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CONFORMAL INVARIANTS OF QED DOMAINS

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Abstract. Given a Jordan domain Ω in the extended complex plane $\bar{\boldsymbol{C}}$, denote by $M_b(\Omega)$, $M(\Omega)$ and $R(\Omega)$ the boundary quasiextremal distance constant, quasiextremal distance constant and quasiconformal reflection constant of Ω, respectively. It is known that $M_b(\Omega) \leq M(\Omega) \leq R(\Omega) + 1$. In this paper, we will give some further relations among $M_b(\Omega)$, $M(\Omega)$ and $R(\Omega)$ by introducing and studying some other closely related constants. Particularly, we will give a necessary and sufficient condition for $M_b(\Omega) = R(\Omega) + 1$ and show that $M(\Omega) < R(\Omega) + 1$ for all asymptotically conformal extension domains other than disks. This gives an affirmative answer to a question asked by Yang, showing that the conjecture $M(\Omega) = R(\Omega) + 1$ by Garnett and Yang is not true for all asymptotically conformal extension domains other than disks. Our discussion relies heavily on the theory of extremal quasiconformal mappings, which in turn gives some interesting results in the extremal quasiconformal mapping theory as well.

1. Introduction. Let Ω be a domain in the extended complex plane $\bar{\boldsymbol{C}}$. Given a pair of disjoint nondegenerate continua A and B in $\bar{\Omega}$, let $\mathrm{mod}(A, B; \Omega)$ denote the modulus of the family $\Gamma(A, B; \Omega)$ of curves that join A and B in Ω. The following so-called quasiextremal distance constant (or QED constant) was introduced in [Y1]:

$$M(\Omega) = \sup\left\{\frac{\mathrm{mod}(A, B; \boldsymbol{C})}{\mathrm{mod}(A, B; \Omega)}\,;\ \text{for all pairs } A \text{ and } B \text{ in } \bar{\Omega}\right\}. \tag{1.1}$$

The domain Ω is a QED domain if its QED constant $M(\Omega)$ is finite. QED domains were introduced by Gehring and Martio [GM] as a useful class of domains in the study of quasiconformal mappings.

In this paper, we will always assume that Ω is a Jordan domain. It was proved in [GM] that Ω is a QED domain if and only if Ω is a quasidisk, meaning as usual that Ω is the image of the unit disk $D = \{|z| < 1\}$ under a quasiconformal self-mapping of the extended complex plane $\bar{\boldsymbol{C}}$, or equivalently, there exists a quasiconformal reflection in $\partial\Omega$ which interchanges Ω and $\Omega^* = \bar{\boldsymbol{C}} - \bar{\Omega}$ and keeps every point of $\partial\Omega$ fixed. Thus, a QED domain Ω determines the quasiconformal reflection constant $R(\Omega)$, defined as

$$R(\Omega) = \inf\{K[f]\,;\ \text{for all quasiconformal reflections } f \text{ in } \partial\Omega\}, \tag{1.2}$$

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where $K[f]$ is the maximal dilatation of f. Clearly, $R(\Omega) = R(\Omega^*)$. The two constants $M(\Omega)$ and $R(\Omega)$ are closely related to one another. For example, it was proved in [Y1] that

$$M(\Omega) \leq R(\Omega) + 1\,, \tag{1.3}$$

and it was conjectured by Garnett and Yang [GY] that the equality in (1.3) holds for all QED domains. But this was disproved by Yang [Y3] for ellipses. In [Y3] the following boundary quasiextremal distance constant (or BQED constant) $M_b(\Omega)$ was also introduced:

$$M_b(\Omega) = \sup\left\{\frac{\mathrm{mod}(A, B; \boldsymbol{C})}{\mathrm{mod}(A, B; \Omega)}\ ;\ \text{for all pairs } A \text{ and } B \text{ in } \partial\Omega\right\}. \tag{1.4}$$

Clearly, $M_b(\Omega) \leq M(\Omega)$. However, the question whether $M_b(\Omega) = M(\Omega)$ still remains open.

We say a Jordan domain Ω is an asymptotically conformal extension domain if the Riemann mapping from D to Ω has a quasiconformal extension to a neighborhood of D whose complex dilatation μ satisfies $|\mu(z)| \to 0$ as $|z| \to 1+$. Ω is a disk if it is the image of D under a Möbius transformation. Clearly, a smooth domain is always an asymptotically conformal extension domain, but the converse is not true. Very recently, Yang [Y4] proved that $M(\Omega) < R(\Omega) + 1$ for all smooth domains other than disks and asked whether $M(\Omega) < R(\Omega) + 1$ for all asypmtotically conformal extension domains other than disks. On the other hand, Wu and Yang [WY] proved that $M_b(\Omega) < R(\Omega) + 1$ for all asypmtotically conformal extension domains other than disks.

In this paper, we will continue to investigate the relations among $M_b(\Omega)$, $M(\Omega)$ and $R(\Omega)$ by introducing and studying some other closely related constants associated to a quasisymmetric homeomorphism. In particular, we will give a necessary and sufficient condition for $M_b(\Omega) = R(\Omega) + 1$ (Theorem 2) and show that $M(\Omega) < R(\Omega) + 1$ for all asypmtotically conformal extension domains other than disks (Theorem 5). This contains the abovementioned results obtained by Wu and Yang and gives an affirmative answer to the question of Yang as well. Consequently, the Garnett-Yang conjecture is not true for all asypmtotically conformal extension domains other than disks. On the other hand, as will be seen, our discussion relies heavily on the theory of extremal quasiconformal mappings, which in turn gives some interesting results in the extremal quasiconformal mapping theory as well (see Sections 2 and 3).

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2. Related quantities. In this section, we will define and discuss some constants associated with a quasisymmetric homeomorphism. The results will be used in the next section to prove some properties of the boundary quasiextremal distance constant.

For a Jordan domain Ω in the extended complex plane $\bar{\boldsymbol{C}}$, let f_1 and f_2 map Ω and Ω^* conformally onto D and D^*, respectively. Extend f_1 and f_2 to the boundary $\partial\Omega = \partial\Omega^*$ and define $h_\Omega = f_2 \circ f_1^{-1}|\partial D$, which is known as the sewing mapping of the domains Ω and Ω^*. Then Ω is a QED domain, or equivalently, a quasidisk, if and only if h_Ω is a quasisymmetric homeomorphism of the unit circle onto itself in the sense of Beurling-Ahlfors [BA]. Conversely, a quasisymmetric homeomorphism h also determines a pair of complementary Jordan domains, which we denote by Ω_h and Ω_h^*, respectively.

Given a quasisymmetric homeomorphism h of the unit circle onto itself, we denote by $Q(h)$ the class of all quasiconformal mappings of the unit disk D with boundary values h. The homeomorphism h then determines the extremal maximal dilatation $K^*(h)$, defined as

$$K^*(h) = \inf_{f \in Q(h)} K[f]. \tag{2.1}$$

Clearly, $K^*(h^{-1}) = K^*(h)$. $f \in Q(h)$ is called extremal if $K[f] = K^*(h)$ (see [St3]). It is well-known that there always exists at least one extremal mapping in the class $Q(h)$. h also determines the boundary dilatation $H(h)$ (see [St3]), defined as

$$H(h) = \inf\{K[f|D - E]\ ;\ \text{for all } f \in Q(h) \text{ and all compact subsets } E \subset D\}, \tag{2.2}$$

where $K[f|D - E]$ is the maximal dilatation of f on $D - E$. Then, $H(h) = H(h^{-1})$ and $H(h) \leq K^*(h)$. The set of all normalized (fixing three boundary points on ∂D) quasisymmetric homeomorphisms of the unit circle onto itself is known as the universal Teichmüller space T of Bers (see [Le], [Na]). Following Earle-Li [EL], a point h is called a Strebel point if $H(h) < K^*(h)$. Then, by a result of Lakic [La], the set of Strebel points is open and dense in the universal Teichmüller space T.

Now the maximal dilatation $K(h)$ of h is defined as

$$K(h) = \sup\left\{\frac{\operatorname{mod}(h(A), h(B); D)}{\operatorname{mod}(A, B; D)}\ ;\ \text{for all pairs } A \text{ and } B \text{ in } \partial D\right\}. \tag{2.3}$$

Clearly, $K(h^{-1}) = K(h)$. By the quasi-invariance property of modulus under quasiconformal mappings, it follows that $K(h) \leq K^*(h)$. It was an open question for a long time to determine whether or not $K(h) = K^*(h)$ always holds before Anderson and Hinkkanen disproved this by giving concrete examples of a family of affine mappings of some parallelograms (see [AH]). Later, a necessary condition for $K(h) = K^*(h)$ was obtained independently by Wu [Wu] and Yang [Y2]. We say h is induced by affine mappings if it is the restriction to ∂D of a map of the form $\phi_2 \circ f_K \circ \phi_1^{-1}$, where $f_K(x + iy) = x + iKy$, while ϕ_1 and ϕ_2 are conformal mappings from a rectangle $\{x + iy; 0 < x < a, 0 < y < b\}$ and its image $\{u + iv; 0 < u < a, 0 < v < Kb\}$ under f_K onto D, respectively. Then the necessary condition for $K(h) = K^*(h)$ obtained by Wu [Wu] and Yang [Y2] can be stated as follows.

THEOREM A ([Wu], [Y2]). *Let $h : \partial D \to \partial D$ be a quasisymmetric homeomorphism. If $K(h) = K^*(h)$, then either h is induced by an affine mapping or $H(h) = K^*(h)$.*

In their papers [Wu] and [Y2], Wu and Yang also asked whether the converse of Theorem A was true. Recently, the author [S2] proved that there exists a family of quasisymmetric homeomorphisms h such that $K(h) < K^*(h) = H(h)$, which gives a negative answer to the question. So the necessary condition in Theorem A is not sufficient for $K(h) = K^*(h)$.

EXAMPLE 1 ([S2]). For convenience sake we use the upper half plane $\boldsymbol{H} = \{z\,;\,\mathrm{Im} z > 0\}$ instead of the unit disk D. For any $K > 1$, we consider the quasisymmetric homeomorphism h of Strebel (see [St1], [St2]), namely, $h = h_K : \partial\boldsymbol{H} \to \partial\boldsymbol{H}$ that is defined to be $h(x) = x$ for $x \le 0$ and $h(x) = Kx$ for $x > 0$. It is easily computed from [St1] that

$$H(h) = K^*(h) = 1 + \frac{1}{2\pi^2}\log^2 K + \frac{1}{\pi}\log K\sqrt{1 + \frac{1}{4\pi^2}\log^2 K}\,.$$

In fact, let

$$f(z) = K^{1-1/\pi \arg z} z\,.$$

Then f is an extremal mapping in $Q(h)$. It was calculated in [S2] that

$$K(h) = \sup\{\Lambda(K\rho)/\Lambda(\rho)\,;\ \rho > 0\},$$

where $\Lambda(\rho)$ is the conformal module of the quadrilateral Q with domain $\boldsymbol{H}$ and vertices $\infty, -1, 0$ and ρ. It was also proved there that, when K is large,

$$K(h) < 1 + \frac{1}{\pi}\log K + \frac{1}{4\pi^2}\log^2 K\,.$$

Another approach due to Reich [Re] and developed in [CC] gave a necessary and sufficient condition for $K(h) = K^*(h)$. Recall that for any pair of disjoint nondegenerate continua A and B in $\bar{\boldsymbol{C}}$, there exists a unique real-valued function $u_{A,B}$, which is continuous in $\bar{\boldsymbol{C}}$ and harmonic in $\bar{\boldsymbol{C}} - A \cup B$, with constant values 0 and 1 in A and B, respectively, such that $\mathrm{mod}(A, B; \boldsymbol{C}) = \mathcal{D}_{\boldsymbol{C}}[u_{A,B}]$. Here $\mathcal{D}_\Omega[u]$ denotes the Dirichlet integral

$$\mathcal{D}_\Omega[u] = \iint_\Omega |\nabla u|^2 = 2\iint_\Omega (|u_z|^2 + |u_{\bar{z}}|^2)\,dxdy\,.$$

In what follows, we will abbreviate $\mathcal{D}_D[u]$ to $\mathcal{D}[u]$ for simplicity. When $A, B \subset \partial D$,

$$u_{A,B}|D = \frac{1}{2}(\phi_{A,B} + \bar{\phi}_{A,B})\,, \tag{2.4}$$

where $\phi_{A,B}$ is the conformal mapping of D onto

$$R_{A,B} = \{w = u + iv\,;\ \ 0 < u < 1, 0 < v < \mathrm{mod}(A, B; D)\}.$$

Note that for any pair $A, B \subset \partial D$,

$$\mathrm{mod}(A, B; D) = \iint_D |\phi'^2_{A,B}| = \mathcal{D}[u_{A,B}]\,.$$

For the relations between moduli and harmonic functions, we refer the reader to Gerhing [Ge] or Ahlfors [Ah, Chapter 4].

Under the above notation, we have

THEOREM B ([Re], [CC]). *Let $h : \partial D \to \partial D$ be a quasisymmetric homeomorphism. Then $K(h) = K^*(h)$ if and only if $Q(h)$ contains an extremal mapping whose complex dilatation μ satisfies*

$$\sup_{A,B\subset\partial D} \frac{\mathrm{Re}\iint_D \mu(z)\phi'^2_{A,B}(z)\,dxdy}{\iint_D |\phi'^2_{A,B}(z)|\,dxdy} = \|\mu\|_\infty\,. \tag{2.5}$$

Now, let $\mathcal{D}$ denote the set of all real-valued functions u which are harmonic in D with finite Dirichlet integral. We also denote by $\mathcal{A}D$ the set of all functions ϕ holomorphic in D with finite Dirichlet integral. For any $u \in \mathcal{D}$, let $P(u \circ h)$ denote the Poission integral of $u \circ h$. We define

$$K_1(h) = \sup_{A,B \subset \partial D} \frac{\mathcal{D}[P(u_{A,B} \circ h)]}{\mathcal{D}[u_{A,B}]} . \tag{2.6}$$

Noting that for all pairs $A, B \subset \partial D$, $u_{A,B}$ is the unique harmonic function with the minimal Dirichlet integral among all harmonic functions on the disk D with boundary values 0 and 1 on A and B, respectively, we conclude that

$$\mathrm{mod}(h^{-1}(A), h^{-1}(B); D) = \mathcal{D}[u_{h^{-1}(A),h^{-1}(B)}] \le \mathcal{D}[P(u_{A,B} \circ h)] ,$$

so it follows that $K(h) = K(h^{-1}) \le K_1(h)$. We also define

$$K_2(h) = \sup_{u \in \mathcal{D}} \frac{\mathcal{D}[P(u \circ h)]}{\mathcal{D}[u]} . \tag{2.7}$$

Note that the constant $K_2(h)$ (more precisely, $\max(K_2(h), K_2(h^{-1}))$) was already introduced by Beurling and Ahlfors in their famous paper [BA] and has been much investigated recently (see [KP], [NS], [P1–P5], [S1–S4]). In particular, it was pointed by the author [S4] that $K_2(h) = K_2(h^{-1})$ for all quasisymmetric homeomorphisms h, which implies that Schober's domain functionals are actually curve functionals (for more details, see [Sc], P. 379). Clearly, $K_1(h) \le K_2(h)$. On the other hand, by the quasi-invariance property of the Dirichlet integral under quasiconformal mappings, it holds that $K_2(h) \le K^*(h)$. Consequently, for any quasisymmetric homeomorphism h, it holds that

$$K(h) \le K_1(h) \le K_2(h) \le K^*(h) .$$

To determine when $K_2(h) = K^*(h)$, the author proved

THEOREM C [S1]. *Let $h : \partial D \to \partial D$ be a quasisymmetric homeomorphism. Then $K_2(h) = K^*(h)$ if and only if $Q(h)$ contains an extremal quasiconformal mapping whose Beltrami differential μ satisfies*

$$\sup_{\phi \in \mathcal{A}D} \frac{\mathrm{Re} \iint_D \mu(z)\phi'^2(z)dxdy}{\iint_D |\phi'^2(z)|dxdy} = \|\mu\|_\infty . \tag{2.8}$$

REMARK. Here it should be appropriate to point out a result of Shiga and Tanigawa. After the paper [S2] was published, the paper [ST] by Shiga and Tanigawa was called to the author's attention. In their paper, among other things, Shiga and Tanigawa proved that there exists a quasisymmetric homeomorphism h such that $H(h) = K^*(h)$, and that the relation (2.8) and consequently the relation (2.5) do not hold for any extremal quasiconformal mapping in $Q(h)$, which implies that $K(h) < K^*(h)$ by Theorem B. Therefore, this has already given an example h for which $H(h) = K^*(h)$ but $K(h) < K^*(h)$. However, this example was abstractly constructed and somewhat complicated (but has some further properties). Note that for this construction, $K_2(h) < K^*(h)$ by Theorem C, while for Strebel's quasisymmetric homeomorphism h in Example 1, it holds that $K_2(h) = K^*(h)$ (see [S2]).

Now we prove the following

THEOREM 1. *Let $h : \partial D \to \partial D$ be a quasisymmetric homeomorphism. Then the following hold.*

(1) *$K_1(h) = K^*(h)$ if and only if $Q(h)$ contains an extremal mapping whose complex dilatation μ satisfies the relation* (2.5).

(2) *If, in addition, $K_1(h)$ is attained by a pair of disjoint nondegenerate continua in ∂D, then $K_1(h) = K^*(h)$ if and only if h is induced by an affine mapping.*

PROOF. (1) Since $K(h) \le K_1(h) \le K^*(h)$, the if part follows directly from Theorem B.

Now suppose $K_1(h) = K^*(h)$. Then there exists a sequence of pairs of disjoint nondegenerate continua A_n and B_n in ∂D such that

$$\lim_{n\to\infty} \frac{\mathcal{D}[P(u_{A_n,B_n} \circ h)]}{\mathcal{D}[u_{A_n,B_n}]} = K^*(h)\,. \tag{2.9}$$

Set $\phi_n = \phi_{A_n,B_n}$, $u_n = u_{A_n,B_n}$. Then

$$D[u_{A_n,B_n}] = \iint_D |{\phi_n'}^2|\,. \tag{2.10}$$

For any extremal quasiconformal mapping $f \in Q(h)$,

$$(u_n \circ f)_z = \frac{1}{2}(\phi_n'(f(z)) f_z + \overline{\phi_n'(f(z))}\,\bar{f}_z)\,,$$
$$(u_n \circ f)_{\bar z} = \frac{1}{2}(\phi_n'(f(z)) f_{\bar z} + \overline{\phi_n'(f(z)) f_z})\,.$$

So we have

$$\begin{aligned}\mathcal{D}[u_n \circ f] &= 2\iint_D (|(u_n \circ f)_z|^2 + |(u_n \circ f)_{\bar z}|^2)dxdy \\ &= \iint_D ((|f_z|^2 + |f_{\bar z}|^2)|{\phi_n'}^2(f(z))| + 2\mathrm{Re}{\phi_n'}^2(f(z)) f_z f_{\bar z})dxdy \\ &= \iint_D \frac{(1+|\nu|^2)|{\phi_n'}^2| - 2\mathrm{Re}\nu{\phi_n'}^2}{1-|\nu|^2}dudv\,,\end{aligned} \tag{2.11}$$

where ν is the Beltrami differential of $g = f^{-1}$. Since $\mathcal{D}[u_n \circ f] \ge \mathcal{D}[P(u_n \circ h)]$, we obtain from (2.9) through (2.11)

$$\lim_{n\to\infty} \iint_D \frac{(1+|\nu|^2)|{\phi_n'}^2| - 2\mathrm{Re}\nu{\phi_n'}^2}{1-|\nu|^2}dudv \Big/ \iint_D |{\phi_n'}^2|dudv = K^*(h)\,,$$

which implies

$$\lim_{n\to\infty} \frac{\mathrm{Re}\iint_D \nu(-{\phi_n'}^2)dudv}{\iint_D |{\phi_n'}^2|dudv} = \|\nu\|_\infty\,. \tag{2.12}$$

Now, for a pair of disjoint nondegenerate continua A and B in ∂D, we denote by $\tilde{A}$ and $\tilde{B}$ the closure of the complementary components of $A \cup B$ on ∂D. Then

$$\phi_{\tilde{A},\tilde{B}} = 1 + \frac{i}{\operatorname{mod}(A, B; D)}\phi_{A,B}\,. \tag{2.13}$$

Consequently, (2.12) and (2.13) imply

$$\lim_{n\to\infty} \frac{\operatorname{Re}\iint_D \nu\phi'^2_{\tilde{A}_n,\tilde{B}_n}dudv}{\iint_D |\phi'^2_{\tilde{A}_n,\tilde{B}_n}|dudv} = \|\nu\|_\infty\,. \tag{2.14}$$

The if part of Theorem B then implies $K(h^{-1}) = K^*(h^{-1})$ and consequently $K(h) = K^*(h)$. By the only if part of Theorem B, we get the required conclusion.

(2) It suffices to prove that if $K_1(h)$ is attained by a pair of disjoint nondegenerate continua, say A and B, and that $K_1(h) = K^*(h)$, then h is induced by an affine mapping.

Indeed, we can deduce from the above proof (see (2.14)) that

$$\frac{\operatorname{Re}\iint_D \nu\phi'^2_{\tilde{A},\tilde{B}}dudv}{\iint_D |\phi'^2_{\tilde{A},\tilde{B}}|dudv} = \|\nu\|_\infty.$$

This forces that $\nu = \|\nu\|_\infty|\phi'^2_{\tilde{A},\tilde{B}}|/\phi'^2_{\tilde{A},\tilde{B}}$, which implies that h^{-1} and consequently that h is induced by an affine mapping.

REMARK. As stated above, for a general quasisymmetric homeomorphism h, it holds that

$$K(h) \le K_1(h) \le K_2(h) \le K^*(h)\,. \tag{2.15}$$

By Theorem C, it is known that there exists a large class of quasisymmetric homeomorphisms h such that the strict inequality $K_2(h) < K^*(h)$ holds. On the other hand, Theorem 4 in the next section implies that the strict inequality $K(h) < K_1(h)$ holds for all sewing mappings h of pairs of complementary asymptotically conformal extension domains other than disks. Now we point out that the strict inequality $K_1(h) < K_2(h)$ also holds for the Strebel's quasymmetric homeomorphism h in Example 1. Indeed, examining the proof in [S2], it is found that $K(h)$ is attained by a pair of disjoint nondegenerate continua, which implies, by Theorem B, Theorem 1, Proposition 1 and Theorem 3 in the next section, that the strict inequality $K(h) < K_1(h) < K_2(h) (= K^*(h))$ holds for all K. It seems that the strict inequality $K(h) < K_1(h) < K_2(h) < K^*(h)$ also holds for a single quasisymmetric homeomorphism h, but no example is known to this author.

3. The BQED constant $M_b(\Omega)$.

Let Ω be a QED domain in the extended complex plane $\bar{\mathbf{C}}$. Recall that $h_\Omega = f_2 \circ f_1^{-1}|\partial D$ is the sewing mapping of the domains Ω and Ω^*. In this section, we will prove some properties of the quantity $M_b(\Omega)$ and some relations of $M_b(\Omega)$ to the constants associated to h_Ω introduced in Section 2.

First we note

PROPOSITION 1. *$R(\Omega) = K^*(h_\Omega)$ and $K(h_\Omega) + 1 \le M_b(\Omega) \le K_1(h_\Omega^{-1}) + 1$.*

PROOF. Since there is a one to one correspondence between quasiconformal extensions of h_Ω and quasiconformal reflections in $\partial\Omega$, it follows easily that $R(\Omega) = K^*(h_\Omega)$. In fact, if f is a quasiconformal reflection in $\partial\Omega$, then $J \circ f_2 \circ f \circ f_1^{-1} \in Q(h_\Omega)$, where J is the conformal reflection in ∂D defined as $J(z) = 1/\bar{z}$.

Now, for any pair of disjoint nondegenerate continua A and B in $\partial\Omega$,

$$\operatorname{mod}(A, B; \boldsymbol{C}) \geq \operatorname{mod}(A, B; \Omega) + \operatorname{mod}(A, B; \Omega^*)\,.$$

So we obtain

$$\begin{aligned}\frac{\operatorname{mod}(A, B; \boldsymbol{C})}{\operatorname{mod}(A, B; \Omega)} &\geq 1 + \frac{\operatorname{mod}(A, B; \Omega^*)}{\operatorname{mod}(A, B; \Omega)} = 1 + \frac{\operatorname{mod}(f_2(A), f_2(B); D^*)}{\operatorname{mod}(f_1(A), f_1(B); D)} \\ &= 1 + \frac{\operatorname{mod}(h_\Omega \circ f_1(A), h_\Omega \circ f_1(B); D)}{\operatorname{mod}(f_1(A), f_1(B); D)}\,,\end{aligned}$$

from which $K(h_\Omega) + 1 \leq M_b(\Omega)$ follows.

On the other hand, by the uniqueness of harmonic functions,

$$\begin{aligned}\operatorname{mod}(A, B; \boldsymbol{C}) &= \iint_{\boldsymbol{C}} |\nabla u_{A,B}|^2 \\ &\leq \iint_\Omega |\nabla(u_{f_1(A), f_1(B)} \circ f_1)|^2 + \iint_{\Omega^*} |\nabla(P(u_{f_1(A), f_1(B)} \circ h_\Omega^{-1}) \circ J \circ f_2)|^2 \\ &= \mathcal{D}[u_{f_1(A), f_1(B)}] + \mathcal{D}[P(u_{f_1(A), f_1(B)} \circ h_\Omega^{-1})]\,.\end{aligned}$$

So it follows that

$$\begin{aligned}\frac{\operatorname{mod}(A, B; \boldsymbol{C})}{\operatorname{mod}(A, B; \Omega)} &= \frac{\operatorname{mod}(A, B; \boldsymbol{C})}{\operatorname{mod}(f_1(A), f_1(B); D)} = \frac{\operatorname{mod}(A, B; \boldsymbol{C})}{\mathcal{D}[u_{f_1(A), f_1(B)}]} \\ &\leq 1 + \frac{\mathcal{D}[P(u_{f_1(A), f_1(B)} \circ h_\Omega^{-1})]}{\mathcal{D}[u_{f_1(A), f_1(B)}]}\,,\end{aligned} \tag{3.1}$$

which implies $M_b(\Omega) \leq K_1(h_\Omega^{-1}) + 1$ as required.

Now we can prove

THEOREM 2. *Let Ω be a QED domain in the extended complex plane. Then the following hold.*

(1) $M_b(\Omega) = R(\Omega) + 1$ *if and only if $Q(h_\Omega)$ contains an extremal mapping whose complex dilatation μ satisfies the relation* (2.5).

(2) *If, in addition, $M_b(\Omega)$ is attained by a pair of disjoint nondegenerate continua in $\partial\Omega$, then $M_b(\Omega) = R(\Omega) + 1$ if and only if h_Ω is induced by an affine mapping.*

PROOF. (1) The assertion follows directly from Theorem B, Theorem 1 and Proposition 1.

(2) We only need to prove that if $M_b(\Omega)$ is attained by a pair of disjoint nondegenerate continua, say A and B, then $M_b(\Omega) = R(\Omega) + 1$ implies that h_Ω is induced by an affine mapping.

By (3.1) we have

$$M_b(\Omega) = \frac{\operatorname{mod}(A, B; \mathbf{C})}{\operatorname{mod}(A, B; \Omega)} \le 1 + \frac{\mathcal{D}[P(u_{f_1(A), f_1(B)} \circ h_\Omega^{-1})]}{\mathcal{D}[u_{f_1(A), f_1(B)}]}$$
$$\le 1 + K_1(h_\Omega^{-1}) \le 1 + K^*(h_\Omega^{-1}) = 1 + R(\Omega) = M_b(\Omega)\,,$$

which implies that $K_1(h_\Omega^{-1})$ is attained by the pair $f_1(A)$ and $f_1(B)$ and that $K_1(h_\Omega^{-1}) = K^*(h_\Omega^{-1})$. We conclude by Theorem 1(2) that h_Ω^{-1} and consequently h_Ω are induced by affine mappings.

An immediate consequence of Theorems B, 1 and 2, and Proposition 1 is the following

COROLLARY 1. *Let Ω be a QED domain in the extended complex plane. Then the following conditions are all equivalent*:

(1) $M_b(\Omega) = R(\Omega) + 1$.

(2) $K(h_\Omega) = K^*(h_\Omega)$.

(3) $K_1(h_\Omega) = K^*(h_\Omega)$.

(4) *$Q(h_\Omega)$ contains an extremal mapping whose complex dilatation μ satisfies the relation* (2.5).

For $n \ge 1$, let A_n and B_n be a pair of disjoint nondegenerate continua in $\bar{\mathbf{C}}$ such that A_n and B_n converge in the Hausdorff metric to continua A and B, respectively. We say (A_n, B_n) is degenerate if the pair (A, B) is degenerate. Now, for any pair of disjoint nondegererate continua A and B in ∂D joined by z_1, z_2 and z_3, z_4, respectively, by the well-known Christoffel-Schwarz formula, we have

$$\psi_{A,B} =: \frac{\phi'^2_{A,B}}{\iint_D |\phi'^2_{A,B}|}$$
$$= -\frac{(z_1 - z_2)(z_3 - z_4)}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)} \Bigg/ \iint_D \frac{|(z_1 - z_2)(z_3 - z_4)|}{|(z - z_1)(z - z_2)(z - z_3)(z - z_4)|}\ .$$

So, if (A_n, B_n) is degenerate, then (ψ_{A_n,B_n}) is degenerate in the sense that $\psi_{A_n,B_n} \to 0$ locally uniformly in D.

The following corollary is an immediate consequence of Theorems A, B and 2. Here, by the extremal quasiconformal mapping theory, we give a simple proof using our Theorem 2.

COROLLARY 2. *Let Ω be a QED domain in the extended complex plane. If $M_b(\Omega) = R(\Omega) + 1$, then either h_Ω is induced by an affine mapping or $H(h_\Omega) = K^*(h_\Omega)$.*

PROOF. Let $M_b(\Omega) = R(\Omega) + 1$. If $M_b(\Omega)$ is attained by a pair of disjoint nondegenerate continua in $\partial\Omega$, then the second part of Theorem 2 implies that h_Ω is induced by an affine mapping. Otherwise, we conclude by Theorem 2(1) that there exists a degenerate sequence (A_n, B_n) of pairs of disjoint nondegenerate continua in ∂D such that

$$\lim_{n\to\infty} \operatorname{Re} \iint_D \mu\psi_{A_n,B_n} = \|\mu\|_\infty\,, \tag{3.2}$$

where μ is the Beltrami differential of an extremal mapping in $Q(h_\Omega)$. Since (A_n, B_n) is degenerate, (ψ_{A_n,B_n}) is degenerate. By the theory of extremal quasiconformal mappings (see Strebel [St3]), we can conclude from (3.2) that $H(h_\Omega) = K^*(h_\Omega)$.

REMARK. Let Ω be a QED domain in the extended complex plane. If $H(h_\Omega) < K^*(h_\Omega)$, and h_Ω is not induced by an affine mapping, then $M_b(\Omega) < R(\Omega) + 1$. So, if h is a Strebel point which is not induced by an affine mapping, then $M_b(\Omega_h) < R(\Omega_h) + 1$. Consequently, by the density of Strebel points [La], we conclude that there exists a large class of domains for which $M_b(\Omega) < R(\Omega) + 1$. On the other hand, Example 1 shows that there still exists a domain Ω for which $H(h_\Omega) = K^*(h_\Omega)$, but $M_b(\Omega) < R(\Omega) + 1$. So the converse of Corollary 2 is not ture.

In the rest of the section, we give some relations between $M_b(\Omega)$ and $K(h_\Omega)$. When h_Ω is induced by an affine mapping, it holds that $M_b(\Omega) = K(h_\Omega) + 1$. The following theorem claims that the converse is true if $K(h_\Omega)$ is in addition attained by a pair of disjoint nondegenerate continua.

THEOREM 3. *Let Ω be a QED domain in the extended complex plane. If A and B is a pair of disjoint nondegenerate continua in ∂D which attains the supremum in* (2.3), *that is, $K(h_\Omega) = \mathrm{mod}(h_\Omega(A), h_\Omega(B); D)/\mathrm{mod}(A, B; D)$, then $M_b(\Omega) = K(h_\Omega) + 1$ if and only if h_Ω is induced by an affine mapping.*

PROOF. It suffices to prove that if $M_b(\Omega) = K(h_\Omega) + 1$, then h_Ω is induced by an affine mapping. For simplicity, we set $h_\Omega = h$.

Since we obtain

$$\begin{aligned} 1 + K(h) = M_b(\Omega) &\geq \frac{\mathrm{mod}(f_1^{-1}(A), f_1^{-1}(B); \boldsymbol{C})}{\mathrm{mod}(f_1^{-1}(A), f_1^{-1}(B); \Omega)} \\ &\geq \frac{\mathrm{mod}(f_1^{-1}(A), f_1^{-1}(B); \Omega) + \mathrm{mod}(f_1^{-1}(A), f_1^{-1}(B); \Omega^*)}{\mathrm{mod}(f_1^{-1}(A), f_1^{-1}(B); \Omega)} \\ &= 1 + \frac{\mathrm{mod}(h(A), h(B); D)}{\mathrm{mod}(A, B; D)} = 1 + K(h)\,, \end{aligned}$$

it follows that

$$M_b(\Omega) = \frac{\mathrm{mod}(f_1^{-1}(A), f_1^{-1}(B); \boldsymbol{C})}{\mathrm{mod}(f_1^{-1}(A), f_1^{-1}(B); \Omega)}\,.$$

Consequently, we obtain

$$\begin{aligned} \mathrm{mod}(f_1^{-1}(A), f_1^{-1}(B); \boldsymbol{C}) &= \iint_{\boldsymbol{C}} |\nabla u_{f_1^{-1}(A), f_1^{-1}(B)}|^2 \\ &= \iint_{\Omega} |\nabla u_{f_1^{-1}(A), f_1^{-1}(B)}|^2 + \iint_{\Omega^*} |\nabla u_{f_1^{-1}(A), f_1^{-1}(B)}|^2 \\ &= \iint_{D} |\nabla (u_{f_1^{-1}(A), f_1^{-1}(B)} \circ f_1^{-1})|^2 + \iint_{D^*} |\nabla (u_{f_1^{-1}(A), f_1^{-1}(B)} \circ f_2^{-1})|^2 \end{aligned}$$

$$\geq \iint_D |\nabla u_{A,B}|^2 + \iint_{D^*} |\nabla u_{h(A),h(B)}|^2$$
$$= \mathrm{mod}(A, B; D) + \mathrm{mod}(h(A), h(B); D)$$
$$= (1 + K(h))\mathrm{mod}(A, B; D) = M_b(\Omega)\mathrm{mod}(f_1^{-1}(A), f_1^{-1}(B); \Omega)$$
$$= \mathrm{mod}(f_1^{-1}(A), f_1^{-1}(B); \boldsymbol{C}) .$$

Then

$$u_{f_1^{-1}(A), f_1^{-1}(B)} \circ f_1^{-1} = u_{A,B} ,$$
$$u_{f_1^{-1}(A), f_1^{-1}(B)} \circ f_2^{-1} = u_{h(A),h(B)} .$$

Noting that $u_{f_1^{-1}(A), f_1^{-1}(B)}$ is continuous in $\bar{\boldsymbol{C}}$, we conclude that $u_{h(A),h(B)} \circ f_2 = u_{A,B} \circ f_1$ on $\partial\Omega$ and so

$$u_{h(A),h(B)} \circ h = u_{A,B} .$$

Thus

$$\mathrm{Re}(\phi_{h(A),h(B)} \circ h - \phi_{A,B}) = 0 .$$

By the mapping properties of $\phi_{h(A),h(B)}$ and $\phi_{A,B}$, it follows that for $w = u + iv \in \phi_{A,B}(\tilde{A} \cup \tilde{B})$,

$$\phi_{h(A),h(B)} \circ h \circ \phi_{A,B}^{-1}(w) = f_{K(h)}(w) = u + iK(h)v , \tag{3.3}$$

where, as before, $\tilde{A}$ and $\tilde{B}$ are the closure of the complementary components of $A \cup B$ on ∂D.

On the other hand, since

$$K(h) = \frac{\mathrm{mod}(h(A), h(B); D)}{\mathrm{mod}(A, B; D)} = \frac{\mathrm{mod}(\tilde{A}, \tilde{B}; D)}{\mathrm{mod}(h(\tilde{A}), h(\tilde{B}); D)}$$
$$= \frac{\mathrm{mod}(h^{-1} \circ h(\tilde{A}), h^{-1} \circ h(\tilde{B}); D)}{\mathrm{mod}(h(\tilde{A}), h(\tilde{B}); D)} ,$$

repeating the above reasoning, we obtain for $w \in \phi_{h(\tilde{A}),h(\tilde{B})}(h(A), h(B))$,

$$\phi_{\tilde{A},\tilde{B}} \circ h^{-1} \circ \phi_{h(\tilde{A}),h(\tilde{B})}^{-1}(w) = f_{K(h^{-1})}(w) = f_{K(h)}(w) . \tag{3.4}$$

Noting that

$$\phi_{\tilde{A},\tilde{B}} = 1 + \frac{i}{\mathrm{mod}(A, B; D)}\phi_{A,B} ,$$
$$\phi_{h(\tilde{A}),h(\tilde{B})} = 1 + \frac{i}{K(h)\mathrm{mod}(A, B; D)}\phi_{h(A),h(B)} ,$$

a direct computation from (3.4) yields for $w \in \phi_{A,B}(A \cup B)$,

$$\phi_{h(A),h(B)} \circ h \circ \phi_{A,B}^{-1}(w) = f_{K(h)}(w) = u + iK(h)v . \tag{3.5}$$

Consequently, (3.3) and (3.5) imply that $h_\Omega = h$ is induced by an affine mapping.

Now we give an application of Theorem 3 to asymptotically conformal extension domains. As will be seen in Theorem 4, we have a rather satisfactory description of these various invariants in this case. Before observing this, we establish some preliminary results.

We first prove

LEMMA 1. *Let $h : \partial D \to \partial D$ be a quasisymmetric homeomorphism. If (A_n, B_n) is a degenerate sequence of pairs of disjoint nondegenerate continua in ∂D, then*

$$\limsup_{n\to\infty} \frac{\mathcal{D}[P(u_{A_n,B_n} \circ h)]}{\mathcal{D}[u_{A_n,B_n}]} \le H(h).$$

PROOF. Since (A_n, B_n) is degenerate, $\psi_{A_n,B_n} = \phi'^2_{A_n,B_n} / \iint_D |\phi'^2_{A_n,B_n}| \to 0$ locally uniformly in D.

For any $\varepsilon > 0$, choose some quasiconformal mapping $f \in Q(h)$ and some compact subset E of D such that $K[f|D-E] \le H(h) + \varepsilon$. Then, by (2.11),

$$\begin{aligned}
\mathcal{D}[u_n \circ f] &= \iint_D ((|f_z|^2 + |f_{\bar z}|^2)|\phi'^2_{A_n,B_n}(f(z))| + 2\mathrm{Re}\phi'^2_{A_n,B_n}(f(z)) f_z f_{\bar z}) dxdy \\
&\le \iint_D (|f_z| + |f_{\bar z}|)^2 |\phi'^2_{A_n,B_n}(f(z))| dxdy \\
&= \iint_D \frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|} \circ f^{-1} |\phi'^2_{A_n,B_n}| dudv \\
&\le K[f] \iint_{f(E)} |\phi'^2_{A_n,B_n}| dudv + (H(h)+\varepsilon) \iint_{D-f(E)} |\phi'^2_{A_n,B_n}| dudv .
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\frac{\mathcal{D}[P(u_{A_n,B_n} \circ h)]}{\mathcal{D}[u_{A_n,B_n}]} &\le \frac{\mathcal{D}[u_{A_n,B_n} \circ f]}{\mathcal{D}[u_{A_n,B_n}]} \\
&\le \frac{(H(h)+\varepsilon) \iint_{D-f^{-1}(E)} |\phi'^2_{A_n,B_n}| + K[f] \iint_{f^{-1}(E)} |\phi'^2_{A_n,B_n}|}{\iint_D |\phi'^2_{A_n,B_n}|} \\
&\le H(h) + \varepsilon + K[f] \iint_{f^{-1}(E)} |\psi_{A_n,B_n}| \to H(h) + \varepsilon .
\end{aligned}$$

Since ε is arbitrary, it follows that

$$\limsup_{n\to\infty} \frac{\mathcal{D}[P(u_{A_n,B_n} \circ h)]}{\mathcal{D}[u_{A_n,B_n}]} \le H(h) .$$

Some immediate consequences of Lemma 1 are the following propositions. The first one was the main result proved by Wu [Wu, Theorem 1] and implicit in Yang [Y2], and it was used to derive their necessary condition Theorem A. Both discussions in [Wu] and [Y2] are somewhat complicated.

PROPOSITION 2 ([Wu], [Y2]). *Let $h : \partial D \to \partial D$ be a quasisymmetric homeomorphism. Then either there exists a pair of disjoint nondegenerate continua A and B in ∂D such that $K(h) = \mathrm{mod}(h(A), h(B); D)/\mathrm{mod}(A, B; D)$ or $K(h) \le H(h)$.*

PROOF. For $n \geq 1$, let A_n and B_n be a pair of disjoint nondegenerate continua on ∂D such that

$$K(h) = \lim_{n\to\infty} \frac{\mathrm{mod}(h(A_n), h(B_n); D)}{\mathrm{mod}(A_n, B_n; D)} \tag{3.6}$$

and that A_n and B_n converge in the Hausdorff metric to continua A and B, respectively.

If A and B is a pair of disjoint nondegenerate continua, by the continuity of moduli, it follows from (3.6) that $K(h) = \mathrm{mod}(h(A), h(B); D)/\mathrm{mod}(A, B; D)$.

If (A_n, B_n) is degenerate, by Lemma 1 we get

$$\limsup_{n\to\infty} \frac{\mathcal{D}[P(u_{A_n,B_n} \circ h^{-1})]}{\mathcal{D}[u_{A_n,B_n}]} \leq H(h^{-1}) = H(h)\,. \tag{3.7}$$

On the other hand, since

$$\frac{\mathrm{mod}(h(A_n), h(B_n); D)}{\mathrm{mod}(A_n, B_n; D)} \leq \frac{\mathcal{D}[P(u_{A_n,B_n} \circ h^{-1})]}{\mathcal{D}[u_{A_n,B_n}]}\,,$$

it follows from (3.6) and (3.7) that $K(h) \leq H(h)$.

PROPOSITION 3. *Let Ω be a QED domain in the extended complex plane. Then either there exists a pair of disjoint nondegenerate continua A an B in $\partial\Omega$ such that $M_b(\Omega) = \mathrm{mod}(A, B; \mathbf{C})/\mathrm{mod}(A, B; \Omega)$ or $M_b(\Omega) \leq H(h_\Omega) + 1$.*

PROOF. For $n \geq 1$, let A_n and B_n be a pair of disjoint nondegenerate continua on $\partial\Omega$ such that

$$M_b(\Omega) = \lim_{n\to\infty} \frac{\mathrm{mod}(A_n, B_n; \mathbf{C})}{\mathrm{mod}(A_n, B_n; \Omega)} \tag{3.8}$$

and that A_n and B_n converge in the Hausdorff metric to continua A and B, respectively.

If (A_n, B_n) is nondegenerate, it follows from the continuity of moduli that

$$M_b(\Omega) = \frac{\mathrm{mod}(A, B; \mathbf{C})}{\mathrm{mod}(A, B; \Omega)}\,.$$

Now we suppose that (A_n, B_n) is degenerate. Then $(f_1(A_n), f_1(B_n))$ is also degenerate. So Lemma 1 implies that

$$\limsup_{n\to\infty} \frac{\mathcal{D}[P(u_{f_1(A_n), f_1(B_n)} \circ h_\Omega^{-1})]}{\mathcal{D}[u_{f_1(A_n), f_1(B_n)}]} \leq H(h_\Omega)\,. \tag{3.9}$$

On the other hand, by (3.1) we have

$$\frac{\mathrm{mod}(A_n, B_n; \mathbf{C})}{\mathrm{mod}(A_n, B_n; \Omega)} \leq 1 + \frac{\mathcal{D}[P(u_{f_1(A_n), f_1(B_n)} \circ h_\Omega^{-1})]}{\mathcal{D}[u_{f_1(A_n), f_1(B_n)}]}\,. \tag{3.10}$$

It then follows from (3.8), (3.9) and (3.10) that $M_b(\Omega) \leq H(h_\Omega) + 1$.

PROPOSITION 4. *Let $h : \partial D \to \partial D$ be a quasisymmetric homeomorphism. Then either there exists a pair of disjoint nondegenerate continua A and B in ∂D such that $K_1(h) = \mathcal{D}[P(u_{A,B} \circ h)]/\mathcal{D}[u_{A,B}]$ or $K_1(h) \leq H(h)$.*

PROOF. For $n \geq 1$, let A_n and B_n be a pair of disjoint nondegenerate continua on ∂D such that

$$K_1(h) = \lim_{n\to\infty} \frac{\mathcal{D}[P(u_{A_n,B_n} \circ h)]}{\mathcal{D}[u_{A_n,B_n}]} \tag{3.11}$$

and that A_n and B_n converge in the Hausdorff metric to continua A and B, respectively.

If A and B is a pair of disjoint nondegenerate continua, by the continuity of moduli, $\mathrm{mod}(A_n, B_n; D) \to \mathrm{mod}(A, B; D)$, that is, $\mathcal{D}[u_{A_n,B_n}] \to \mathcal{D}[u_{A,B}]$. So we obtain

$$\lim_{n\to\infty} \mathcal{D}[u_{A_n,B_n} - u_{A,B}] = \lim_{n\to\infty} \mathcal{D}[u_{A_n,B_n}] - \mathcal{D}[u_{A,B}] = 0\,.$$

Since

$$\mathcal{D}[P((u_{A_n,B_n} - u_{A,B}) \circ h)] \leq K^*(h)\mathcal{D}[u_{A_n,B_n} - u_{A,B}]\,,$$

we obtain $\mathcal{D}[P((u_{A_n,B_n} - u_{A,B}) \circ h)] \to 0$, which implies that $\mathcal{D}[P(u_{A_n,B_n} \circ h)] \to \mathcal{D}[P(u_{A,B} \circ h)]$. Consequently, by (3.11) we obtain $K_1(h) = \mathcal{D}[P(u_{A,B} \circ h)]/\mathcal{D}[u_{A,B}]$.

If (A_n, B_n) is degenerate, by Lemma 1 and (3.11) we get $K_1(h) \leq H(h)$.

For completeness, we recall the following analogous result for the quantity $K_2(h)$ proved by the author [S4].

PROPOSITION 5 [S4]. Let $h : \partial D \to \partial D$ be a quasisymmetric homeomorphism. Then either there exists an element $u \in \mathcal{D}$ such that $K_2(h) = \mathcal{D}[P(u \circ h)]/\mathcal{D}[u]$ or $K_2(h) \leq H(h)$.

Recall that a Jordan domain Ω is an asymptotically conformal extension domain if the conformal mapping $f_1^{-1} : D \to \Omega$ has a quasiconformal extension to a neighborhood of D whose complex dilatation μ satisfies $|\mu(z)| \to 0$ as $|z| \to 1+$. It is known from Gardiner-Sullivan [GS] that Ω is an asymptotically conformal extension domain if and only if $H(h_\Omega) = 1$, namely, h_Ω is symmetric. Note that for an asymptotically conformal extension domain Ω, h_Ω can not be induced by affine mappings unless Ω is a disk.

By means of Corollary 1, Theorem 3 and Propositions 1 through 5, we obtain

THEOREM 4. *Let Ω be an asymptotically conformal extension domain. Then all the supremums in* (1.4), (2.3), (2.6) *and* (2.7) *can be attained. Namely, the following hold.*

(1) *There exists a pair of disjoint nondegenerate continua A_1 and B_1 in ∂D such that* $K(h_\Omega) = \mathrm{mod}(h_\Omega(A_1), h_\Omega(B_1); D)/\mathrm{mod}(A_1, B_1; D)$.

(2) *There exists a pair of disjoint nondegenerate continua A_2 and B_2 in $\partial\Omega$ such that* $M_b(\Omega) = \mathrm{mod}(A_2, B_2; \mathbf{C})/\mathrm{mod}(A_2, B_2; \Omega)$.

(3) *There exists a pair of disjoint nondegenerate continua A_3 and B_3 in ∂D such that* $K_1(h_\Omega) = \mathcal{D}[P(u_{A_3,B_3} \circ h_\Omega)]/\mathcal{D}[u_{A_3,B_3}]$.

(4) *There exists an element $u \in \mathcal{D}$ such that* $K_2(h_\Omega) = \mathcal{D}_D[P(u \circ h_\Omega)]/\mathcal{D}_D[u]$.

Furthermore, $K(h_\Omega) < M_b(\Omega) - 1 \leq K_1(h_\Omega) < K^*(h_\Omega) = R(\Omega)$ unless Ω is a disk.

REMARK. As stated in Section 1, Wu and Yang [WY, Theorem 2.3] proved that $M_b(\Omega) < R(\Omega) + 1$ for all asymptotically conformal extension domains other than disks. On the other hand, Wu [Wu, Theorem 4] and Yang [Y2, Corollary 2.6] proved independently that $K(h_\Omega) < K^*(h_\Omega)$ for all asymptotically conformal extension domains other than disks. Theorem 4 implies the stronger result that $K(h_\Omega) < M_b(\Omega)-1 < K^*(h_\Omega)$ for such domains.

4. The QED constant $M(\Omega)$. In this section, we will prove, among other things, that $M(\Omega) < R(\Omega)+1$ for all asymptotically conformal extension domains other than disks. Recall that for a pair of disjoint nondegenerate compact subsets (which need not be connected) A and B of the extended complex plane, we may define $\operatorname{mod}(A, B; \boldsymbol{C})$ as before. Furthermore, there still exists a real-valued function $u_{A,B}$, which is continuous in $\bar{\boldsymbol{C}}$, harmonic in $\bar{\boldsymbol{C}} - A \cup B$, with constant values 0 and 1 in A and B, respectively, such that $\operatorname{mod}(A, B; \boldsymbol{C}) = \mathcal{D}_{\boldsymbol{C}}[u_{A,B}]$.

First we note the following

THEOREM 5. *Let Ω be a QED domain in the extended complex plane. If $M(\Omega)$ is attained by a pair of disjoint nondegenerate continua A and B in $\bar{\Omega}$, then $M(\Omega) = R(\Omega)+1$ if and only if h_Ω is induced by an affine mapping.*

PROOF. We need to prove that if $M(\Omega) = R(\Omega) + 1$, then h_Ω is induced by an affine mapping. As done in [Y3], let $f : \bar{\boldsymbol{C}} \to \bar{\boldsymbol{C}}$ be a quasiconformal mapping such that f is conformal from Ω onto D and is $R(\Omega)$-quasiconformal in Ω^*. Set $A' = f(A)$, $B' = f(B)$ and $g = f^{-1}$. By the uniqueness of harmonic functions, it follows that

$$
\begin{aligned}
&\operatorname{mod}(A, B; \boldsymbol{C}) \le \operatorname{mod}(A \cup g \circ J(A'), B \cup g \circ J(B'); \boldsymbol{C}) \\
&\quad = \iint_{\boldsymbol{C}} |\nabla u_{A \cup g\circ J(A'), B \cup g \circ J(B')}|^2 \le \iint_{\boldsymbol{C}} |\nabla (u_{A' \cup J(A'), B' \cup J(B')} \circ f)|^2 \\
&\quad \le \iint_{D} |\nabla (u_{A' \cup J(A'), B' \cup J(B')})|^2 + R(\Omega) \iint_{D^*} |\nabla (u_{A' \cup J(A'), B' \cup J(B')})|^2 \\
&\quad = (1 + R(\Omega)) \operatorname{mod}(A', B'; D) = (1 + R(\Omega)) \operatorname{mod}(A, B; \Omega)\,.
\end{aligned}
\tag{4.1}
$$

Since $\operatorname{mod}(A, B; \boldsymbol{C}) = (1 + R(\Omega))\operatorname{mod}(A, B; \Omega)$, it follows from (4.1) that $A = A \cup g \circ J(A')$, $B = B \cup g \circ J(B')$. So A and B must lie on the boundary $\partial\Omega$. We conclude by Theorem 2(2) that h_Ω is induced by an affine mapping.

Now we can state the main result of this section.

THEOREM 6. *Let Ω be an asymptotically conformal extension domain. Then there exists a pair of disjoint nondegenerate continua A and B in $\bar{\Omega}$ such that $M(\Omega) = \operatorname{mod}(A, B; \boldsymbol{C}) / \operatorname{mod}(A, B; \Omega)$. Furthermore, $M(\Omega) < R(\Omega) + 1$ unless Ω is a disk.*

As stated in Introduction, Yang [Y4] proved Theorem 6 in the case when Ω is a smooth domain and asked whether it still holds when Ω is a general asymptotically conformal extension domain. Our Theorem 6 gives this an affirmative answer, showing that the conjecture $M(\Omega) = R(\Omega) + 1$ by Garnett and Yang [GY] is not true for all asymptotically conformal extension domains other than disks.

Theorem 6 is an immediate consequence of Theorem 5 and the following Theorem 7, a generalization of Theorem 6. The proof of Theorem 7 relies on some well-known facts which may be stated as follows. For any $\rho > 0$, let $\Lambda(\rho)$ denote, as before, the conformal module of the quadrilateral Q with domain the upper half plane $\boldsymbol{H}$ and vertices $\infty, -1, 0$ and ρ. It is well-known (see [BA]) that $\Lambda(\rho)\Lambda(\rho^{-1}) = 1$. Furthermore, when $\rho \geq 1$,

$$\Lambda(\rho) = 1 + \theta(\rho)\log\rho\,, \tag{4.2}$$

where $\theta(\rho)$ increases monotonically from $\theta(1) = 0.2284$ to $\theta(\infty) = 1/\pi = 0.3183$. Now, for a ring domain R, its conformal module $M(R)$ is defined as $M(R) = \log(r_2/r_1)$, if R is mapped conformally onto $\{r_1 < |z| < r_2\}$. In particular, for the Teichmüller ring domain $R(\rho)$ bounded by the segment $[-1, 0]$ and by the ray $\{z = x\,;\ \rho \leq x < \infty\}$, we have $M(R(\rho)) = \pi\Lambda(\rho)$. On the other hand, for a ring domain R with complementay components A and B, we have $\text{mod}(A, B; \boldsymbol{C}) = 2\pi/M(R)$.

Now we prove

THEOREM 7. *Let Ω be a QED domain in the extended complex plane. Then either there exists a pair of disjoint nondegenerate continua A an B in $\bar{\Omega}$ such that $M(\Omega) = \text{mod}(A, B; \boldsymbol{C})/\text{mod}(A, B; \Omega)$ or $M(\Omega) \leq 2H(h_\Omega)$.*

PROOF. For $n \geq 1$, let A_n and B_n be a pair of disjoint nondegenerate continua on $\bar{\Omega}$ such that

$$M(\Omega) = \lim_{n\to\infty} \frac{\text{mod}(A_n, B_n; \boldsymbol{C})}{\text{mod}(A_n, B_n; \Omega)} \tag{4.3}$$

and that A_n and B_n converge in the Hausdorff metric to continua A and B, respectively.

If (A_n, B_n) is not degenerate, then by the continuity of moduli, it follows from (4.3) that

$$M(\Omega) = \frac{\text{mod}(A, B; \boldsymbol{C})}{\text{mod}(A, B; \Omega)}\,.$$

Now we suppose that (A_n, B_n) is degenerate. Then, as in [Y3], there are the following five possibilities, depending on the sizes and relative positions of A and B:

(1) A is a single point, B is a nondegenerate continua and $A \cap B = \emptyset$.

(2) A, B both are single points and $A \cap B = \emptyset$.

(3) A is a single point, B is nondegenerate and $A \cap B \neq \emptyset$.

(4) A, B both are single points and $A \cap B \neq \emptyset$.

(5) A, B both are nondegenerate and $A \cap B \neq \emptyset$.

In all these cases, we will show that $M(\Omega) \leq 2H(h_\Omega)$. We adapt the strategy used in [Y3] (see also [WY], [Y4]). For simplicity, we also set $H = H(h_\Omega)$.

In what follows, $D(z_0, r)$ will denote the disk with center z_0 and radius $r > 0$, $D_r = D(0, r)$. For any $\varepsilon > 0$, there exist a $R > 1$ and a quasiconformal mapping g from D_R onto a Jordan domain Ω' such that $K[g] < H + \varepsilon$, and g is conformal from D onto Ω. Set $f = g^{-1}$, $A'_n = f(A_n)$ and $B'_n = f(B_n)$, where A_n and B_n are as in (4.3).

Case (1) Since

$$\text{mod}(A_n, B_n; \Omega) = \text{mod}(A'_n, B'_n; D) \geq \frac{1}{2}\text{mod}(A'_n, B'_n; \boldsymbol{C})\,,$$

we have

$$\frac{\operatorname{mod}(A_n, B_n; \boldsymbol{C})}{\operatorname{mod}(A_n, B_n; \Omega)} \le 2\frac{\operatorname{mod}(A_n, B_n; \boldsymbol{C})}{\operatorname{mod}(A_n', B_n'; \boldsymbol{C})}\,. \tag{4.4}$$

Now we show

$$\limsup_{n\to\infty} \frac{\operatorname{mod}(A_n, B_n; \boldsymbol{C})}{\operatorname{mod}(A_n', B_n'; \boldsymbol{C})} \le H + \varepsilon\,. \tag{4.5}$$

Choose $a_n, b_n \in A_n$ and $c_n, d_n \in B_n$ such that

$$|b_n - c_n| = d(A_n, B_n),\quad |a_n - b_n| = \max_{z\in A_n}|z - b_n|,\quad |c_n - d_n| = \max_{z\in B_n}|c_n - z|\,. \tag{4.6}$$

Using the basic properties of the modulus, we obtain

$$\operatorname{mod}(A_n', B_n'; \boldsymbol{C}) = \frac{2\pi}{M(\bar{\boldsymbol{C}} - A_n' \cup B_n')} \ge \frac{2}{\Lambda[f(a_n), f(b_n), f(c_n), f(d_n)]}\,, \tag{4.7}$$

$$\operatorname{mod}(A_n, B_n; \boldsymbol{C}) = \frac{2\pi}{M(\bar{\boldsymbol{C}} - A_n \cup B_n)} \le \frac{2\pi}{\log(|b_n - c_n|/|b_n - a_n|)}\,, \tag{4.8}$$

where

$$[z_1, z_2, z_3, z_4] = \frac{|z_4 - z_1||z_3 - z_2|}{|z_4 - z_3||z_2 - z_1|}\,.$$

Noting that $\Lambda(\rho) \sim (1/\pi)\log\rho$ as $\rho \to \infty$, we obtain as $n \to \infty$

$$\frac{\operatorname{mod}(A_n, B_n; \boldsymbol{C})}{\operatorname{mod}(A_n', B_n'; \boldsymbol{C})} \le \frac{\pi\Lambda[f(a_n), f(b_n), f(c_n), f(d_n)]}{\log(|b_n - c_n|/|b_n - a_n|)} \sim \frac{\log(1/|f(b_n) - f(a_n)|)}{\log(1/|b_n - a_n|)}\,. \tag{4.9}$$

Now, since f is $H + \varepsilon$-quasiconformal in Ω', by the Hölder continuity of quasiconformal mappings, f is Hölder continuous in Ω' with Hölder index $1/(H + \varepsilon)$ and a coefficient depending only on Ω' and f. We deduce from (4.9) that

$$\frac{\operatorname{mod}(A_n, B_n; \boldsymbol{C})}{\operatorname{mod}(A_n', B_n'; \boldsymbol{C})} \le H + \varepsilon$$

as $n \to \infty$, which yields (4.5). By the arbitrariness of ε, we get from (4.4), (4.5) that $M(\Omega) \le 2H$.

Case (2) This case can be treated similarly as in Case 1. In this case, to establish (4.5), we need the following estimates instead of (4.8):

$$\begin{aligned}\operatorname{mod}(A_n, B_n; \boldsymbol{C}) &= \frac{2\pi}{M(\bar{\boldsymbol{C}} - A_n \cup B_n)}\\ &\le \frac{2\pi}{\log(|b_n - c_n|/(2|b_n - a_n|)) + \log(|b_n - c_n|/(2|d_n - c_n|))}\,.\end{aligned} \tag{4.10}$$

So

$$\begin{aligned}\frac{\operatorname{mod}(A_n, B_n; \boldsymbol{C})}{\operatorname{mod}(A'_n, B'_n; \boldsymbol{C})} &\le \frac{\pi \Lambda[f(a_n), f(b_n), f(c_n), f(d_n)]}{\log(|b_n - c_n|/(2|b_n - a_n|)) + \log(|b_n - c_n|/(2|d_n - c_n|))} \\ &\sim \frac{\log(1/|f(b_n) - f(a_n)|) + \log(1/|f(d_n) - f(c_n)|)}{\log(1/|b_n - a_n|) + \log(1/|d_n - c_n|)} \\ &\to H + \varepsilon\,. \end{aligned} \tag{4.11}$$

Case (3) Choose $a'_n, b'_n \in A'_n$ and $c'_n, d'_n \in B'_n$ such that

$$|b'_n - c'_n| = d(A'_n, B'_n), \quad |a'_n - b'_n| = \max_{z \in A'_n} |z - b'_n|, \quad |c'_n - d'_n| = \max_{z \in B'_n} |c'_n - z|\,. \tag{4.12}$$

We divide our argument into two subcases:

Subcase 3.1 There exists some constant ρ_0 such that $[a'_n, b'_n, c'_n, d'_n] < \rho_0$.

In this case, we have

$$\operatorname{mod}(A'_n, B'_n; \boldsymbol{C}) = \frac{2\pi}{M(\bar{\boldsymbol{C}} - A'_n \cup B'_n)} \ge \frac{2}{\Lambda[a'_n, b'_n, c'_n, d'_n]} \ge \frac{2}{\Lambda(\rho_0)}\,.$$

Since the quasiconformal mapping $g : D_R \to \Omega'$ can be extended to a quasiconformal mapping G on the whole plane, by the quasi-invariance property of modulus under quasiconformal mappings (see [Va]), we get

$$\operatorname{mod}(A_n, B_n; \boldsymbol{C}) \ge \frac{1}{K[G]} \operatorname{mod}(A'_n, B'_n; \boldsymbol{C}) \ge \frac{2}{K[G]\Lambda(\rho_0)}\,. \tag{4.13}$$

Now, choose $\delta > 0$ such that $A'_n \subset D(b'_n, \delta) \subset D_R$ for large n. Noting that for each curve $\gamma \in \Gamma(A_n, B_n; \boldsymbol{C})$, either $\gamma \in \Gamma(A_n, B_n; \Omega')$ or γ contains a subarc which joins $g(\partial D(b'_n, |b'_n - a'_n|))$ and $g(\partial D(b'_n, \delta))$, and that g is $H+\varepsilon$-quasiconformal in D_R, we obtain (see [Va])

$$\operatorname{mod}(A_n, B_n; \boldsymbol{C}) \le \operatorname{mod}(A_n, B_n; \Omega') + \frac{2\pi(H + \varepsilon)}{\log(\delta/|b'_n - a'_n|)}\,. \tag{4.14}$$

It follows from (4.13) and (4.14) that

$$\begin{aligned} 1 \ge \frac{\operatorname{mod}(A_n, B_n; \Omega')}{\operatorname{mod}(A_n, B_n; \boldsymbol{C})} &\ge \frac{\operatorname{mod}(A_n, B_n; \boldsymbol{C}) - 2\pi(H + \varepsilon)/\log(\delta/|b'_n - a'_n|)}{\operatorname{mod}(A_n, B_n; \boldsymbol{C})} \\ &\ge 1 - \frac{\pi(H + \varepsilon)K[G]\Lambda(\rho_0)}{\log(\delta/|b'_n - a'_n|)} \to 1\,, \end{aligned}$$

which implies

$$\frac{\operatorname{mod}(A_n, B_n; \boldsymbol{C})}{\operatorname{mod}(A_n, B_n; \Omega')} \to 1\,. \tag{4.15}$$

Now it follows that

$$\operatorname{mod}(A_n, B_n; \Omega') \le (H + \varepsilon)\operatorname{mod}(A'_n, B'_n; D_R) \le 2(H + \varepsilon)\operatorname{mod}(A_n, B_n; \Omega)\,,$$

which together with (4.15) implies that

$$\frac{\operatorname{mod}(A_n, B_n; \boldsymbol{C})}{\operatorname{mod}(A_n, B_n; \Omega)} \le 2(H + \varepsilon)\frac{\operatorname{mod}(A_n, B_n; \boldsymbol{C})}{\operatorname{mod}(A_n, B_n; \Omega')} \to 2(H + \varepsilon)\,,$$

and hence $M(\Omega) \le 2H$ as required.

Subcase 3.2 $\limsup_{n\to\infty}[a_n', b_n', c_n', d_n'] = \infty$.

Without loss of generality, we may assume that $\lim_{n\to\infty}[a_n', b_n', c_n', d_n'] = \infty$. Then it follows that

$$\operatorname{mod}(A_n', B_n'; \boldsymbol{C}) = \frac{2\pi}{M(\bar{\boldsymbol{C}} - A_n' \cup B_n')} \ge \frac{2}{\Lambda[a_n', b_n', c_n', d_n']},$$

and

$$\begin{aligned}\operatorname{mod}(A_n, B_n; \boldsymbol{C}) &= \frac{2\pi}{M(\bar{\boldsymbol{C}} - A_n \cup B_n)} \\ &\le \frac{2\pi}{M(g(\{|b_n' - a_n'| < |z - b_n'| < |b_n' - c_n'|\}))} \le \frac{2\pi(H+\varepsilon)}{\log(|b_n' - c_n'|/|b_n' - a_n'|)}.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\frac{\operatorname{mod}(A_n, B_n; \boldsymbol{C})}{\operatorname{mod}(A_n', B_n'; \boldsymbol{C})} &\le \frac{\pi(H+\varepsilon)\Lambda[a_n', b_n', c_n', d_n']}{\log(|b_n' - c_n'|/|b_n' - a_n'|)} \\ &\sim (H+\varepsilon)\frac{\log((|b_n' - c_n'||d_n' - a_n'|)/(|b_n' - a_n'||d_n' - c_n'|))}{\log(|b_n' - c_n'|/|b_n' - a_n'|)} \\ &\to H + \varepsilon .\end{aligned}$$

So (4.5) also holds in this subcase, which implies that $M(\Omega) \le 2H$ as required.

Case (4) We consider the same two subcases as in Case 3.

Subcase 4.1 There exists some constant ρ_0 such that $[a_n', b_n', c_n', d_n'] < \rho_0$.

In this case, the argument in Subcase 3.1 is valid here as well. So $M(\Omega) \le 2H$.

Subcase 4.2 Without loss of generality, we can assume $\lim_{n\to\infty}[a_n', b_n', c_n', d_n'] = \infty$.

In this case, we need a different approach. Let $A = B = \{w_0\}$ and $z_0 = f(w_0)$. First we suppose that $w_0 \in \Omega$. Then there exists some $\delta > 0$ such that $A_n \cup B_n \subset D(w_0, \delta) \subset \Omega$ when n is large. Since f is conformal in Ω, there exists two constants $\lambda_1 > 0$ and $\lambda_2 > 0$ not depending on n such that

$$\lambda_1|w_1 - w_2| \le |f(w_1) - f(w_2)| \le \lambda_2|w_1 - w_2|$$

for all $w_1, w_2 \in A_n \cup B_n$ when n is large. Therefore, using the discrete form of an equivalent definition for modulus due to Bagby [Ba, Theorem 5], it is easy to show that, for large n,

$$\frac{2\pi}{\operatorname{mod}(A_n', B_n'; \boldsymbol{C})} \le \lambda(\lambda_1, \lambda_2) + \frac{2\pi}{\operatorname{mod}(A_n, B_n; \boldsymbol{C})}, \tag{4.16}$$

where λ is a constant depending only on λ_1 and λ_1. Now, since $[a_n', b_n', c_n', d_n'] \to \infty$, so $\operatorname{mod}(A_n', B_n'; \boldsymbol{C}) \to 0$, and so $\operatorname{mod}(A_n, B_n; \boldsymbol{C}) \to 0$. It follows from (4.16) that

$$\limsup_{n\to\infty} \frac{\operatorname{mod}(A_n, B_n; \boldsymbol{C})}{\operatorname{mod}(A_n', B_n'; \boldsymbol{C})} \le 1 .$$

This together with (4.4) implies that $M(\Omega) \le 2$.

Now we suppose that $w_0 \in \partial\Omega$ and so that $z_0 \in \partial D$. By definition of modulus, we may assume, without loss of generality, that both $\bar{\boldsymbol{C}} - A_n$ and $\bar{\boldsymbol{C}} - B_n$ are connected. Set for

$z \in \bar{\boldsymbol{C}} - A'_n \cup J(A'_n) \cup B'_n \cup J(B'_n)$, $u_n = u_{A'_n \cup J(A'_n), B'_n \cup J(B'_n)}$ and $\psi_n = u_{nz}^2 / \iint_{\boldsymbol{C}} |u_{nz}^2|$. Then ψ_n is holomorphic in $\bar{\boldsymbol{C}} - A'_n \cup J(A'_n) \cup B'_n \cup J(B'_n)$ with $\iint_{\boldsymbol{C} - A'_n \cup J(A'_n) \cup B'_n \cup J(B'_n)} |\psi_n| = 1$. Since each of A'_n, $J(A'_n)$, B'_n and $J(B'_n)$ shrinks to $\{z_0\}$ as $n \to \infty$, without loss of generality, we assume that there exists some function ψ which is holomorphic in $\bar{\boldsymbol{C}} - \{z_0\}$ such that $\psi_n \to \psi$ locally uniformly in $\bar{\boldsymbol{C}} - \{z_0\}$. By Fatou's lemma, we obtain $\iint_{\boldsymbol{C}} |\psi| \leq 1$, which can happen only if $\psi = 0$.

Now, let $F : \bar{\boldsymbol{C}} \to \bar{\boldsymbol{C}}$ be a quasiconformal extension of f to the whole plane. Since $z_0 \in \partial D$ and $A'_n \cup J(A'_n) \cup B'_n \cup J(B'_n) \subset D_R$ for large n. By the uniqueness of harmonic functions, we have

$$
\begin{aligned}
\operatorname{mod}(A_n, B_n; \boldsymbol{C}) &\leq \operatorname{mod}(A_n \cup g \circ J(A'_n), B_n \cup g \circ J(B'_n); \boldsymbol{C}) \\
&= \iint_{\boldsymbol{C}} |\nabla u_{A_n \cup g \circ J(A'_n), B_n \cup g \circ J(B'_n)}|^2 \leq \iint_{\boldsymbol{C}} |\nabla (u_n \circ F)|^2 \\
&\leq \iint_D |\nabla (u_n)|^2 + (H + \varepsilon) \iint_{D_R - D} |\nabla (u_n)|^2 \\
&\quad + K[F] \iint_{\boldsymbol{C} - D_R} |\nabla (u_n)|^2 \\
&\leq 4 \left(\iint_D |u_{nz}^2| + (H + \varepsilon) \iint_{D^*} |u_{nz}^2| + K[F] \iint_{\boldsymbol{C} - D_R} |u_{nz}^2| \right) .
\end{aligned} \tag{4.17}
$$

On the other hand, we see

$$
\begin{aligned}
\operatorname{mod}(A_n, B_n; \Omega) &= \operatorname{mod}(A'_n, B'_n; D) \\
&= 4 \iint_D |u_{nz}^2| = 4 \iint_{D^*} |u_{nz}^2| = 2 \iint_{\boldsymbol{C}} |u_{nz}^2| .
\end{aligned} \tag{4.18}
$$

Noting that $\psi_n \to 0$ locally uniformly in $\bar{\boldsymbol{C}} - \{z_0\}$, from (4.17) and (4.18) we obtain

$$
\frac{\operatorname{mod}(A_n, B_n; \boldsymbol{C})}{\operatorname{mod}(A_n, B_n; \Omega)} \leq 1 + H + \varepsilon + 2K[F] \iint_{\boldsymbol{C} - D_R} |\psi_n| \to 1 + H + \varepsilon ,
$$

which implies that $M(\Omega) \leq 1 + H \leq 2H$ as required.

Case (5) In this case, $\operatorname{mod}(A_n, B_n; \boldsymbol{C}) \to \infty$, and so $\operatorname{mod}(A_n, B_n; \Omega) \to \infty$. Now

$$
\begin{aligned}
\operatorname{mod}(A_n, B_n; \boldsymbol{C}) &\leq \operatorname{mod}(A_n, B_n; \Omega') + \operatorname{mod}(\partial\Omega, \partial\Omega'; \boldsymbol{C}) \\
&\leq 2(H + \varepsilon)\operatorname{mod}(A_n, B_n; \Omega) + \operatorname{mod}(\partial\Omega, \partial\Omega'; \boldsymbol{C}) ,
\end{aligned}
$$

which implies that

$$
\limsup_{n \to \infty} \frac{\operatorname{mod}(A_n, B_n; \boldsymbol{C})}{\operatorname{mod}(A_n, B_n; \Omega)} \leq 2(H + \varepsilon) .
$$

It then follows that $M(\Omega) \leq 2H$.

Now the proof of Theorem 7 is complete.

REMARK. An interesting question is to determine whether the bound $2H(h_\Omega)$ in Theorem 7 can be replaced by $1 + H(h_\Omega)$. If the answer to the question were affirmative, then there would be a large class of domains Ω for which $M(\Omega) < R(\Omega) + 1$, namely, the domains

Ω whose associated sewing mappings h_Ω are Strebel points and are not induced by affine mappings.

Finally, as stated in Section 1, we point out that the question whether $M_b(\Omega) = M(\Omega)$ still remains open, even for asymptotically conformal extension domains. Theorems 4 and 6 may shed some new light on this problem for asymptotically conformal extension domains. We hope to attack this problem in the near future.

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