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HARDY INEQUALITY FOR CENSORED STABLE PROCESSES

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Abstract. A Hardy inequality is established for censored stable processes on a large class of bounded domains including bounded Lipschitz domains in $\boldsymbol{R}^n$ with $n \geq 2$.

1. Introduction. In order to state the classical Hardy inequality (for Brownian motion) due to Ancona [2], we first recall the following definition.

DEFINITION 1.1. A domain D in $\boldsymbol{R}^n$, $n \geq 3$, is said to be *uniformly Δ-regular* if there is a constant $c > 0$ such that for each $x \in \partial D$ and all $r > 0$,

$$\mathrm{Cap}(B(x, r) \cap D^c) \geq c\, r^{n-2}\,, \tag{1.1}$$

where Cap is the Newtonian capacity in $\boldsymbol{R}^n$.

In [2], Ancona showed that if D is uniformly Δ-regular in $\boldsymbol{R}^n$, $n \geq 3$, there is a constant $C > 0$ depending only on n and the constant c in the definition of Δ-regularity (1.1) such that the following Hardy inequality holds

$$\int_D \frac{u^2(x)}{\delta_D(x)^2}\, dx \leq C \int_D |\nabla u|^2 dx \quad \text{for all } u \in W_0^{1,2}(D)\,, \tag{1.2}$$

where $\delta_D(x)$ is the Euclidean distance between x and D^c. Note that the Dirichlet integral in the right hand side of (1.2) represents the bilinear form associated with a Brownian motion killed upon leaving D. Hardy inequality plays an important role in probability and analysis, see, for example, Bañuelos [3] and Davies [10].

This paper is concerned with obtaining Hardy inequalities for censored stable processes in D (see Theorem 2.3, Corollary 2.4 and Theorem 3.1 below) and for killed symmetric stable processes in D (see Theorem 3.2 below).

Recall that a symmetric α-stable process X on $\boldsymbol{R}^n$ is a Lévy process whose transition density $p(t, x-y)$ with respect to the Lebesgue measure is uniquely determined by its Fourier transform

$$\int_{\boldsymbol{R}^n} e^{ix\cdot\xi} p(t, x) dx = e^{-t|\xi|^\alpha}\,.$$

Here α must be in the interval $(0, 2]$ and $n \geq 1$. When $\alpha = 2$, we get a Brownian motion running with a time clock twice as fast as the standard one. Let X be a symmetric α-stable

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process in $\boldsymbol{R}^n$ with $\alpha \in (0, 2)$. It is well-known that the Dirichlet form $(\mathcal{E}^{(\alpha)}, W^{\alpha/2,2}(\boldsymbol{R}^n))$ associated with X is given by

$$\mathcal{E}^{(\alpha)}(u, v) = \frac{1}{2}\mathcal{A}(n, -\alpha) \int_{\boldsymbol{R}^n} \int_{\boldsymbol{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+\alpha}}\, dxdy\,, \tag{1.3}$$

$$W^{\alpha/2,2}(\boldsymbol{R}^n) = \left\{u \in L^2(\boldsymbol{R}^n)\,;\, \int_{\boldsymbol{R}^n} \int_{\boldsymbol{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}}\, dxdy < \infty\right\}\,, \tag{1.4}$$

where

$$\mathcal{A}(n, -\alpha) = \frac{\alpha\,\Gamma((n + \alpha)/2)}{2^{1-\alpha}\,\pi^{n/2}\,\Gamma(1 - (\alpha/2))}\,. \tag{1.5}$$

For $\alpha = 2$, define $W^{1,2}(\boldsymbol{R}^n) = \{u \in L^2(\boldsymbol{R}^n, dx)\,;\ \nabla u \in L^2(\boldsymbol{R}^n, dx)\}$.

Given an open set $D \subset \boldsymbol{R}^n$, define $\tau_D = \inf\{t > 0\,;\ X_t \notin D\}$. Let $X_t^D(\omega) = X_t(\omega)$ if $t < \tau_D(\omega)$ and set $X_t^D(\omega) = \partial$ if $t \geq \tau_D(\omega)$, where ∂ is a coffin state added to $\boldsymbol{R}^n$. The process X^D is called the killed symmetric α-stable process in D. Note that when $0 < \alpha < 2$, X^D is irreducible even when D is disconnected. The Dirichlet space of X^D on $L^2(D, dx)$ is $(\mathcal{E}^{(\alpha)}, H_0^{\alpha/2}(D))$ (cf. Theorem 4.4.3 of [14]), where

$$H_0^{\alpha/2}(D) = \{f \in W^{\alpha/2,2}(\boldsymbol{R}^n)\,;\ f = 0 \text{ q.e. on } D^c\}\,.$$

Here q.e. is the abbreviation for quasi-everywhere with respect to the Riesz capacity determined by $(\mathcal{E}^{(\alpha)}, W^{\alpha/2,2}(\boldsymbol{R}^n))$ (cf. [14]). The space $H_0^{\alpha/2}(D)$ can also be characterized as the $\mathcal{E}^{(\alpha)}$-closure of $C_c^\infty(D)$, the space of smooth functions with compact supports in D. For $u \in H_0^{\alpha/2}(D)$, by (1.3),

$$\begin{aligned}\mathcal{E}^{(\alpha)}(u, v) =& \frac{1}{2}\mathcal{A}(n,\ -\alpha) \int_D \int_D \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+\alpha}}\, dxdy \\ &+ \int_D u(x)v(x)\kappa_D^{(\alpha)}(x)dx\,,\end{aligned} \tag{1.6}$$

where

$$\kappa_D^{(\alpha)}(x) = \mathcal{A}(n,\ -\alpha) \int_{D^c} \frac{1}{|x - y|^{n+\alpha}} dy \tag{1.7}$$

is the density of the killing measure of X^D.

A domain D in $\boldsymbol{R}^n$ is said to satisfy a uniform exterior volume condition if there is a constant $c > 0$ such that for any $x \in \partial D$ and any $r > 0$, $m(B(x, r) \cap D^c) \geq cr^n$. Here m denotes the Lebesgue measure on $\boldsymbol{R}^n$. It is elementary to see that, if D satisfies a uniform exterior volume condition, then $\kappa_D^{(\alpha)}(x) \geq c\,\delta_D(x)^{-\alpha}$ for $x \in D$ and so

$$\int_D \frac{u^2(x)}{\delta_D(x)^\alpha} dx \leq C(D, \alpha) \int_D u^2(x)\kappa_D^{(\alpha)}(x)dx\,, \quad u \in C_c^\infty(D)\,.$$

Therefore, in this case, we trivially have the following Hardy inequality for killed symmetric stable processes in D

$$\int_D \frac{u^2(x)}{\delta_D(x)^\alpha} dx \le C(D,\alpha)\, \mathcal{E}^{(\alpha)}(u,u), \quad u \in C_c^\infty(D)\,. \tag{1.8}$$

In this paper we are mainly concerned with the following type of Hardy inequalities

$$\int_D \frac{u(x)^2}{\delta_D(x)^\alpha} dx \le c(D,\alpha) \left(\int_D \int_D \frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha}} dx dy + \int_D u(x)^2\, dx \right) \quad \text{for } u \in C_c^\infty(D)\,, \tag{1.9}$$

and

$$\int_D \frac{u(x)^2}{\delta_D(x)^\alpha} dx \le c(D,\alpha) \int_D \int_D \frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha}} dx dy \quad \text{for } \ u \in C_c^\infty(D)\,, \tag{1.10}$$

for a certain class of domains D including bounded Lipschitz domains. In fact we will show that for a bounded Lipschitz domain $D \subset \boldsymbol{R}^n$, (1.9) holds for $\alpha \in (0,\,1) \cup (1,\,2)$ and (1.10) holds for $\alpha \in (1,2)$. The Hardy inequality (1.9) was known for bounded C^∞-smooth domains D and for $\alpha \ne 1$, see 4.3.2(9) of Triebel [23] and the references therein. We remark here that Hardy inequality (1.10) can not hold on a bounded Lipschitz domain D for $\alpha \le 1$, since it is proved in [6] that in this case constant functions are in the space spanned by $C_c^\infty(D)$ under the metric

$$\left(\int_D \int_D \frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha}}\, dx dy + \int_D u(x)^2\, dx \right)^{1/2}.$$

We call inequality (1.9) or (1.10) a Hardy inequality for censored α-stable process in D. We now explain the reason behind this.

Using Fourier transforms, the Sobolev space $W^{s,2}(\boldsymbol{R}^n)$ can be defined for any $s \in \boldsymbol{R}$. For $f \in L^2(\boldsymbol{R}^n)$, define

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\boldsymbol{R}^n} f(x) e^{i\,\xi\cdot x}\, dx\,, \quad \xi \in \boldsymbol{R}^n\,.$$

Then one defines

$$W^{s,2}(\boldsymbol{R}^n) = \left\{ u \in L^2(\boldsymbol{R}^n)\,;\, \int_{\boldsymbol{R}^n} \left(1+|\xi|^2\right)^s |\hat{u}(\xi)|^2\, d\xi < \infty \right\}, \tag{1.11}$$

$$\|u\|_{s,2} = \left(\int_{\boldsymbol{R}^n} \left(1+|\xi|^2\right)^s |\hat{u}(\xi)|^2\, d\xi \right)^{1/2}. \tag{1.12}$$

It is well-known that $C_c^\infty(\boldsymbol{R}^n)$ is a dense subspace of $W^{s,2}(\boldsymbol{R}^n)$. By the Plancherel theorem,

$$\int_{\boldsymbol{R}^n} |\hat{f}(\xi)|^2\, d\xi = \int_{\boldsymbol{R}^n} |f(x)|^2\, dx\,.$$

It is easy to see that (cf. Example 1.4.1 of [14]) the Sobolev spaces defined by (1.4) and (1.11) are the same for $0 < s < 1$ and that

$$\mathcal{E}^{(\alpha)}(u, v) = \int_{\boldsymbol{R}^n} |\xi|^\alpha \, \hat{u}(\xi) \cdot \overline{\hat{v}}(\xi) \, d\xi \quad \text{for } u, v \in W^{\alpha/2,2}(\boldsymbol{R}^n) .$$

By this alternative characterization, it is clear that $W^{1,2}(\boldsymbol{R}^n) \subset W^{\alpha/2,2}(\boldsymbol{R}^n)$ for $0 < \alpha < 2$ and that for $u, v \in W^{1,2}(\boldsymbol{R}^n)$,

$$\lim_{\alpha \to 2} \mathcal{E}^{(\alpha)}(u,\ v) = \int_{\boldsymbol{R}^n} \nabla u \cdot \nabla v \, dx .$$

For any open set $D \subset \boldsymbol{R}^n$ and $s \in \boldsymbol{R}$, define $H_0^s(D)$ as the closure of $C_c^\infty(D)$ under $\| \cdot \|_{s,2}$. So $H_0^s(D) \subset W^{s,2}(\boldsymbol{R}^n)$ is equipped with the norm $\| \cdot \|_{s,2}$ from $W^{s,2}(\boldsymbol{R}^n)$.

For a symmetric α-stable process X on $\boldsymbol{R}^n$ with $0 < \alpha < 2$, typically $\lim_{t \uparrow \tau_D} X_t$ exists and belongs to D. We would like to extend X^D beyond its lifetime τ_D by the Ikeda-Nagasawa-Watanabe piecing together procedure described as follows. Let $Y_t(\omega) = X_t^D(\omega)$ for $t < \tau_D(\omega)$. If $X^D_{\tau_D -}(\omega) \notin D$, set $Y_t(\omega) = \partial$ for $t \geq \tau_D(\omega)$. If $X^D_{\tau_D -}(\omega) \in D$, let $Y_{\tau_D}(\omega) = X^D_{\tau_D -}(\omega)$ and glue an independent copy of X^D starting from $X^D_{\tau_D -}(\omega)$ to $Y_{\tau_D}(\omega)$. Iterating this procedure countably many times, we obtain a strong Markov process on D. This process Y is called a censored α-stable process in [6] and is proved there that its Dirichlet form is $(\mathcal{C}^{(\alpha)}, \mathcal{F}_D^{(\alpha)})$, where

$$\mathcal{C}^{(\alpha)}(u, v) = \frac{1}{2} \mathcal{A}(n,\ -\alpha) \int_D \int_D \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+\alpha}} \, dx dy , \qquad u, v \in C_c^\infty(D) , \tag{1.13}$$

and $\mathcal{F}_D^{(\alpha)}$ is the closure of $C_c^\infty(D)$ under Hilbert inner product $\mathcal{C}_1^{(\alpha)} = \mathcal{C}^{(\alpha)} + (\,\cdot\,,\ \cdot\,)_{L^2(D)}$.

It was observed in [6] that the space $\mathcal{F}_D^{(\alpha)}$ is in fact the Sobolev (or Besov) space of fractional order $W_0^{\alpha/2,2}(D)$, whose definition we now recall. To simplify notations, let $s = \alpha/2$.

For an open set $D \subset \boldsymbol{R}^n$, define

$$W^{s,2}(D) = \{ u \in L^2(D, dx) \, ; \ u = v \text{ a.e. on } D \text{ for some } v \in W^{s,2}(\boldsymbol{R}^n) \} ,$$
$$\|u\|_{s,2;D} = \inf \{ \|v\|_{s,2} \, ; \ v \in W^{s,2}(\boldsymbol{R}^n) \text{ and } v = u \text{ a.e. on } D \} .$$

It is known (cf. [23]) that $(W^{s,2}(D),\ \| \cdot \|_{s,2;D})$ is a Hilbert space. Let $(W_0^{s,2}(D),\ \| \cdot \|_{s,2;D})$ be the smallest closed subspace of $W^{s,2}(D)$ containing $C_c^\infty(D)$.

For $0 < d \leq n$, we will use $\mathcal{H}^d$ to denote the d-dimensional Hausdorff measure in $\boldsymbol{R}^n$.

DEFINITION 1.2. A Borel set $\Gamma \subset \boldsymbol{R}^n$ is called a *d-set* for some $0 < d \leq n$ if there exist positive constants c_1 and c_2 such that for all $x \in \Gamma$ and $r \in (0,\ 1]$,

$$c_1 r^d \leq \mathcal{H}^d(\Gamma \cap B(x, r)) \leq c_2 r^d .$$

The notion of d-sets arises both in the theory of function spaces and in fractal geometry. It is known (see Proposition 1 in Chapter VIII of [16]) that if Γ is a d-set, then its Euclidean closure $\overline{\Gamma}$ is a d-set and $\overline{\Gamma} \setminus \Gamma$ has zero $\mathcal{H}^d$-measure.

If an open set D is an n-set and $0 < s < 1$, then by Theorem 1 on page 103 of [16],

$$W_0^{s,2}(D) = \mathcal{F}_D^{(2s)} \text{ and the Sobolev norm } \|\cdot\|_{s,2;D} \text{ is equivalent to } \sqrt{\mathcal{C}_1^{(2s)}}\,. \tag{1.14}$$

This explains why we call inequality (1.9), in particular, (1.10) a *Hardy inequality for censored α-stable process in D*. We will prove Hardy inequality (1.9) and (1.10) by using complex interpolation methods.

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2. Complex interpolation. We first recall the following definition from [18]. Let $\boldsymbol{C}$ denote the field of complex numbers.

DEFINITION 2.1. Let $X_1 \subset X_0$ be a continuous dense injection of two Banach spaces. Let $S = \{z \in \boldsymbol{C};\ 0 \le \mathrm{Re} z \le 1\}$, and S^0 denote the interior of S. Let $H(X_0, X_1)$ denote all the continuous bounded functions $f : S \to X_0$ which are holomorphic in S^0 with $f(iy) \in X_0$ and $f(1+iy) \in X_1$ for each fixed $y \in \boldsymbol{R}$, and such that the following norm is finite:

$$\|f\| := \max\left\{\max_{y\in\boldsymbol{R}} \|f(iy)\|_{X_0},\ \max_{y\in\boldsymbol{R}} \|f(1+iy)\|_{X_1}\right\}.$$

For $0 \le \theta \le 1$, the *interpolation space X_θ with weight θ* is defined by

$$X_\theta := [X_0, X_1]_\theta = \{f(\theta)\,;\ f \in H(X_0, X_1)\}\,.$$

with norm

$$\|u\|_\theta := \inf\{\|f\|\,;\ f \in H(X_0, X_1) \text{ with } f(\theta) = u\}\,.$$

The Banach spaces X_0 and X_1 are called an *interpolation couple*.

Using the result from [20] or page 211 of [21], we see from (1.11)–(1.12) that $W^{s,2}(\boldsymbol{R}^n) = [L^2(\boldsymbol{R}^n),\ W^{1,2}(\boldsymbol{R}^n)]_s$ for $0 < s < 1$.

We record two basic facts about interpolation in the following Proposition, which are stated as **4** in [8] and Theorem 1.2.4 in [23] respectively. Suppose that $X_1 \subset X_0$ and $Y_1 \subset Y_0$ are two interpolation couples.

PROPOSITION 2.1. (1) *If T is a bounded linear operator from X_0 to Y_0 and from X_1 to Y_1 with $\|Tx\|_{Y_i} \le M_i\|x\|_{X_i}$ for $i = 0, 1$. Then T is a bounded linear operator from $X_\theta := [X_0, X_1]_\theta$ to $Y_\theta := [Y_0, Y_1]_\theta$, where $0 < \theta < 1$ and*

$$\|Tx\|_{Y_\theta} \le M_0^{1-\theta} M_1^\theta\, \|x\|_{X_\theta} \quad \text{for } x \in X_\theta\,.$$

(2) *Suppose that S is a bounded linear operator from Y_0 to X_0 and from Y_1 to X_1, and R is a bounded linear operator from X_0 to Y_0 and from X_1 to Y_1 such that $SR = I$, the identity map, when restricted to X_0 and X_1 respectively. Then for any $0 < \theta < 1$, RS is a projection in space $[Y_0, Y_1]_\theta$ in the sense that $(RS)^2 = RS$. Furthermore R is an isomorphic mapping from $[Y_0, Y_1]_\theta$ onto the range of RS of $[X_0, X_1]_\theta$, which is a closed subspace of $[X_0, X_1]_\theta$.*

The main object of this section is to show that, under certain conditions on D which are satisfied when D is a bounded Lipschitz domain, the complex interpolation between the spaces $L^2(D)$ and $W_0^{1,2}(D)$ with weight s is $W_0^{s,2}(D)$.

A domain D is called an extension domain if there is a bounded linear map $E : L^2(D) \to L^2(\boldsymbol{R}^n)$ such that $Ef = f$ a.e. on D, and when restricted to $W^{1,2}(D) \subset L^2(D)$, it is a bounded linear map from $W^{1,2}(D)$ into $W^{1,2}(\boldsymbol{R}^n)$. Jones [15] showed that any (ε, δ)-domain is an extension domain. Here

DEFINITION 2.2. An open connected subset Γ of $\boldsymbol{R}^n$ is an (ε, δ)-*domain* for some $\varepsilon > 0$ and $\delta \in (0, \infty]$, if for all $x, y \in \Gamma$ with $|x-y| < \delta$, there exists a rectifiable arc $\gamma \subset \Gamma$ with length l_γ joining x to y such that $l_\gamma \le |x-y|/\varepsilon$ and

$$\mathrm{dist}(z, \partial\Gamma) \ge \frac{\varepsilon|x-z|\cdot|y-z|}{|x-y|} \quad \text{for all } z \in \gamma .$$

Recall that for an open set $D \subset \boldsymbol{R}^n$ and $0 < s < 1$,

$$H_0^s(D) = \{u \in W^{s,2}(\boldsymbol{R}^n) ;\ u = 0 \text{ q.e. on } D^c\} ,$$

equipped with the norm inherited from $(W^{s,2}(\boldsymbol{R}^n), \|\cdot\|_{s,2})$. So for $u \in H_0^s(D)$,

$$\|u\|_{H_0^s(D)} = \left(\frac{1}{2}\mathcal{A}(n,-\alpha)\int_D\int_D \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}dxdy + \int_D u(x)^2\kappa_D^{(2s)}(x)dx\right)^{1/2} ,$$

where $\kappa_D^{(2s)}$ is given by (1.7).

PROPOSITION 2.2. *Suppose that D is an open set such that its boundary ∂D has zero Lebesgue measure and the interior of D^c is a finite union of disjoint extension domains. Then*

$$[L^2(D), W_0^{1,2}(D)]_s = H_0^s(D) \quad \textit{for all } 0 < s < 1 .$$

PROOF. The proof is the same as that of Corollary III.1.6 in [22], though the result there was proved only for Lipschitz domains. For reader's convenience we spell out the details. Denote by D' the interior of D^c. Let $1_{D'} : L^2(\boldsymbol{R}^n) \to L^2(D')$ be the restriction map, and let E be the extension map from $W^{1,2}(D')$ to $W^{1,2}(\boldsymbol{R}^n)$. Then $E \circ 1_{D'}$ is a bounded linear operator in both $L^2(\boldsymbol{R}^n)$ and $W^{1,2}(\boldsymbol{R}^n)$. Thus by Proposition 2.1(1),

$$\|(E\circ 1_{D'})f\|_{s,2} \le c\|f\|_{s,2}$$

for any $f \in W^{s,2}(\boldsymbol{R}^n)$. By the definition of $W^{s,2}(D')$-space, we have

$$\|Eu\|_{s,2} \le c\|u\|_{s,2;D'} \quad \text{for } u \in W^{s,2}(D') .$$

The reverse inequality holds by definition. It follows from Proposition 2.1(2) that E is an embedding of $W^{s,2}(D')$ into $W^{s,2}(\boldsymbol{R}^n)$, the projection $E \circ 1_{D'}$ and $I - (E \circ 1_{D'})$ give the direct sum decomposition of $W^{s,2}(\boldsymbol{R}^n) = W^{s,2}(D') \oplus H_0^s(D)$, and that $W^{s,2}(D') = [L^2(D'), W^{1,2}(D')]_s$. Therefore $H_0^s(D) = [L^2(D), W_0^{s,2}(D)]_s$. □

Under the following condition on the domain $D \subset \boldsymbol{R}^n$, Strichartz (Corollary II.4.2 of [22]) proved that the multiplier 1_D is a bounded linear operator from $W^{s,2}(\boldsymbol{R}^n)$ into $W^{s,2}(\boldsymbol{R}^n)$ for $0 \le s < 1/2$.

Condition A: There exist a coordinate system and an integer N such that almost every line parallel to the axes intersects D in at most N components, or even in at most N components in the unit cube about every lattice point,

Note that a bounded Lipschitz domain satisfies both Condition A and the conditions of Proposition 2.2.

THEOREM 2.3. *Suppose that D is a bounded domain in $\boldsymbol{R}^n$ satisfying Condition A and the conditions of Proposition* 2.2. *Furthermore assume that D is an n-set and ∂D is an $(n-1)$-set. Then*

$$W_0^{s,2}(D) = H_0^s(D) \quad \text{for } s \in (-1/2,\ 1/2) \cup (1/2,\ 1)\,,$$

and the norms of these two spaces are equivalent. In particular, *for $\alpha \in (0,\ 1) \cup (1,\ 2)$*, *there is a constant $c_0 = c_0(D, \alpha) > 0$ such that*

$$\int_D u(x)^2 \kappa_D^{(\alpha)}(x)\, dx \le c_0 \left(\int_D \int_D \frac{(u(x) - u(y))^2}{|x-y|^{n+\alpha}}\, dx dy + \int_D u(x)^2\, dx \right)$$

$$\text{for } u \in W_0^{\alpha/2,2}(D)\,,$$

where $\kappa_D^{(\alpha)}$ is given by (1.7).

PROOF. Clearly $H_0^s(D) \subset W_0^{s,2}(D)$, as $C_c^\infty(D)$ is a dense subset in both spaces and $\|u\|_{H_0^s(D)} \ge \|u\|_{s,2;D}$ for $u \in C_c^\infty(D)$. We now show that the converse $W_0^{1,2}(D) \subset H_0^s(D)$ holds for $-1/2 < s < 1$ except for $s = 1/2$.

When $0 \le s < 1/2$, as mentioned above, 1_D is a bounded linear operator from $W^{s,2}(\boldsymbol{R}^n)$ into $W^{s,2}(\boldsymbol{R}^n)$ by [22]. The same map is thus continuous for $-1/2 < s < 0$, in view of the duality between $W^{s,2}(\boldsymbol{R}^n)$ and $W^{-s,2}(\boldsymbol{R}^n)$ (cf. [23]). Thus for $-1/2 < s < 1/2$, there is a constant $c = c(s) > 0$ such that

$$\|1_D u\|_{s,2} \le c\|u\|_{s,2} \quad \text{for every } u \in W^{s,2}(\boldsymbol{R}^n)\,.$$

By the definition for $W^{s,2}(D)$, we have for any $f \in C_c^\infty(D) \subset W^{s,2}(D)$,

$$\|1_D f\|_{s,2} \le c\|f\|_{s,2;D}$$

and $1_D f \in H_0^s(D)$ with $\|f\|_{H_0^2(D)} = \|1_D f\|_{s,2}$. As $W_0^{s,2}(D)$ is the $\|\cdot\|_{s,2;D}$-closure of $C_c^\infty(D)$, we have $W_0^{s,2}(D) \subset H_0^s(D)$. Clearly, $\|u\|_{H_0^2(D)} = \|u\|_{s,2} \ge \|u\|_{s,2;D}$ for $u \in W_0^{s,2}(D)$. Therefore these two norms are equivalent and $H_0^s(D) = W_0^{s,2}(D)$ for $-1/2 < s < 1/2$.

For $1/2 < s < 1$, we know from Proposition 3.6 of Caetano [7] that the trace operator $\mathrm{tr}|_{\partial D} : u \to u|_{\partial D}$ is a bounded linear operator from $W^{s,2}(D)$ onto $L^2(\partial D, \mathcal{H}^{n-1})$. We refer the definitions of trace operator $\mathrm{tr}|_{\partial D}$ to [7]. As any $u \in W_0^{s,2}(D)$ can be $\|\cdot\|_{s,2;D}$-approximated by a sequence of functions in $C_c^\infty(D)$, we have $\mathrm{tr}|_{\partial D} u = 0$ for $u \in W_0^{s,2}(D)$.

Note that by almost the identical proof, Lemma 2.1(ii) of [18] holds with $W^{s,2}(D)$ and $H_0^s(D)$ in place of spaces $L_s^2(\mathbf{R}_+^n)$ and $L_{0s}^2(\mathbf{R}_+^n)$ there. That is, if $v \in W^{s,2}(D)$ with $\mathrm{tr}|_{\partial D} v = 0$, then $v \in H_0^s(D)$. Therefore we conclude $u \in H_0^s(D)$ and so $W_0^{s,2}(D) = H_0^s(D)$.

Since the identity map is a continuous map from $H_0^s(D)$ onto $W_0^{s,2}(D)$, by the inverse operator theorem, its inverse is also a continuous map. This says that these two spaces have equivalent norms. □

COROLLARY 2.4 (Hardy inequality). *Let D be a bounded Lipschitz domain in $\mathbf{R}^n$ with $n \geq 2$, and let $\delta_D(x)$ is the Euclidean distance between x and ∂D.*

(1) *For $\alpha \in (0, 1) \cup (1, 2)$, there is a constant $c = c(D, \alpha) > 0$ such that for every $u \in C_c^\infty(D)$,*

$$\int_D \frac{u(x)^2}{\delta_D(x)^\alpha} dx \leq c \left(\int_D \int_D \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}}\, dx dy + \int_D u(x)^2\, dx \right). \tag{2.1}$$

(2) *For $\alpha \in (1, 2)$, there is a constant $c = c(D, \alpha) > 0$ such that for every $u \in C_c^\infty(D)$,*

$$\int_D \frac{u(x)^2}{\delta_D(x)^\alpha} dx \leq c \int_D \int_D \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}}\, dx dy\,. \tag{2.2}$$

PROOF. Note that a bounded Lipschitz domain satisfies the condition of Theorem 2.3. As $\kappa_D^{(\alpha)}(x) \approx \delta_D(x)^{-\alpha}$, it follows from Theorem 2.3 that for $\alpha \in (0, 1) \cup (1, 2)$, there is a constant $c = c(D, \alpha) > 0$ so that inequality (2.1) holds for every $u \in C_c^\infty(D)$.

We claim that for $\alpha \in (1, 2)$, there is a constant $c_1 = c_1(D, \alpha)$ such that

$$\int_D u(x)^2\, dx \leq c_1 \int_D \int_D \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}}\, dx dy \quad \text{for} \quad u \in W_0^{\alpha/2,2}(D)\,. \tag{2.3}$$

Since D is a bounded n-set, it is well known (e.g. see Chapter VI of [16]) that there is a bounded extension operator $E : W^{\alpha/2,2}(D) \to W^{\alpha/2,2}(\mathbf{R}^n)$ such that $Eu = u$ a.e. on D for $u \in W^{\alpha/2,2}(D)$. On the other hand, Adams' embedding theorem (see Lemma 1 on page 214 of [16]) says that $W^{\alpha/2,2}(\mathbf{R}^n)$ is compactly embedded in $L^p(D, dx)$ for every $p < 2n/(n - \alpha)$. Thus $W^{\alpha/2,2}(D)$ is compactly embedded in $L^2(D, dx)$. Suppose that (2.3) is not true. Then there is a sequence $\{u_k,\ k \geq 1\}$ in $W_0^{\alpha/2,2}(D)$ with $\|u_k\|_{L^2(D,dx)} = 1$ such that $\mathcal{C}^{(\alpha)}(u_k, u_k) \to 0$ as $k \to \infty$, where $\mathcal{C}^{(\alpha)}$ is given by (1.13). From the compact embedding of $W_0^{\alpha/2,2}(D) \subset W^{\alpha/2,s}(D)$ in $L^2(D, dx)$, there is a subsequence $\{k_j,\ j \geq 1\}$ such that $u_{k_j} \to u$ in $L^2(D, dx)$. On the other hand, it is easy to see that the Cesáro mean of a subsequence of $\{u_{k_j}, j \geq 1\}$ converges to u in the space $(W^{\alpha/2,2}(D), \|\cdot\|_{\alpha/2,2})$ and so $u \in W_0^{\alpha/2,2}(D)$. Fatou lemma implies that $\mathcal{C}^{(\alpha)}(u, u) = 0$ and so u is a non-zero constant. However it is shown in [6] that for $\alpha \in (1, 2)$, $1 \notin W_0^{\alpha/2,2}(D)$, a contradiction. Thus we have established (2.3), which combines with the previously derived (2.1) proves (2.2) for $\alpha \in (1, 2)$. □

An interesting and important question is: how does the constant $c = c(D, \alpha)$ depend on α? More explicitly, is $c(D, \alpha)$ some interpolation between 1 and $c(D, 2)$, where $c(D, 2)$ is the constant C in (1.2) whenever it holds? For example, is it true that there is some $A > 0$ such that $c(D,\alpha) \le A\, c(D,2)^{\alpha/2}$? A positive answer to this question will give useful information on the first eigenvalue for censored stable processes in domain D as well as for killed stable processes in D (cf. [4]) via the first Dirichlet eigenvalue for Δ in D.

3. Second Approach. To help answer the question at the end of last section, the following approach to Hardy inequalities might be useful. In this section we assume $n \ge 3$, and in Theorem 3.1 we will assume that for some $s \in (0,1)$, we have

$$W_0^{s,2}(D) = [L^2(D), W_0^{1,2}(D)]_s \,. \tag{3.1}$$

Note that in section 2 we have proved (3.1) for $s \neq 1/2$ and a class of domains including bounded Lipschitz domains. It is plausible that (3.1) should hold for a much larger class of domains. Let $\||\cdot|\|_{s,2;D}$ denote the norm inherited from complex interpolation between the two Hilbert spaces $L^2(D)$ and $W_0^{1,2}(D)$.

THEOREM 3.1. *Suppose that D is an uniformly Δ-regular domain in $\boldsymbol{R}^n$ and that for some $0 < s < 1$ condition* (3.1) *is satisfied. Then*

$$\int_D \frac{u(x)^2}{\delta_D(x)^{2s}} dx \le C^s\, \||u|\|_{s,2}^2 \quad \text{for all} \quad u \in W_0^{s,2}(D)\,, \tag{3.2}$$

where $C > 0$ is the constant in inequality (1.2) *that depends only on n and the constant c in the definition of Δ-regularity* (1.1).

PROOF. Let $F_0 = L^2(D,\ \delta_D(x)^2 dx)$ and $F_1 = L^2(D,\ dx)$. It is known from [20] or page 211 of [21] that $L^2(D,\ \delta_D(x)^{2(1-s)}dx)$ is the complex linear interpolation space between Banach spaces F_0 and F_1 with weight s. Let T be a linear map defined on $W_0^{1,2}(D) = L^2(D) \cap W_0^{1,2}(D)$ into $F_0 \cap F_1$ by

$$Tu(x) = \frac{u(x)}{\delta_D(x)} \quad \text{for} \quad x \in D\,.$$

Clearly, for $u \in W_0^{1,2}(D)$,

$$\|Tu\|_{F_0} \equiv \|Tu\|_{L^2(D,\, \delta_D(x)^2 dx)} = \|u\|_{L^2(D,\, dx)}\,,$$

and by the Hardy inequality (1.2) there is a $C > 0$, depending only on n and the constant c in the definition of Δ-regularity, such that

$$\|Tu\|_{F_1} \equiv \|Tu\|_{L^2(D,\, dx)} \le \sqrt{C}\|u\|_{1,2}\,. \tag{3.3}$$

Therefore by the interpolation theorem on page 211 of [21], T extends uniquely to $[L^2(D), W_0^{1,2}(D)]_s = W_0^{s,2}(D)$ and

$$\|Tu\|_{L^2(D,\, \delta_D(x)^{2(1-s)}dx)} \le C^{s/2}\, \||u|\|_{s,2;D} \quad \text{for any} \quad u \in W_0^{s,2}(D)\,,$$

where C is the constant in (3.3). This proves the theorem. □

When D is an n-set, we know that $|||\cdot|||_{s,2;D}$ is equivalent to the norm $\sqrt{\mathcal{C}_1^{(2s)}}$ in the sense that there are constants $c(D,s)$ and $C(D,s)$ such that

$$c(D,s)\sqrt{\mathcal{C}_1^{(2s)}(u,u)} \le |||u|||_{s,2;D} \le C(D,s)\sqrt{\mathcal{C}_1^{(2s)}(u,u)} \quad \text{for } u \in W_0^{s,2}(D)\,, \tag{3.4}$$

where $\mathcal{C}^{(2s)}$ is given by (1.13). So in this case (3.2) gives the desired Hardy inequality for censored $(2s)$-stable process in D. A question that is related to the one stated at the end of last section is as follows. Can the constants $c(D,s)$ and $C(D,s)$ in (3.4) (or at least $C(D,s)$) be chosen so that it depends only on D but not on s?

Using the same proof as above, we have by Proposition 2.2 the following Hardy inequality for killed symmetric α-stable process in D. Unlike the one mentioned in (1.8) under the uniform exterior volume condition, the following inequality is far from trivial, as we do not have any easy comparison between the killing measure density $\kappa_D^{(\alpha)}(x)$ and the function $\delta_D(x)^{-\alpha}$.

THEOREM 3.2. *Suppose that D is an uniformly Δ-regular domain in $\boldsymbol{R}^n$ and that it satisfies the condition of Proposition* 2.2*. Then for any* $0<\alpha<2$, *there is a constant* $c(D,\alpha)>0$ *such that*

$$\int_D \frac{u(x)^2}{\delta_D(x)^\alpha}dx \le c(D,\alpha)\mathcal{E}^{(\alpha)}(u,u) \quad \textit{for all} \quad u \in W_0^{s,2}(D)\,, \tag{3.5}$$

where $\mathcal{E}^{(\alpha)}$ *is given by* (1.6)*.*

4. Applications. The Hardy inequality obtained above for censored stable processes should have many implications on the study of censored stable processes. Here we just mention one.

Let D be a bounded Lipschitz domain. It was proved in [6] that the censored α-stable process Y is transient with finite lifetime when $\alpha>1$. Let $\lambda_1(Y)>0$ be the first eigenvalue of Y.

THEOREM 4.1. *Suppose that D is a bounded Lipschitz domain in $\boldsymbol{R}^n$. There is a constant $c_n>0$ such that for* $\alpha\in(1,\,2)$,

$$c(D,\alpha)^{-1}\mathrm{Inr}(D)^{-\alpha} \le \lambda_1(Y) \le c_n\,\mathrm{Inr}(D)^{-\alpha}, \tag{4.1}$$

where $\mathrm{Inr}(D) := \sup\{\delta_D(x)\,;\ x\in D\}$ *is the inner radius of D and $c(D,\alpha)$ is the constant in Corollary* 2.4*.*

PROOF. The lower bound estimate follows from Corollary 2.4 by a similar argument as that for Theorem 1.5.3 in [11]. For the upper bound estimate, note that $W_0^{\alpha/2,2}(D)\supset H_0^{\alpha/2}(D)$ and $\|u\|_{\alpha/2,2;D}\le \mathcal{E}_1^{(\alpha)}(u,u)^{1/2}$. Thus $\lambda_1(Y)\le\lambda_1(X^D)$, where $\lambda_1(X^D)$ is the first eigenvalue for the killed stable process in D. Clearly, $\lambda_1(X^D)$ is no larger than the killed stable process in a ball with radius $\mathrm{Inr}(D)$, which is $c_n\,\mathrm{Inr}(D)^{-\alpha}$ by scaling. □

Similarly, one can get bounds on the first eigenvalue $\lambda_1(X^D)$ for killed symmetric α-stable process in D.

REMARK. The lower bound estimate in (4.1) is not of much use unless we know how the constant $c(D, \alpha)$ depends on D and α. See the problems posed at the end of last two sections.

In [9], the Hardy inequality will be applied, together with the Harnack and boundary Harnack inequalities established in [6], to obtain sharp estimates on the Green functions of censored α-stable processes in bounded $C^{1,1}$-domains for $\alpha \in (1, 2)$.

The paper [13] by Fitzsimmons contains the equivalent characterization of Hardy inequality in the context of general symmetric Markov processes.

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