g(y) dy,$$

$j = 0, 1, \dots, k-1$, on $L^{p,a}(dx)$, $1 \le p < \infty$, $a > -p(\alpha + 1/2) - 1$. This leads to the inequalities

$$\int_0^\infty \left| \int_x^\infty h(y) dy \right|^p x^{a+p(2j+\alpha+1/2)} dx \le C_{p,a,j} \int_0^\infty |h(x)|^p x^{a+p(2j+\alpha+3/2)} dx,$$

$j = 0, 1, \dots, k-1$, that follow from Hardy's inequality (1.8), since $a + p(\alpha + 1/2) > -1$.

We will frequently use the bounds

$$J_\alpha(t) = O(t^\alpha), \quad t \to 0^+, \tag{1.2}$$

and

$$J_\alpha(t) = O(t^{-1/2}), \quad t \to \infty. \tag{1.3}$$

A more precise description of behaviour of the Bessel function $J_\alpha(t)$ at infinity is given by the asymptotic

$$\sqrt{t}J_\alpha(t) = \sqrt{2/\pi}\left(\cos(t+a_\alpha) + b_\alpha\frac{\sin(t+c_\alpha)}{t} + O(t^{-2})\right), \quad t\to\infty. \tag{1.4}$$

The functions defined by those series in Theorem 1.1 and Theorem 1.2 are understood as pointwise sums of the series (they are everywhere convergent, cf. Lemma 2.2). A bit of comment is, perhaps, necessary on the question why $(\mathcal{H}_\alpha\circ\mathcal{H}_\gamma)f$ is well-defined for $f\in C_c^\infty(0,\infty)$. If $\alpha\geq -1/2$, then a natural assumption to make the integral defining $\mathcal{H}_\alpha g(x)$ convergent is to assume g to be Lebesgue integrable (the kernels $\varphi_x^\alpha(y)$, $x>0$, are (uniformly) bounded on $0<y<\infty$). Assume $\alpha,\gamma>-1$ and $f\in C_c^\infty(0,\infty)$. Then, $\mathcal{H}_\gamma f(y)$ is a continuous function of $0<y<\infty$ and, by using (1.2),

$$\mathcal{H}_\gamma f(y) = O(y^{\gamma+1/2}), \quad y\to 0^+. \tag{1.5}$$

Moreover, by using (1.4),

$$\mathcal{H}_\gamma f(y) = O(y^{-2}), \quad y\to\infty, \tag{1.6}$$

(using higher order asymptotics, better than (1.4), allows to get $\mathcal{H}_\gamma f(y) = O(y^{-k})$ with arbitrarily large k). Note that (1.5) and (1.6) ensure $\mathcal{H}_\gamma f(y)$ to be integrable, and hence, for $\alpha\geq -1/2$, $\mathcal{H}_\alpha(\mathcal{H}_\gamma f)(x)$, $0<x<\infty$, makes sense. In the general case, $\alpha,\gamma>-1$, (1.5) and (1.6) show that the function $y\to(xy)^{1/2}J_\alpha(xy)\mathcal{H}_\gamma f(y)$ is integrable and again the integral defining $\mathcal{H}_\alpha(\mathcal{H}_\gamma f)(x)$, $0<x<\infty$, makes sense.

Finally, we recall the following two forms of Hardy's inequality:
if $a<-1$ and $1\leq p<\infty$, then

$$\int_0^\infty\left|\int_0^x f(t)dt\right|^p x^a dx \leq C\int_0^\infty |f(x)|^p x^{a+p}dx; \tag{1.7}$$

if $a>-1$ and $1\leq p<\infty$, then

$$\int_0^\infty\left|\int_x^\infty f(t)dt\right|^p x^a dx \leq C\int_0^\infty |f(x)|^p x^{a+p}dx. \tag{1.8}$$

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2. Preliminaries. This section contains five lemmas and their proofs; the lemmas will be used in the proofs of Theorems 1.1 and 1.2.

LEMMA 2.1. *Let $\gamma,\delta>-1$ and $f\in C_c^\infty(0,\pi)$ or $f\in C_c^\infty(0,\infty)$. Then*

$$\langle f,\phi_n^{(\gamma,\delta)}\rangle = -(n(n+\gamma+\delta+1))^{-1/2}\langle F,\phi_{n-1}^{(\gamma+1,\delta+1)}\rangle, \tag{2.1}$$

where $F=f'+f\cdot\omega$, $\omega(x)=-B_\gamma\cot(x/2)+B_\delta\tan(x/2)$ and $B_\gamma=\gamma/2+1/4$. Similarly,

$$\langle f,\psi_n^\gamma\rangle = -\frac{1}{2n^{1/2}}\langle F,\psi_{n-1}^{\gamma+1}\rangle, \tag{2.2}$$

where $F = f' + f \cdot \omega_\gamma$ *and* $\omega_\gamma(x) = x - (\gamma + 1/2)/x$.

PROOF. (2.1) is proved by applying the differential identity, [Sz, (4.9.1)],

$$\frac{d}{dx}\left[P_{n-1}^{(\gamma+1,\delta+1)}(\cos x)\left(\sin\frac{x}{2}\right)^{2\gamma+2}\left(\cos\frac{x}{2}\right)^{2\delta+2}\right] = nP_n^{(\gamma,\delta)}(\cos x)\left(\sin\frac{x}{2}\right)^{2\gamma+1}\left(\cos\frac{x}{2}\right)^{2\delta+1}.$$

Indeed, integration by parts gives (we denote $w_{a,b}(x) = (\sin(x/2))^a(\cos(x/2))^b$)

$$\begin{aligned}
\langle f, \phi_n^{(\gamma,\delta)}\rangle &= t_n^{(\gamma,\delta)}\int_0^\pi f(x)P_n^{(\gamma,\delta)}(\cos x)w_{\gamma+1/2,\delta+1/2}(x)dx \\
&= \frac{t_n^{(\gamma,\delta)}}{n}\int_0^\pi \frac{f(x)}{w_{\gamma+1/2,\delta+1/2}(x)}\cdot nP_n^{(\gamma,\delta)}(\cos x)w_{2\gamma+1,2\delta+1}(x)dx \\
&= -\frac{t_n^{(\gamma,\delta)}}{n}\int_0^\pi \left(\frac{f(x)}{w_{\gamma+1/2,\delta+1/2}(x)}\right)' P_{n-1}^{(\gamma+1,\delta+1)}(\cos x)w_{2\gamma+2,2\delta+2}(x)dx \\
&= -\frac{t_n^{(\gamma,\delta)}}{nt_{n-1}^{(\gamma+1,\delta+1)}}\int_0^\pi F(x)\phi_{n-1}^{(\gamma+1,\delta+1)}(x)dx\,,
\end{aligned}$$

which is (2.1). Similarly, by applying the differential identity

$$\frac{d}{dx}(L_{n-1}^{\gamma+1}(x^2)\,e^{-x^2}x^{2(\gamma+1)}) = 2nL_n^\gamma(x^2)e^{-x^2}x^{2\gamma+1}\,,$$

which is easily verified by using well-known differential properties of Laguerre polynomials, we get with $A(n,\gamma) = \left(2n!/\Gamma(n+\gamma+1)\right)^{1/2}$

$$\begin{aligned}
\langle f, \psi_n^\gamma\rangle &= A(n,\gamma)\int_0^\infty f(x)e^{-x^2/2}L_n^\gamma(x^2)x^{\gamma+1/2}dx \\
&= A(n,\gamma)\frac{1}{2n}\int_0^\infty f(x)e^{x^2/2}x^{-(\gamma+1/2)}\cdot 2nL_n^\gamma(x^2)e^{-x^2}x^{2\gamma+1}dx \\
&= -\frac{A(n,\gamma)}{2n}\int_0^\infty (f(x)e^{x^2/2}x^{-(\gamma+1/2)})'\cdot L_{n-1}^{\gamma+1}(x^2)e^{-x^2}x^{2(\gamma+1)}dx \\
&= -\frac{A(n,\gamma)}{A(n-1,\gamma+1)}\cdot\frac{1}{2n}\int_0^\infty F(x)\psi_{n-1}^{\gamma+1}(x)dx\,,
\end{aligned}$$

which gives (2.2).

LEMMA 2.2. *Let* $\alpha, \beta, \gamma, \delta > -1$ *and* $f \in C_c^\infty(0,\pi)$ (*or* $f \in C_c^\infty(0,\infty)$). *Then the series*

$$\sum_0^\infty \langle f, \phi_n^{(\gamma,\delta)}\rangle\phi_n^{(\alpha,\beta)}(y)\,, \quad \left(or \sum_0^\infty \langle f, \psi_n^\gamma\rangle\psi_n^\alpha(y)\right) \tag{2.3}$$

converges absolutely for every given y, $0 < y < \pi$ (*or* $0 < y < \infty$).

PROOF. We have (cf. [M1, (2.8)]) for arbitrarily fixed positive constant c

$$|\phi_n^{(\alpha,\beta)}(x)| \le C \begin{cases} (nx)^{\alpha+1/2}, & 0 < x \le cn^{-1}, \\ 1, & cn^{-1} \le x \le \pi - cn^{-1}, \\ (n(\pi - x))^{\beta+1/2}, & \pi - cn^{-1} < x < \pi, \end{cases} \tag{2.4}$$

with $C > 0$ independent of $n = 1, 2, \ldots$ (it follows that for $\alpha, \beta \ge -1/2$ the sequence $\{\phi_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ is uniformly bounded). Let k be an arbitrary positive integer. Using (2.1) k times, we get

$$\langle f, \phi_n^{(\gamma,\delta)} \rangle = O(n^{-k}) \langle F_k, \phi_{n-k}^{(\gamma+k,\delta+k)} \rangle,$$

where F_k is a C^∞ function compactly supported in $(0, \pi)$. Hence, by Schwarz's inequality, $\langle f, \phi_n^{(\gamma,\delta)} \rangle = O(n^{-k})$, $n \to \infty$. By taking $k = 2$, the absolute convergence of the first series in (2.3) is now clear, since, given y, $0 < y < \pi$, we have $|\phi_n^{(\alpha,\beta)}(y)| \le C$ for sufficiently large n. Similarly, for the system $\{\psi_n^\alpha\}_{n=0}^\infty$ we have

$$|\psi_n^\alpha(x)| \le C \begin{cases} x^{\alpha+1/2} n^{\alpha/2}, & 0 < x < n^{-1/2}/2, \\ n^{-1/12}, & n^{-1/2}/2 \le x \le 2n^{-1/2}, \\ \exp(-\tau x), & 2n^{-1/2} \le x < \infty, \end{cases} \tag{2.5}$$

for a $\tau > 0$ with C independent of $n = 1, 2, \ldots$ (it follows that for $\alpha \ge -1/2$ the sequence $\{\psi_n^\alpha\}_{n=0}^\infty$ is uniformly bounded; more precisely, $\|\psi_n^\alpha\|_\infty = O(n^{-1/12})$). This estimate is easily implied by [M2, (2.5)]; moreover, the first interval in (2.5) can be taken as $0 < x < cn^{-1/2}$ with arbitrarily fixed $c > 0$, cf. [Sz, (7.6.8)]. The argument for the convergence of the second series in (2.3) is now analogous to that just given (we use (2.2) as an equivalent of (2.1)).

LEMMA 2.3. *Let $\alpha, \beta, \gamma, \delta > -1$, $1 \le p < \infty$, $f \in C_c^\infty(0, \pi)$ (or $f \in C_c^\infty(0, \infty)$) and $-p(\alpha + 1/2) - 1 < a$, $-p(\beta + 1/2) - 1 < b$ (or $-p(\alpha + 1/2) - 1 < a$). Then the series*

$$\sum_0^\infty \langle f, \phi_n^{(\gamma,\delta)} \rangle \phi_n^{(\alpha,\beta)}, \quad \left(or \sum_0^\infty \langle f, \psi_n^\gamma \rangle \psi_n^\alpha \right)$$

converges in $L^{p,a,b}(dx)$ (or $L^p(dx)$) to the limit given by the pointwise convergent series in (2.3).

PROOF. It follows from (2.4) that $\|\phi_n^{(\alpha,\beta)}\|_{p,a,b} = O(n^A)$ with a constant A. On the other hand, $\langle f, \phi_n^{(\gamma,\delta)} \rangle = O(n^{-k})$, $n \to \infty$, for arbitrarily large k. Hence the sum $\sum_{n=0}^\infty |\langle f, \phi_n^{(\gamma,\delta)} \rangle| \cdot \|\phi_n^{(\alpha,\beta)}\|_{p,a,b}$ is finite and this gives the convergence in $L^{p,a,b}$. Choosing a subsequence, converging almost everywhere, of the sequence of the partial sums of the series $\sum_0^\infty \langle f, \phi_n^{(\gamma,\delta)} \rangle \phi_n^{(\alpha,\beta)}$, we may identify the sum of this series with the sum of the pointwise convergent series in (2.3) (in particular, the sum of the series in (2.3) is in $L^{p,a,b}(dx)$). The argument for the Laguerre series is analogous and uses (2.5).

LEMMA 2.4. *Let $\alpha, \beta, \gamma, \delta > -1$ and $f \in C_c^\infty(0, \infty)$. Let $f_\lambda(x) = f(\lambda x)$, $\lambda > 0$, and, in the Jacobi case, consider λ so large that the support of f_λ is contained in $(0, \pi)$.*

Given $N = 1, 2, \dots$ and $K > 0$, there is a constant $C = C_{N,K}$ such that for $0 < x < K$ and large λ

$$\Big|\sum_0^{N[\lambda]} \langle f_\lambda, \phi_n^{(\gamma,\delta)}\rangle \phi_n^{(\alpha,\beta)}\Big(\frac{x}{\lambda}\Big)\Big| \le Cx^{\alpha+1/2}, \qquad \Big|\sum_0^{N[\lambda^2]} \langle f_\lambda, \psi_n^{\gamma}\rangle \psi_n^{\alpha}\Big(\frac{x}{\lambda}\Big)\Big| \le Cx^{\alpha+1/2}.$$

PROOF. We consider only the case of Laguerre expansions (the argument for the Jacobi expansions is analogous). We have

$$\langle f_\lambda, \psi_n^\gamma\rangle = \frac{1}{\lambda}\int_0^\infty f(u)\psi_n^\gamma(u/\lambda)du.$$

If f is supported in (m, M), $0 < m < M < \infty$, then for $u \le M$, $x \le K$ and $n \le N[\lambda^2]$, we have $u/\lambda \le cn^{-1/2}$ and $x/\lambda \le cn^{-1/2}$ with $c = \max\{M, K\}N^{1/2}$. Hence, by (2.5) (cf. a remark following (2.5)),

$$|\langle f_\lambda, \psi_n^\gamma\rangle| \le C\lambda^{-\gamma-3/2}n^{\gamma/2}$$

and

$$\Big|\psi_n^\alpha\Big(\frac{x}{\lambda}\Big)\Big| \le Cx^{\alpha+1/2}\lambda^{-\alpha-1/2}n^{\alpha/2}.$$

Therefore,

$$\begin{aligned}\Big|\sum_0^{N[\lambda^2]} \langle f_\lambda, \psi_n^{\gamma}\rangle \psi_n^{\alpha}\Big(\frac{x}{\lambda}\Big)\Big| &\le Cx^{\alpha+1/2}\lambda^{-(\alpha+\gamma+2)}\sum_0^{N[\lambda^2]} n^{(\alpha+\gamma)/2}\\ &\le Cx^{\alpha+1/2}.\end{aligned}$$

LEMMA 2.5. *Let $\alpha, \beta, \gamma, \delta > -1$ and $f \in C_c^\infty(0,\infty)$. Let $f_\lambda(x) = f(\lambda x)$, $\lambda > 0$, and consider λ so large that the support of f_λ is contained in $(0, \pi)$. Given $N = 1, 2, \dots$ and $0 < r < s < \infty$, there is a constant $C = C_{N,r,s}$ such that for $r < x < s$ and large λ*

$$\Big|\sum_{N[\lambda]+1}^{\infty} \langle f_\lambda, \phi_n^{(\gamma,\delta)}\rangle \phi_n^{(\alpha,\beta)}\Big(\frac{x}{\lambda}\Big)\Big| \le C. \tag{2.6}$$

PROOF. Consider λ so large that $s/\lambda < \pi/2$. If $n \ge N[\lambda] + 1$ and $r < x < s$, then $cn^{-1} < x/\lambda < \pi/2$, where $c = rN$. Hence, by (2.4), $|\phi_n^{(\alpha,\beta)}(x/\lambda)| \le C$ and, by using (2.1) twice,

$$\langle f_\lambda, \phi_n^{(\gamma,\delta)}\rangle = O(n^{-2})\langle G_\lambda, \phi_{n-2}^{(\gamma+2,\delta+2)}\rangle,$$

where $G_\lambda(x) = \lambda^2 f''(\lambda x) + 2\lambda f'(\lambda x)\omega(x) + f(\lambda x)\omega'(x) + f(\lambda x)\omega(x)^2 = G_\lambda^1(x) + \dots + G_\lambda^4(x)$. We have, assuming again that f is supported in (m, M),

$$|\langle G_\lambda^1, \phi_{n-2}^{(\gamma+2,\delta+2)}\rangle| \le C\|G_\lambda^1\|_1\|\phi_{n-2}^{(\gamma+2,\delta+2)}\|_\infty \le C\lambda\int_m^M |f''(u)|du \le C\lambda.$$

Similarly, for $i = 2, 3, 4$, by using the bounds

$$\omega(x) \le C\left(\frac{1}{x}+\frac{1}{\pi - x}\right), \quad \omega'(x) \le C\left(\frac{1}{x^2}+\frac{1}{(\pi - x)^2}\right), \quad \omega(x)^2 \le C\left(\frac{1}{x^2}+\frac{1}{(\pi - x)^2}\right),$$

we obtain $|\langle G^i_\lambda, \phi^{(\gamma+2,\delta+2)}_{n-2}\rangle| \le C\lambda$ for $i = 2, 3, 4$. Therefore, the left side of (2.6) is bounded by

$$C\sum_{i=1}^{4}\sum_{n=N[\lambda]+1}^{\infty}\frac{1}{n^2}|\langle G^i_\lambda, \phi^{(\gamma+2,\delta+2)}_{n-2}\rangle| \le C\lambda\sum_{n=N[\lambda]+1}^{\infty}\frac{1}{n^2} \le CN^{-1}.$$

3. Proof of Theorem 1.1. Choose any function f in $C_c^\infty(0,\infty)$ and assume its support is contained in the interval (m, M), $0 < m < M < \infty$. Let $f_\lambda(x) = f(\lambda x)$, $\lambda > 0$. By the assumption, we have

$$\left\| \sum_0^\infty \langle f_\lambda, \psi_n^\gamma\rangle \psi_n^\alpha \right\|_{p,a} \le C\|f_\lambda\|_{p,a}.$$

Since $\lambda^{(a+1)/p}\|f_\lambda\|_{p,a} = \|f\|_{p,a}$, by multiplying the above inequality by $\lambda^{(a+1)/p}$, we obtain $\|F_\lambda\|_{p,a} \le C\|f\|_{p,a}$, where we denote

$$F_\lambda(x) = \sum_0^\infty \langle f_\lambda, \psi_n^\gamma\rangle \psi_n^\alpha\left(\frac{x}{\lambda}\right).$$

By using Parseval's identity, we also easily obtain $\|F_\lambda\|_2 = \|f\|_2$. Hence, we can choose a sequence $\lambda_1 < \lambda_2 < \cdots, \lambda_j \to \infty$, such that F_{λ_j} is weakly convergent in $L^{p,a}(dx)$ to an F in $L^{p,a}(dx)$, and is also weakly convergent in $L^2(dx)$ to an $\tilde{F}$ in $L^2(dx)$. In fact, $F(x) = \tilde{F}(x)$ a. e. since, by the weak convergence in both L^2 and $L^{p,a}$, $\langle F, \chi_{(r,s)}\rangle = \langle \tilde{F}, \chi_{(r,s)}\rangle$ for every interval (r, s), $0 < r < s < \infty$. Clearly, $\|F\|_{p,a} \le C\|f\|_{p,a}$. To finish the proof of Theorem 1.1, we shall show that

$$F(x) = (\mathcal{H}_\alpha \circ \mathcal{H}_\gamma)f(x) \tag{3.1}$$

for almost every x in $(0,\infty)$.

Given N, $N = 1, 2, \ldots$, separating the series $\sum_0^\infty \langle f_\lambda, \psi_n^\gamma\rangle\psi_n^\alpha(x/\lambda)$ at the point $N[\lambda^2]$, we write

$$G^N(x,\lambda) = \sum_{n=0}^{N[\lambda^2]}\langle f_\lambda, \psi_n^\gamma\rangle\psi_n^\alpha\left(\frac{x}{\lambda}\right), \quad H^N(x,\lambda) = \sum_{n=N[\lambda^2]+1}^{\infty}\langle f_\lambda, \psi_n^\gamma\rangle\psi_n^\alpha\left(\frac{x}{\lambda}\right).$$

We claim that, repeating a fairly general functional analysis argument from [I], in order to prove (3.1) it is sufficient to establish the following: For every fixed $N = 1, 2, \ldots$ and $x > 0$

$$\lim_{\lambda\to\infty}\sum_{n=0}^{N[\lambda^2]}\langle f_\lambda, \psi_n^\gamma\rangle\psi_n^\alpha\left(\frac{x}{\lambda}\right) = \int_0^{2\sqrt{N}}\sqrt{xu}\,J_\alpha(xu)\mathcal{H}_\gamma f(u)du \tag{3.2}$$

and

$$\int_0^\infty |H^N(u,\lambda)|^2 du = O(N^{-1}) \tag{3.3}$$

uniformly in $\lambda \to \infty$.

For the sake of completeness we now recall Igari's argument. Using (3.3) and the diagonal method of choice, we choose a subsequence of $\{\lambda_j\}$ (call it again $\{\lambda_j\}$) such that for every $N = 1, 2, \dots$, $H^N(\cdot, \lambda_j)$ converges weakly in $L^2(dx)$, say to an H^N, $H^N \in L^2(dx)$. Again by (3.3) we have $\|H^N\|_2 = O(N^{-1/2})$. Since $F_{\lambda_j} \to F$ and $H^N(\cdot,\lambda_j) \to H^N$ weakly in $L^2(dx)$, $G^N(\cdot,\lambda_j) = F_{\lambda_j} - H^N(\cdot,\lambda_j)$ converges weakly in L^2, say to a G^N, $G^N \in L^2(dx)$. Therefore

$$F = G^N + H^N, \quad N = 1, 2, \dots .$$

By using $\|H^N\|_2 = O(N^{-1/2})$, from the sequence $\{H^N\}$ we choose a subsequence $\{H^{N(k)}\}$ such that $H^{N(k)}(x) \to 0$ a. e. as $k \to \infty$. Therefore, $F(x) = \lim_{k\to\infty} G^{N(k)}(x)$ a. e. and for every $N(k)$, $k = 1, 2, \dots$, we have: $G^{N(k)}(\cdot,\lambda_j) \to G^{N(k)}$ weakly in $L^2(dx)$ as $j \to \infty$ and, by (3.2), $G^{N(k)}(x,\lambda_j)$ converges for every x as $j \to \infty$, to

$$B_k(x) = \int_0^{2\sqrt{N(k)}} \sqrt{xu}\, J_\alpha(xu)\mathcal{H}_\gamma f(u)du\,.$$

The dominated convergence theorem is now used (this is possible by Lemma 2.4) to show that $\langle G^{N(k)}, \chi_{(r,s)}\rangle = \langle B_k, \chi_{(r,s)}\rangle$ for every $0 < r < s < \infty$. This gives $G^{N(k)}(x) = B_k(x)$ a. e. It is now clear that $F(x) = \lim_{k\to\infty} B_k(x) = (\mathcal{H}_\alpha \circ \mathcal{H}_\gamma)(x)$ a. e.

To prove (3.2) we will use Hilb's asymptotic formula, [Sz, Theorem 8.22.4], written in the form (cf. comments in Section 5)

$$\psi_n^\alpha(t) = \sqrt{2t}\, J_\alpha(2n^{1/2}t) + \begin{cases} O(tn^{-3/4}), & cn^{-1/2} \le t \le \omega, \\ O(t^{\alpha+1/2}n^{\alpha/2-1}), & 0 < t < cn^{-1/2}. \end{cases} \tag{3.4}$$

Here $\alpha > -1$ and C and ω are arbitrarily fixed positive constants. In the case $\alpha = 0$, the last bound is to be replaced by $O(t^{\alpha+1/2}n^{\alpha/2-1}) + O(t^4(1 + |\log(t^{-2}n^{-1})|))$. We remark that in the case $\alpha, \gamma > -1/2$, the analysis is slightly easier, since Hilb's formula then takes the simpler form (5.5). In what follows, the case when at least one of the parameters α, γ is zero requires separate argument; we will not discuss it here.

Fix $N = 1, 2, \dots$ and $x > 0$. Since the first summand of $G^N(x,\lambda)$ (corresponding to $n = 0$) tends to zero as $\lambda \to 0$, we drop it below. If $0 < n \le N[\lambda^2]$ and $0 < y < M$, then $y/\lambda \le cn^{-1/2}$ with $c = MN^{-1/2}$. Hence, using the last line of (3.4) gives for $0 < n \le N[\lambda^2]$ and large λ

$$\begin{aligned} \langle f_\lambda, \psi_n^\gamma\rangle &= \frac{1}{\lambda}\int_0^M f(y)\left[\sqrt{\frac{2y}{\lambda}} J_\gamma\left(\frac{2n^{1/2}y}{\lambda}\right) + O\left(\left(\frac{y}{\lambda}\right)^{\gamma+1/2} n^{\gamma/2-1}\right)\right]dy \\ &= \frac{1}{n^{1/4}\lambda}\mathcal{H}_\gamma f\left(\left(\frac{4n}{\lambda^2}\right)^{1/2}\right) + O(\lambda^{-\gamma-3/2}n^{\gamma/2-1}) \end{aligned} \tag{3.5}$$

and

$$\psi_n^\alpha\left(\frac{x}{\lambda}\right) = \frac{1}{n^{1/4}}\left(\left(\frac{4n}{\lambda^2}\right)^{1/2} x\right)^{1/2} J_\alpha\left(\left(\frac{4n}{\lambda^2}\right)^{1/2} x\right) + O(\lambda^{-\alpha-1/2} n^{\alpha/2-1}) . \tag{3.6}$$

Summing the terms that come from the product of the main parts of (3.5) and (3.6), we obtain

$$\sum_{n=1}^{N[\lambda^2]} \left(\left(\frac{4n}{\lambda^2}\right)^{1/2} x\right)^{1/2} J_\alpha\left(\left(\frac{4n}{\lambda^2}\right)^{1/2} x\right) \cdot \mathcal{H}_\gamma f\left(\left(\frac{4n}{\lambda^2}\right)^{1/2}\right) \cdot \frac{1}{n^{1/2}\lambda}$$
$$= \frac{1}{2}\sum_{n=1}^{N[\lambda^2]} \left(\left(\frac{4n}{\lambda^2}\right)^{1/2} x\right)^{1/2} J_\alpha\left(\left(\frac{4n}{\lambda^2}\right)^{1/2} x\right) \cdot \mathcal{H}_\gamma f\left(\left(\frac{4n}{\lambda^2}\right)^{1/2}\right) \cdot \left(\frac{4n}{\lambda^2}\right)^{-1/2} \cdot \frac{4}{\lambda^2} ,$$

and this, when $\lambda \to \infty$, approaches

$$\frac{1}{2}\int_0^{4N} \sqrt{t^{1/2}x}\, J_\alpha(t^{1/2}x)\mathcal{H}_\gamma f(t^{1/2}) t^{-1/2} dt ,$$

which, after a change of variable, becomes the right side of (3.2).

It remains to check that the sum of products of any other combinations of summands in (3.5) and (3.6) is $o(1)$ as $\lambda \to \infty$. We first take remainders in both (3.5) and (3.6). Their combination gives a sum which is bounded by (here and for the next cases we choose $0 < \eta < 1$ such that $(\alpha+\gamma)/2 + 1 - \eta > 0$; in places where we can do it, we use the estimate $\sum_1^A n^\tau = O(A^{\tau+1})$, $\tau > -1$)

$$C\sum_{n=1}^{N[\lambda^2]} \lambda^{-(\alpha+\gamma)-2} n^{(\alpha+\gamma)/2-2} \le \frac{C}{\lambda^{2\eta}} \sum_{n=1}^{N[\lambda^2]} \left(\frac{n}{\lambda^2}\right)^{(\alpha+\gamma)/2+1-\eta} \cdot \frac{1}{n^{3-\eta}} \le C_N \lambda^{-2\eta} .$$

We now consider combinations of a main part and a remainder. Taking the main part in (3.6) and the remainder in (3.5) and using (1.2) give the bound

$$C\sum_{n=1}^{N[\lambda^2]} \left(\left(\frac{4n}{\lambda^2}\right)^{1/2} x\right)^{1/2} \left| J_\alpha\left(\left(\frac{4n}{\lambda^2}\right)^{1/2} x\right)\right| \cdot \lambda^{-\gamma-3/2} n^{\gamma/2-5/4}$$
$$\le C\sum_{n=1}^{N[\lambda^2]} \lambda^{-(\alpha+\gamma)-2} n^{(\alpha+\gamma)/2-1}$$
$$\le C\frac{1}{\lambda^{2\eta}} \sum_{n=1}^{N[\lambda^2]} \left(\frac{n}{\lambda^2}\right)^{(\alpha+\gamma)/2+1-\eta} \frac{1}{n^{2-\eta}}$$
$$\le C_N \lambda^{-2\eta} .$$

Taking the main part in (3.5) and the remainder in (3.6) and using (1.5) give the bound

$$C\sum_{n=1}^{N[\lambda^2]} \left|\mathcal{H}_\gamma f\left(\left(\frac{4n}{\lambda^2}\right)^{1/2}\right)\right| \cdot \lambda^{-\alpha-3/2} n^{\alpha/2-1-1/4} \le C\sum_{n=1}^{N[\lambda^2]} \lambda^{-(\alpha+\gamma)-2} n^{(\alpha+\gamma)/2-1} ,$$

and this type of sum has been just treated. All possible sums occured to be $o(1)$ as $\lambda \to \infty$.

To prove (3.3), we use (2.2) to get

$$\langle f_\lambda, \psi_n^\gamma \rangle = -\frac{1}{2n^{1/2}} \langle F^\lambda, \psi_{n-1}^{\gamma+1} \rangle \,,$$

where $F^\lambda(x) = \lambda f'(\lambda x) + f(\lambda x)(x - (\gamma + 1/2)/x)$. Next, Lemma 2.3 for $p = 2$ and Parseval's identity give

$$\begin{aligned}
\int_0^\infty |H^N(u,\lambda)|^2 du &= \lambda \int_0^\infty \Big| \sum_{N[\lambda^2]+1}^\infty \langle f_\lambda, \psi_n^\gamma \rangle \psi_n^\alpha(u) \Big|^2 du \\
&= \lambda \sum_{N[\lambda^2]+1}^\infty |\langle f_\lambda, \psi_n^\gamma \rangle|^2 \\
&= \lambda \sum_{N[\lambda^2]+1}^\infty \frac{1}{4n} |\langle F^\lambda, \psi_{n-1}^{\gamma+1} \rangle|^2 \\
&\le C \frac{1}{\lambda N} \sum_0^\infty |\langle F^\lambda, \psi_{n-1}^{\gamma+1} \rangle|^2 \\
&= \frac{C}{N} \int_0^\infty \Big| \frac{\lambda u f'(\lambda u) + f(\lambda u) u^2 - (\gamma + 1/2) f(\lambda u)}{\lambda u} \Big|^2 \lambda du \\
&= \frac{C}{N} \int_0^\infty \Big| \frac{y f'(y) + f(y) y^2/\lambda^2 - (\gamma + 1/2) f(y)}{y} \Big|^2 dy \,.
\end{aligned}$$

This finishes the proof of (3.3). The proof of Theorem 1.1 is now complete.

4. Proof of Theorem 1.2. Let f, f_λ be as in the proof of Theorem 1.1. From now on, we assume λ to be sufficiently large, so large that the support of f_λ is contained in $(0,\pi)$. By the assumption

$$\Big\| \sum_0^\infty \langle f_\lambda, \phi_n^{(\gamma,\delta)} \rangle \phi_n^{(\alpha,\beta)} \Big\|_{p,a,b} \le C \| f_\lambda \|_{p,a,b} \,.$$

Multiplying the above by $2^{a/p} \lambda^{(a+1)/p}$ and changing variable give

$$\begin{aligned}
&\bigg(\int_0^{\lambda\pi} \Big| \sum_0^\infty \langle f_\lambda, \phi_n^{(\gamma,\delta)} \rangle \phi_n^{(\alpha,\beta)} \Big(\frac{y}{\lambda} \Big) \Big|^p v_{a,b,\lambda}(y) y^a dy \bigg)^{1/p} \\
&\qquad \le C \bigg(\int_0^M |f(y)|^p v_{a,b,\lambda}(y) y^a dy \bigg)^{1/p} ,
\end{aligned} \tag{4.1}$$

where we denote

$$v_{a,b,\lambda}(y) = \bigg(\Big(\sin \frac{y}{2\lambda} \Big) \Big/ (y/2\lambda) \bigg)^a \Big(\cos \frac{y}{2\lambda} \Big)^b .$$

If $p = 2$ and $a = b = 0$, then we have

$$\left\| \sum_0^\infty \langle f_\lambda, \phi_n^{(\gamma,\delta)} \rangle \phi_n^{(\alpha,\beta)} \right\|_2 = \| f_\lambda \|_2 ,$$

$\| \cdot \|_2$ denoting the L^2-norm in $L^2((0,\pi), dx)$, which gives

$$\left(\int_0^{\lambda\pi} \left| \sum_0^\infty \langle f_\lambda, \phi_n^{(\gamma,\delta)} \rangle \phi_n^{(\alpha,\beta)} \left(\frac{y}{\lambda} \right) \right|^2 dy \right)^{1/2} = \left(\int_0^M |f(y)|^2 dy \right)^{1/2} .$$

Let

$$F_\lambda(y) = \sum_0^\infty \langle f_\lambda, \phi_n^{(\gamma,\delta)} \rangle \phi_n^{(\alpha,\beta)} \left(\frac{y}{\lambda} \right)$$

and $\tilde{F}_\lambda(y) = F_\lambda(y) v_{a,b,\lambda}(y)^{1/p}$ when $0 < y < \lambda\pi$, and $\tilde{F}_\lambda(y) = F_\lambda(y) = 0$ otherwise. Choose a sequence $\lambda_1 < \lambda_2 < \dots, \lambda_j \to \infty$, such that F_{λ_j} is weakly convergent in $L^2(dy)$. Let F be its weak limit. To complete the proof of Theorem 1.2, we shall show that

$$\|F\|_{p,a} \le C \|f\|_{p,a} \tag{4.2}$$

and

$$F(y) = (\mathcal{H}_\alpha \circ \mathcal{H}_\gamma)(y) \tag{4.3}$$

for almost every y, $0 < y < \infty$. Choose any $\varepsilon > 0$. For sufficiently large λ_j and $0 < y < M$, we have $v_{a,b,\lambda}(y) \le (1+\varepsilon)^p$. Hence, from (4.1),

$$\|\tilde{F}_{\lambda_j}\|_{p,a} \le C(1+\varepsilon)\|f\|_{p,a} .$$

Therefore, from the sequence $\{\lambda_j\}$ we can choose a subsequence (call it again $\{\lambda_j\}$) such that $\tilde{F}_{\lambda_j}$ is weakly convergent in $L^{p,a}$. Let $\tilde{F}$ be its weak limit. Clearly, $\|\tilde{F}\|_{p,a} \le C(1+\varepsilon)\|f\|_{p,a}$, and arbitrariness of ε gives $\|\tilde{F}\|_{p,a} \le C\|f\|_{p,a}$. To finish the proof of (4.2), it is now sufficient to note that $\tilde{F}(y) = F(y)$ a. e., and this is implied by $\langle \tilde{F}, \chi_{(r,s)} \rangle = \langle F, \chi_{(r,s)} \rangle$ for every $0 < r < s < \infty$. The last identity is proved by using the fact that

$$\lim_{j\to\infty} \langle \tilde{F}_{\lambda_j}, \chi_{(r,s)} \rangle = \lim_{j\to\infty} \langle F_{\lambda_j}, \chi_{(r,s)} \rangle ,$$

which is a consequence of Lemma 2.4, Lemma 2.5 and the dominated convergence theorem. The rest of the argument will concern the proof of (4.3).

Given N, $N = 1, 2, \dots$, we now separate the series $\sum_0^\infty \langle f_\lambda, \phi_n^{(\gamma,\delta)} \rangle \phi_n^{(\alpha,\beta)}(x/\lambda)$ at the point $N[\lambda]$, and write

$$G^N(x,\lambda) = \sum_{n=0}^{N[\lambda]} \langle f_\lambda, \phi_n^{(\gamma,\delta)} \rangle \phi_n^{(\alpha,\beta)} \left(\frac{x}{\lambda} \right)$$

and

$$H^N(x,\lambda) = \sum_{n=N[\lambda]+1}^{\infty} \langle f_\lambda, \phi_n^{(\gamma,\delta)} \rangle \phi_n^{(\alpha,\beta)} \left(\frac{x}{\lambda} \right) .$$

As in the proof of Theorem 1.1 it is now sufficient to show that

$$\lim_{\lambda\to\infty} G^N(x,\lambda) = \int_0^N \sqrt{xu}\, J_\alpha(xu)\mathcal{H}_\gamma f(u)du \tag{4.4}$$

for every fixed $N = 1, 2, \dots$ and $x > 0$, and

$$\int_0^\pi |H^N(u,\lambda)|^2 du = O(N^{-2}) \tag{4.5}$$

uniformly in $\lambda \to \infty$.

To prove (4.4) we will use Hilb's asymptotic formula, [Sz, Theorem 8.21.12], written in the form (cf. comments in Section 5)

$$\phi_n^{(\alpha,\beta)}(t) = (nt)^{1/2} J_\alpha(nt) + \begin{cases} O(t)\,, & cn^{-1} \le t \le \pi - \varepsilon\,, \\ O(t^{\alpha+1/2} n^{\alpha-1/2})\,, & 0 < t < cn^{-1}\,, \end{cases} \tag{4.6}$$

where c and $\varepsilon < \pi$ are arbitrarily fixed positive constants.

Fix $N = 1, 2, \dots$ and $0 < x < K$, where K is given and large. Neglecting $n = 0$ (for the same reasons as explained in the proof of Theorem 1.1) and using the second line in (4.6) give, for $0 < n \le N[\lambda]$,

$$\begin{aligned} \langle f_\lambda, \phi_n^{(\gamma,\delta)}\rangle &= \frac{1}{\lambda}\int_0^M f(y)\left[\left(n\frac{y}{\lambda}\right)^{1/2} J_\gamma\left(n\frac{y}{\lambda}\right) + O\left(\left(\frac{y}{\lambda}\right)^{\gamma+1/2} n^{\gamma-1/2}\right)\right]dy \\ &= \frac{1}{\lambda}\mathcal{H}_\gamma f\left(\frac{n}{\lambda}\right) + O(\lambda^{-\gamma-3/2} n^{\gamma-1/2}) \end{aligned} \tag{4.7}$$

and

$$\phi_n^{(\alpha,\beta)}\left(\frac{x}{\lambda}\right) = \left(n\frac{x}{\lambda}\right)^{1/2} J_\alpha\left(n\frac{x}{\lambda}\right) + O(\lambda^{-\alpha-1/2} n^{\alpha-1/2})\,. \tag{4.8}$$

Summing the terms that come from the product of the main parts of (4.7) and (4.8), we obtain

$$\sum_{n=1}^{N[\lambda]} \left(x\frac{n}{\lambda}\right)^{1/2} J_\alpha\left(x\frac{n}{\lambda}\right)\mathcal{H}_\gamma f\left(\frac{n}{\lambda}\right)\cdot\frac{1}{\lambda}\,,$$

and this, when $\lambda \to \infty$, approaches the right side of (4.4). We now claim that summing any other products of summands in (4.7) and (4.8) gives a quantity that approaches zero when $\lambda \to \infty$. We start with considering the remainders in (4.7) and (4.8). We take $0 < \eta < 2$ such that $\alpha + \gamma + 2 - \eta > 0$. Then, for the relevant sum, we have the bound

$$C\sum_{n=1}^{N[\lambda]} \lambda^{-(\alpha+\gamma)-2} n^{\alpha+\gamma-1} \le C\frac{1}{\lambda^\eta}\sum_{n=1}^{N[\lambda]}\left(\frac{n}{\lambda}\right)^{\alpha+\gamma+2-\eta}\frac{1}{n^{3-\eta}} \le C_N\lambda^{-\eta}\,.$$

We now consider sums that occur by taking a main part and a remainder. Taking the main part in (4.7) and the remainder in (4.8) gives a sum which is bounded by (again we use an η such that $0 < \eta < 2$ and $\alpha + \gamma + 2 - \eta > 0$)

$$C\sum_{n=1}^{N[\lambda]}\left|\mathcal{H}_\gamma f\left(\frac{n}{\lambda}\right)\right|\lambda^{-\alpha-3/2} n^{\alpha-1/2} = \frac{C}{\lambda^\eta}\sum_{n=1}^{N[\lambda]}\left(\frac{n}{\lambda}\right)^{\alpha+\gamma+2-\eta}\frac{1}{n^{2-\eta}} \le C_N\lambda^{-\eta}\,.$$

Taking the main part in (4.8) and the remainder in (4.7) leads to a similar estimate. This finishes the proof of (3.2).

To prove (4.5) we use (2.1) to get

$$\langle f_\lambda, \phi_n^{(\gamma,\delta)}\rangle = -(n(n+\gamma+\delta+1))^{-1/2}\langle F^\lambda, \phi_{n-1}^{(\gamma+1,\delta+1)}\rangle\,,$$

where $F^\lambda(x) = \lambda f'(\lambda x) + f(\lambda x)(-B_\gamma \cot(x/2) + B_\delta \tan(x/2))$, $B_\gamma = \gamma/2 + 1/4$. Consequently, by Parseval's identity,

$$\begin{aligned}\int_0^\pi |H^N(u,\lambda)|^2 du &= \lambda \int_0^\pi \left| \sum_{N[\lambda]+1}^{\infty} \langle f_\lambda, \phi_n^{(\gamma,\delta)}\rangle \phi_n^{(\gamma,\delta)}(u)\right|^2 du \\ &= \lambda \sum_{N[\lambda]+1}^{\infty} |\langle f_\lambda, \phi_n^{(\gamma,\delta)}\rangle|^2 \\ &\le \frac{\lambda}{n^2} \sum_{N[\lambda]+1}^{\infty} |\langle F^\lambda, \phi_{n-1}^{(\gamma+1,\delta+1)}\rangle|^2 \\ &\le \frac{C}{\lambda N^2} \sum_{0}^{\infty} |\langle F^\lambda, \phi_{n-1}^{(\gamma+1,\delta+1)}\rangle|^2 \\ &= \frac{C}{N^2}\int_0^\pi \left| f'(\lambda u) - \frac{1}{\lambda} B_\gamma f(\lambda u) \cot\frac{u}{2} + \frac{1}{\lambda} B_\delta f(\lambda u)\tan\frac{u}{2}\right|^2 \lambda du \\ &= \frac{C}{N^2}\int_0^\infty \left| f'(y) - \frac{1}{\lambda} B_\gamma f(y) \cot\frac{y}{2\lambda} + \frac{1}{\lambda} B_\delta f(y)\tan\frac{y}{2\lambda}\right|^2 dy\,.\end{aligned}$$

This finishes the proof of (4.5) and hence the proof of Theorem 1.2.

5. Asymptotic formulas of Hilb's type. Hilb's asymptotic formula for Laguerre polynomials, as stated in Szegö's monograph [Sz, 8.22.4], is

$$\begin{aligned}e^{-t^2/2}\, t^\alpha\, L_n^\alpha(t^2) =& N^{-\alpha/2}\frac{\Gamma(n+\alpha+1)}{n!}\, J_\alpha(2N^{1/2}t) \\ &+\begin{cases} O(t^{5/2} n^{\alpha/2-3/4})\,, & cn^{-1/2} \le t \le \omega\,,\\ O(t^{\alpha+4} n^\alpha)\,, & 0 < t < cn^{-1/2}\,.\end{cases}\end{aligned} \tag{5.1}$$

Here $\alpha > -1$, $N = n + (\alpha+1)/2$, c and ω are arbitrarily fixed positive constants. In the case $\alpha = 0$, the last bound is to be replaced by $O(t^4(1 + |\log(t^2 n)|))$.

Rewriting (5.1) gives

$$\begin{aligned}\psi_n^\alpha(t) =& N^{-\alpha/2}\left(\frac{\Gamma(n+\alpha+1)}{n!}\right)^{1/2} \sqrt{2t}\, J_\alpha(2N^{1/2}t) \\ &+\begin{cases} O(t^3 n^{-3/4})\,, & cn^{-1/2} \le t \le \omega\,,\\ O(t^{\alpha+9/2} n^{\alpha/2})\,, & 0 < t < cn^{-1/2}\,.\end{cases}\end{aligned} \tag{5.2}$$

In the case $\alpha = 0$, the last bound is to be replaced by $O(t^{9/2}(1 + |\log(t^2 n)|))$.

We have

$$\left| N^{-\alpha/2}\left(\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\right)^{1/2} - 1\right| \le Cn^{-2}\,.$$

This easily follows by using

$$\frac{\Gamma(n+\alpha+1)}{n^\alpha \Gamma(n+1)} = 1 + \frac{\alpha(\alpha+1)}{2n} + O(n^{-2})\,, \tag{5.3}$$

cf. [Le, p. 15] (note that the above also gives $\Gamma(n+\alpha+1)/\Gamma(n+1) \sim n^\alpha$), and

$$\left(1+\frac{\alpha+1}{2n}\right)^\alpha = 1 + \frac{\alpha(\alpha+1)}{2n} + O(n^{-2})\,.$$

Hence, for $n^{1/2}t \ge c$, by using (1.3) we obtain

$$\begin{aligned}\left(N^{-\alpha/2}\left(\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\right)^{1/2} - 1\right)\sqrt{2t}\,J_\alpha(2N^{1/2}t) &= O(n^{-2}t^{1/2}(n^{1/2}t)^{-1/2}) \\ &= O(n^{-9/4})\,.\end{aligned}$$

Similarly, for $0 < n^{1/2}t \le c$, by using (1.2) we get

$$\begin{aligned}\left(N^{-\alpha/2}\left(\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\right)^{1/2} - 1\right)\sqrt{2t}\,J_\alpha(2N^{1/2}t) &= O(n^{-2}t^{1/2}(n^{1/2}t)^{\alpha}) \\ &= O(t^{\alpha+1/2}n^{\alpha/2-2})\,.\end{aligned}$$

Therefore, (5.2) becomes

$$\psi_n^\alpha(t) = \sqrt{2t}\;J_\alpha(2N^{1/2}t) + \begin{cases} O(t^3n^{-3/4})\,, & cn^{-1/2} \le t \le \omega\,, \\ O(t^{\alpha+1/2}n^{\alpha/2-2})\,, & 0 < t < cn^{-1/2}\,. \end{cases} \tag{5.4}$$

(Note that $O(n^{-9/4})$ is absorbed by $O(t^3n^{-3/4})$ while $O(t^{\alpha+9/2}n^{\alpha/2})$ is included in $O(t^{\alpha+1/2}n^{\alpha/2-2})$). In the case $\alpha = 0$, the last bound is to be replaced by $O(t^{9/2}(1 + |\log(t^2n)|))$ (the process of changing $N^{-\alpha/2}(\Gamma(n+\alpha+1)/n!)^{1/2}$ onto 1 does not concern the case $\alpha = 0$).

In the second step, for $n^{1/2}t \ge c$, by using the mean value theorem and $(d/ds)J_\alpha(s) = O(s^{-1/2})$ for s large, we obtain

$$\begin{aligned}\sqrt{2t}(J_\alpha(2N^{1/2}t) - J_\alpha(2n^{1/2}t)) &= O(t^{1/2}\cdot tn^{-1/2}\cdot(n^{1/2}t)^{-1/2}) \\ &= O(tn^{-3/4})\,,\end{aligned}$$

and, for $0 < n^{1/2}t \le c$, by using $(d/ds)J_\alpha(s) = O(s^{\alpha-1})$ for $s \to 0^+$, we obtain

$$\begin{aligned}\sqrt{2t}(J_\alpha(2N^{1/2}t) - J_\alpha(2n^{1/2}t)) &= O(t^{1/2}\cdot tn^{-1/2}\cdot(n^{1/2}t)^{\alpha-1}) \\ &= O(t^{\alpha+1/2}n^{\alpha/2-1})\,.\end{aligned}$$

In conclusion, we now change (5.4) into the form (note that $O(t^{\alpha+1/2}n^{\alpha/2-2})$ is better than $O(t^{\alpha+1/2}n^{\alpha/2-1})$, $O(t^3n^{-3/4})$ is absorbed by $O(tn^{-3/4})$ while $O(t^{9/2}(1+|\log(t^2n)|))$

is included in $O(t^{1/2}n^{-1})$)

$$\psi_n^\alpha(t) = \sqrt{2t}\, J_\alpha(2n^{1/2}t) + \begin{cases} O(tn^{-3/4}), & cn^{-1/2} \le t \le \omega, \\ O(t^{\alpha+1/2}n^{\alpha/2-1}), & 0 < t < cn^{-1/2}. \end{cases}$$

This is (3.4), the asymptotics we used in Section 3. If $\alpha \ge -1/2$, then the above asymptotics simplifies to

$$\psi_n^\alpha(t) = \sqrt{2t} J_\alpha(2n^{1/2}t) + O(n^{-3/4}) \tag{5.5}$$

with the O-bound uniform in $0 \le t \le \omega$, for any given $\omega > 0$.

Hilb's asymptotic formula for Jacobi polynomials, as stated in Szegö's monograph [Sz, 8.21.17], is

$$\left(\sin\frac{t}{2}\right)^\alpha \left(\cos\frac{t}{2}\right)^\beta P_n^{(\alpha,\beta)}(\cos t) = N^{-\alpha}\frac{\Gamma(n+\alpha+1)}{n!}\left(\frac{t}{\sin t}\right)^{1/2} J_\alpha(Nt)$$

$$+ \begin{cases} O(t^{1/2}n^{-3/2}), & cn^{-1} \le t \le \pi - \varepsilon, \\ O(t^{\alpha+2}n^\alpha), & 0 < t < cn^{-1}. \end{cases}$$

Here $\alpha > -1$, β is any real number, $N = n + (\alpha+\beta+1)/2$, $c > 0$, and $0 < \varepsilon < \pi$ are arbitrarily fixed constants. Rewriting the above gives

$$\begin{aligned} \phi_n^{(\alpha,\beta)}(t) = &\frac{N^{-(\alpha+1/2)}}{\sqrt{2}}\left(\frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+1)\Gamma(n+\beta+1)}\right)^{1/2} \\ &\times (Nt)^{1/2}J_\alpha(Nt) + \begin{cases} O(tn^{-1}), & cn^{-1} \le t \le \pi - \varepsilon, \\ O(t^{\alpha+5/2}n^{\alpha+1/2}), & 0 < t < cn^{-1}. \end{cases} \end{aligned} \tag{5.6}$$

We have

$$\left|\frac{1}{\sqrt{2}}N^{-(\alpha+1/2)}\left(\frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+1)\Gamma(n+\beta+1)}\right)^{1/2} - 1\right| \le Cn^{-2}.$$

As in the case of a similar estimate in the beginning of this section, the above inequality is easily checked by using the asymptotics: (5.3),

$$\frac{\Gamma(n+\alpha+\beta+1)}{n^\alpha\Gamma(n+\beta+1)} = 1 + \frac{\alpha(\alpha+2\beta+1)}{2n} + O(n^{-2}),$$

$$\left(1+\frac{\alpha+\beta+1}{2n}\right)^{2\alpha+1} = 1 + \frac{(2\alpha+1)(\alpha+\beta+1)}{2n} + O(n^{-2}),$$

and the identity $(2n+\alpha+\beta+1)/(2n) = 1 + (\alpha+\beta+1)/(2n)$.

Hence, for $nt \ge c$, by using (1.3) we obtain

$$\begin{aligned} &\left(\frac{1}{\sqrt{2}}N^{-(\alpha+1/2)}\left(\frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+1)\Gamma(n+\beta+1)}\right)^{1/2} - 1\right) \\ &\quad\times (Nt)^{1/2}J_\alpha(Nt) = O(n^{-2}). \end{aligned}$$

Similarly, for $0 < nt \le c$, using (1.2) gives

$$\left(\frac{1}{\sqrt{2}}N^{-(\alpha+1/2)}\left(\frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+1)\Gamma(n+\beta+1)}\right)^{1/2}-1\right) \times (Nt)^{1/2}J_\alpha(Nt) = O(t^{\alpha+1/2}n^{\alpha-3/2}).$$

Therefore, in the first step, (5.6) can be replaced by

$$\phi_n^{(\alpha,\beta)}(t) = (Nt)^{1/2}J_\alpha(Nt) + \begin{cases} O(tn^{-1}), & cn^{-1} \le t \le \pi-\varepsilon, \\ O(t^{\alpha+1/2}n^{\alpha-3/2}), & 0 < t < cn^{-1}. \end{cases} \tag{5.7}$$

In the second step, for $nt \ge c$, by using the mean value theorem and $(d/ds)J_\alpha(s) = O(s^{-1/2})$ for s large, we obtain

$$(Nt)^{1/2}(J_\alpha(Nt) - J_\alpha(nt)) = O(t),$$

and, for $0 < nt \le c$, by using $(d/ds)J_\alpha(s) = O(s^{\alpha-1})$ for $s \to 0^+$, we obtain

$$(Nt)^{1/2}(J_\alpha(Nt) - J_\alpha(nt)) = O(t^{\alpha+1/2}n^{\alpha-1/2}).$$

This allows to change (5.7) into the form

$$\phi_n^{(\alpha,\beta)}(t) = (Nt)^{1/2}J_\alpha(nt) + \begin{cases} O(t), & cn^{-1} \le t \le \pi-\varepsilon, \\ O(t^{\alpha+1/2}n^{\alpha-1/2}), & 0 < t < cn^{-1}. \end{cases}$$

Finally, the very last N may be replaced by n by writing $(Nt)^{1/2} - (nt)^{1/2} = n^{-1/2}(N^{1/2} + n^{1/2})^{-1}(N-n)(nt)^{1/2}$ and then using (1.2) and (1.3). This gives (4.6), the asymptotics we used in Section 4.

6. Estimates based on explicit kernel formula. In the case $\alpha, \gamma \ge -1/2$, Schindler [Sch] found an explicit integral kernel representation of the transplantation operator $T = \mathcal{H}_\alpha \circ \mathcal{H}_\gamma$: for any $f \in \mathcal{H}_\gamma(C_c^\infty)$

$$Tf(x) = \int_0^\infty K(x,y)f(y)dy + C_{\alpha,\gamma}f(x),$$

where $C_{\alpha,\gamma} = \cos((\alpha-\gamma)\pi/2)$ and, for $0 < y < x$,

$$K(x,y) = \frac{2\Gamma((\alpha+\gamma+2)/2)}{\Gamma(\gamma+1)\Gamma((\alpha-\gamma)/2)}x^{-(\gamma+3/2)}y^{\gamma+1/2} \cdot {}_2F_1\left(\frac{\alpha+\gamma+2}{2}, \frac{\gamma-\alpha+2}{2}; \gamma+1; \left(\frac{y}{x}\right)^2\right)$$

while, for $x < y < \infty$,

$$K(x,y) = \frac{2\Gamma((\alpha+\gamma+2)/2)}{\Gamma(\alpha+1)\Gamma((\gamma-\alpha)/2)}x^{\alpha+1/2}y^{-(\alpha+3/2)} \cdot {}_2F_1\left(\frac{\alpha+\gamma+2}{2}, \frac{\alpha-\gamma+2}{2}; \alpha+1; \left(\frac{x}{y}\right)^2\right).$$

The above integral is understood in the principal value sense. Moreover, it was shown that the singularity along the diagonal is of the following form: For the constant $D_{\alpha,\gamma} = 4/(\Gamma((\alpha-\gamma)/2)\Gamma((\gamma-\alpha)/2)(\gamma-\alpha))$

$$K(x,y) = D_{\alpha,\gamma}\frac{x}{x^2-y^2} + O\left(\frac{1}{x}\log\frac{x^2}{x^2-y^2}\right), \quad x/2 \le y < x\,,$$

and

$$K(x,y) = D_{\gamma,\alpha}\frac{y}{y^2-x^2} + O\left(\frac{1}{y}\log\frac{y^2}{y^2-x^2}\right), \quad x < y \le 2x\,.$$

We use these results to furnish another proof of Corollary 1.4 in the restricted parameter range $\alpha, \gamma \ge -1/2$.

To show the bound

$$\int_0^\infty \left|\int_0^\infty K(x,y)f(y)dy\right|^p x^a dx \le C\int_0^\infty |f(x)|^p x^a dx \tag{6.1}$$

for $1 < p < \infty$ and $-p(\alpha+1/2)-1 < a < p(\gamma+3/2)-1$, split the inner integration on the left side of (6.1) onto the intervals $(0, x/2)$, $(x/2, 3x/2)$, $(3x/2, \infty)$, and consider each integral separately. On the first interval, using the assumption $a - p(\gamma+3/2) < -1$ and Hardy's inequality (1.7) gives

$$\begin{aligned}\int_0^\infty \left|\int_0^{x/2} K(x,y)f(y)dy\right|^p x^a dx &\le C\int_0^\infty \left(\int_0^{x/2} x^{-(\gamma+3/2)}y^{\gamma+1/2}|f(y)|dy\right)^p x^a dx\\ &\le C\int_0^\infty \left(\int_0^{x/2} |y^{\gamma+1/2}f(y)|dy\right)^p x^{a-p(\gamma+3/2)}dx\\ &\le C\int_0^\infty |f(x)|^p x^a dx.\end{aligned}$$

Similarly, on $(3x/2, \infty)$, using the assumption $a + p(\alpha+1/2) > -1$ and Hardy's inequality (1.8) shows

$$\begin{aligned}\int_0^\infty \left|\int_{3x/2}^\infty K(x,y)f(y)dy\right|^p x^a dx &\le C\int_0^\infty \left(\int_{3x/2}^\infty x^{\alpha+1/2}y^{-(\alpha+3/2)}|f(y)|dy\right)^p x^a dx\\ &\le C\int_0^\infty \left(\int_{3x/2}^\infty |y^{-(\alpha+3/2)}f(y)|dy\right)^p x^{a+p(\alpha+1/2)}dx\\ &\le C\int_0^\infty |f(x)|^p x^a dx.\end{aligned}$$

The integration over the interval $(x/2, 3x/2)$ requires additional lemmas. We will use the following local version of the Hardy-Littlewood maximal operator and the Hilbert transform. For $x > 0$, let

$$M_o f(x) = \sup_{|x-y|\le x/2}\frac{1}{y-x}\int_x^y |f(t)|dt$$

and

$$H_o f(x) = \int_{x/2}^{3x/2}\frac{f(t)}{x-t}dt\,.$$

The following is a version of Lemma (9.6) in [M1].

LEMMA 6.1. *Let $1 < p < \infty$. Assume that the non-negative weight $w(x)$ on $(0, \infty)$ satisfies*

$$\left(\int_u^v w(x)^p dx\right)^{1/p}\left(\int_u^v w(x)^{-p'} dx\right)^{1/p'} \le k(v-u) \tag{6.2}$$

for $0 < u < v < 8u$ with k independent of u and v (in particular, one can take $w(x) = x^a$, $a \in \boldsymbol{R}$). If T_o is either M_o or H_o, then

$$\left(\int_0^\infty |T_o f(x) w(x)|^p dx\right)^{1/p} \le C\left(\int_0^\infty |f(x)w(x)|^p dx\right)^{1/p}$$

with C depending only on k.

The proof of this lemma is a straightforward modification of Muckenhoupt's argument in [M1, p. 31]. We use the diadic decomposition of $(0, \infty)$ onto the union of the intervals $I_n = (2^{n-3}, 2^n)$, $n \in \boldsymbol{Z}$. Then we define a weight $w_n(x)$ on $\boldsymbol{R}$ to be equal $w(x)$ on I_n, periodic with period $2|I_n|$ and symmetric around $x = 2^n$, and check that (6.2) holds with w replaced by w_n and k by $2k$ for $-\infty < u < v < \infty$. Then we apply classical L^p-bounds for the Hardy-Littlewood maximal operator and the Hilbert transform.

The next lemma is a version of Theorem (9.9) in [M1].

LEMMA 6.2. *Assume that $k(x, y)$ is a nonnegative kernel such that for any $x > 0$, $k(x, y)$ is nondecreasing for $0 < y < x$ and nonincreasing for $x < y < \infty$, and*

$$\int_{x/2}^{3x/2} k(x, y)dy \le C$$

with C independent of $x > 0$. Then

$$\int_{x/2}^{3x/2} k(x, y)|f(y)|dy \le C M_o f(x)$$

with the same constant C.

Using the asymptotics of $K(x, y)$ along the diagonal, the identities $x/(x^2 - y^2) = [1/(x + y) + 1/(x - y)]/2$, $y/(x^2 - y^2) = [1/(x - y) - 1/(x + y)]/2$ combined with Lemma 6.2 imply

$$\begin{aligned}\left|\int_{x/2}^{3x/2} K(x, y) f(y)dy\right| &\le C\Bigg(\left|\int_{x/2}^{3x/2} \frac{f(y)}{x-y}dy\right| + \frac{1}{x}\int_{x/2}^{3x/2} |f(y)|dy \\ &\quad + \frac{1}{x}\int_{x/2}^{x} |f(y)| \log\frac{x^2}{x^2-y^2}dy + \frac{1}{x}\int_{x}^{3x/2} |f(y)| \log\frac{y^2}{y^2-x^2}dy\Bigg) \\ &\le C\big(M_o f(x) + H_o f(x)\big).\end{aligned}$$

Hence, by Lemma 6.1, the bound

$$\int_0^\infty \left|\int_{x/2}^{3x/2} K(x, y) f(y)dy\right|^p x^a dx \le C\int_0^\infty |f(x)|^p x^a dx$$

follows. This finishes the proof of (6.1).

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INSTYTUT MATEMATYKI
POLITECHNIKA WROCŁAWSKA
WYB. WYSPIAŃSKIEGO 27
50–370 WROCŁAW
POLAND

E-mail address: stempak@im.pwr.wroc.pl