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A DIVERGENCE-LIKE CHARACTERIZATION OF ADMISSIBLE FUNCTIONS ON DIGRAPHS

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Abstract. In previous papers, the author defined a notion of admissible functions for digraphs and studied its properties. The notion of admissible functions naturally comes from the study of mean curvature functions of codimension-one foliations, and admissible functions of foliated manifolds are represented by a divergence formula. In this paper, we show that the similar divergence-like formula characterizes admissible functions of digraphs.

1. Introduction. Let $D = (V(D), A(D))$ be a finite digraph, which may have loops and parallel arcs. In [7], the author defined a notion of an admissible function $f : V(D) \to \boldsymbol{R}$ and studied its properties. The notion of admissible functions naturally comes from the study of mean curvature functions of codimension-one foliations (see [9], [13]). Indeed, let $(M, \mathcal{F})$ be a codimension-one foliation $\mathcal{F}$ of a closed manifold M. We assume that M and $\mathcal{F}$ are oriented. A smooth function $f : M \to \boldsymbol{R}$ is said to be admissible if there is a Riemannian metric g on M such that $-f$ is a mean curvature function of $\mathcal{F}$ with respect to the metric g. In this case, we have

$$f = \mathrm{div}_g(N)\,,$$

where N is the unit vector field on M orthogonal to $\mathcal{F}$, and $\mathrm{div}_g(N)$ is the divergence of N with respect to the metric g. From the view point of the paper [9], this situation can be interpreted by digraphs, that is, we have the following characterization of admissible functions on digraphs.

THEOREM. *Let D be a finite digraph. f is an admissible function if and only if there is a labelling g_D of D such that $f = \delta_{g_D} 1$, $1 : A(D) \to \boldsymbol{R}$ being identically* 1.

We shall give some definitions and preliminary results in Section 2, and shall prove this theorem in Section 3.

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2. Preliminaries. Let $D = (V(D), A(D))$ be a not necessarily strict digraph, that is, D may have loops and parallel arcs. In this paper, we consider only finite digraphs. An element $e = (u, v) \in A(D)$, which is an ordered pair of vertices in $V(D)$, is called an arc of D. The vertex u (resp. v) of $e = (u, v)$ is called a tail (resp. head) and is denoted by $\alpha(e)$ (resp. $\omega(e)$). (see [2] for generalities on graphs). A labelling g_D of D is defined by a pair (g_V, g_A) with $g_V : V(D) \to \boldsymbol{R}_+$ and $g_A : A(D) \to \boldsymbol{R}_+$, where $\boldsymbol{R}_+ = \{x \in \boldsymbol{R} \mid x > 0\}$. Set

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$C^0(D) = \{f \mid f : V(D) \to \boldsymbol{R}\}$ and $C^1(D) = \{\phi \mid \phi : A(D) \to \boldsymbol{R}\}$. Define the coboundary operator $\delta_{g_D} : C^1(D) \to C^0(D)$ by

$$\delta_{g_D}\phi(v) = \frac{1}{g_V(v)}\left(\sum_{\substack{e\in A(D)\\ \alpha(e)=v}} g_A(e)\phi(e) - \sum_{\substack{e'\in A(D)\\ \omega(e')=v}} g_A(e')\phi(e')\right).$$

Note that if the labelling g_D is given by $g_V \equiv 1 \equiv g_A$, then the corresponding coboundary operator δ_{g_D} is the usual one (cf. [1], [12]). Note also that δ_{g_D} of a labeled digraph (D, g_D) corresponds to the divergence of a Riemannian manifold (M, g). Indeed, define an integration of $f \in C^0(D)$ over a subset $W \subset V(D)$ by

$$\int_W f = \sum_{v\in W} g_V(v) f(v).$$

To define an integration of $\phi \in C^1(D)$ for our purpose, we need some more definitions. For subsets $X, Y \subset V(D)$ with $X \cap Y = \emptyset$, we set

$$\Gamma^+(X, Y) = \{e \in A(D) \mid \alpha(e) \in X,\ \omega(e) \in Y\},$$

$$\Gamma^-(X, Y) = \{e \in A(D) \mid \omega(e) \in X,\ \alpha(e) \in Y\}.$$

If $Y = V(D) \setminus X$, then $\Gamma^\pm(X, Y)$ is simply denoted by $\Gamma^\pm(X)$. For a subdigraph $H \subset D$, we define the boundary ∂H of H by $\partial H = \Gamma^+(V(H)) \cup \Gamma^-(V(H))$, and an integration of $\phi \in C^1(D)$ over ∂H by

$$\int_{\partial H} \phi = \sum_{e\in\Gamma^+(V(H))} g_A(e)\phi(e) - \sum_{e'\in\Gamma^-(V(H))} g_A(e')\phi(e').$$

Then we have the following Stokes' Theorem like formula.

PROPOSITION. *For a subdigraph H of a labeled digraph (D, g_D), we have*

$$\int_H \delta_{g_D}\phi = \int_{\partial H} \phi \text{ for } \phi \in C^1(D).$$

PROOF. As the loops of D give no contributions to the sums in the formula, we may assume that D has no loops.

$$\begin{aligned}\int_H \delta_{g_D}\phi &= \sum_{v\in V(H)} g_V(v)\delta_{g_D}\phi(v)\\ &= \sum_{v\in V(H)}\left(\sum_{\substack{e\in A(D)\\ \alpha(e)=v}} g_A(e)\phi(e) - \sum_{\substack{e'\in A(D)\\ \omega(e')=v}} g_A(e')\phi(e')\right).\end{aligned}$$

For $e \in V(H)$, as the term $g_A(e)\phi(e)$ appears with the '$+$' sign in the summation of $\alpha(e) \in V(H)$ and with the '$-$' sign in the summation of $\omega(e) \in V(H)$, the terms $g_A(e)\phi(e)$ cancel each other and disappear in the summation. Thus the summation of the last formula is taken for e with the '$+$' sign for $\alpha(e) \in V(H)$, $\omega(e) \notin V(H)$, and with the '$-$' sign for $\omega(e) \in V(H)$, $\alpha(e) \notin V(H)$. It then follows that

$$\begin{aligned}\int_H \delta_{g_D}\phi &= \sum_{e\in\Gamma^+(V(H))} g_A(e)\phi(e) - \sum_{e'\in\Gamma^-(V(H))} g_A(e')\phi(e') \\ &= \int_{\partial H}\phi\,. \end{aligned}$$ □

A non-empty proper full subdigraph K of a digraph D is called a $(+)$-subdigraph (resp. $(-)$-subdigraph) if $\Gamma^-(V(K)) = \emptyset$ (resp. $\Gamma^+(V(K)) = \emptyset$). Recall the definition of admissible functions on a digraph D (cf. [7]). We call a function $f : V(D) \to \boldsymbol{R}$ admissible if every minimal $(+)$-subdigraph contains a vertex v with $f(v) > 0$, and every minimal $(-)$-subdigraph contains a vertex w with $f(w) < 0$. Here "minimal" means the usual set theoretical sense, that is, being non-empty and containing no non-empty proper $(+)$-subdigraphs (resp. $(-)$-subdigraphs). In case D has no $(\pm)$-subdigraphs, any function f with $f(v) > 0$ and $f(w) < 0$ for some $v, w \in V(D)$ or $f \equiv 0$ is called admissible.

A digraph D is said to be strongly connected if there is a directed path from u to v for every distinct vertices $u, v \in V(D)$. If D is not strongly connected, then there are u and $v \in V(D)$ so that there is no directed path from u to v. Set $W_u = \{w \in V(D)|$ there is a directed path from u to $w\}$. It is easy to see that $D[W_u]$, the subdigraph of D generated by the vertices W_u, is a $(-)$-subdigraph. Thus we have the following (cf. [7]).

LEMMA 1. *A digraph D is strongly connected if and only if D has no $(\pm)$-subdigraphs.*

Now we recall some relevant facts on foliations (see [3] for generalities on foliations and [11] for differential geometric aspects of foliations). Let $(M, \mathcal{F})$ be a transversely oriented codimension-one foliation $\mathcal{F}$ of a closed oriented manifold M. The transverse orientation of $\mathcal{F}$ determines a vector field N on M transverse to $\mathcal{F}$. A compact domain $C \subset M$ is called foliated if C is a union of leaves of $\mathcal{F}$. A foliated compact domain C is said to be $(+)$-fcd (resp. $(-)$-fcd) if N points outwards (resp. inwards) everywhere on ∂C (cf. [8], [9]). We obtain, in a unique way, a digraph $\Gamma(M, \mathcal{F})$ from $(M, \mathcal{F})$, and from an arbitrarily given digraph D, we can construct a transversely oriented codimension-one foliation of a closed oriented manifold $(M, \mathcal{F})$ such that $D = \Gamma(M, \mathcal{F})$. Indeed, we have

THEOREM O1 ([9]). *Let $(M, \mathcal{F})$ be as above. For each $(M, \mathcal{F})$ there exist a digraph $\Gamma(M, \mathcal{F})$ and a nice transverse embedding $\psi : \Gamma(M, \mathcal{F}) \to (M, \mathcal{F})$. Furthermore, for each arc $e \in A(D)$, $\psi(Int(e))$ intersects each compact leaf of $\mathcal{F}$ at most once.*

THEOREM O2 ([9]). *For any digraph D, there is a foliated manifold $(M, \mathcal{F})$ so that $D = \Gamma(M, \mathcal{F})$.*

In these theorems, a vertex $v \in V(D) = V(\Gamma(M, \mathcal{F}))$ corresponds to a compact foliated domain $C_v \subset M$ and an arc $e = (u, v) \in A(\Gamma(M, \mathcal{F}))$ to a compact leaf $L \in \mathcal{F}$ contained in $C_u \cap C_v$ with $\psi(e) \cap L \neq \emptyset$. The head $\omega(e)$ and the tail $\alpha(e)$ of an arc e are determined by the direction of N so that N goes from $C_{\alpha(e)}$ to $C_{\omega(e)}$ along L. By the above construction, $(\pm)$-fcd's correspond to $(\pm)$-subdigraphs, respectively.

Let f be a smooth function on M. We call f admissible if there is a Riemannian metric g on M such that $-f$ coincides with the mean curvature function of $\mathcal{F}$ with respect to g (see Walczak [13] or Oshikiri [5], [6]). A characterization of admissible functions, which was conjectured by Walczak and proved affirmatively by the author (see Oshikiri [8]), is the following.

THEOREM O3 ([9]). *Let $\mathcal{F}$ be a transversely oriented codimension-one foliation of a closed connected oriented manifold M. Assume that $\mathcal{F}$ contains at least one $(+)$-fcd. Then f is admissible if and only if $f(x) > 0$ somewhere in any minimal $(+)$-fcd and $f(y) < 0$ somewhere in any minimal $(-)$-fcd. In case $\mathcal{F}$ contains no $(+)$-fcd's, any smooth function f with $f(x) > 0$ and $f(y) < 0$ for some $x, y \in M$ or $f \equiv 0$ is admissible.*

For a smooth function h on M with a volume element dV, define a function $\Gamma_{dV}(h) : V(D) \to \boldsymbol{R}$ by

$$\Gamma_{dV}(h)(v) = \int_{D_v} h dV \quad \text{for } v \in V(D)\,.$$

Then we have

THEOREM O4 ([9]). *For a smooth function f on M, the following two conditions are equivalent.*

(1) *f is admissible on $(M, \mathcal{F})$.*

(2) *There is a volume element dV on M so that $\Gamma_{dV}(f)$ is admissible on $\Gamma(M, \mathcal{F})$.*

3. Proof and Remark. Let $D = (V(D), A(D))$ be a finite digraph and $(M, \mathcal{F})$ be a codimension-one foliation with $\Gamma(M, \mathcal{F}) = D$ obtained in Theorem O2. To prove the theorem, we need the following.

LEMMA 2. *For any admissible function $f : V(D) \to \boldsymbol{R}$ there is a Riemannian metric g of M such that $\Gamma_{dV(M,g)}(h) = f$, where $dV(M, g)$ is the volume element of the Riemannian manifold (M, g) and $-h$ is the mean curvature function of $\mathcal{F}$ with respect to the Riemannian metric g.*

PROOF. Fix an arbitrary volume form dV on M. For each $v \in V(D)$, choose a smooth function k_v on D_v with $\mathrm{supp}(k_v) \subset \mathrm{Int}(D_v)$ and $\int_{D_v} k_v dV = f(v)$. Define $k : M \to \boldsymbol{R}$ by $k(x) = k_v(x)$ for $x \in D_v$, $v \in V(D)$. As k_v is smooth on D_v and $\mathrm{supp}(k_v) \subset \mathrm{Int}(D_v)$, k is a smooth function on M such that $\Gamma_{dV}(k) = f$. Furthermore, as f is admissible on D, k is also admissible on M by Theorem O4. Thus, there is a Riemannian metric $\bar{g}$ of M so that $-k$ is the mean curvature function of $\mathcal{F}$ with respect to $\bar{g}$. We deform k and $\bar{g}$ into h and g so that $\Gamma_{dV(M,g)}(h) = f$. To do this, recall the following fact (see [4], Lemma 3 (ii), where the term $H' = e^{-2\psi}H$ should be corrected by $H' = e^{-\psi}H$):

If $g|T\mathcal{F} \otimes TM = \bar{g}|T\mathcal{F} \otimes TM$ and $g(X, Y) = e^{2\rho}\bar{g}(X, Y)$ for X and Y orthogonal to $\mathcal{F}$, then $h = e^{-\rho}k$, where $-h$ (resp. $-k$) is the mean curvature function of $\mathcal{F}$ with respect to the metric g (resp. $\bar{g}$).

By this change of the metric, as $dV(M,g) = e^{-\rho}dV(M,\bar{g})$, it follows that $\int_D h dV(M,g) = \int_D e^{-2\rho} k dV(M,\bar{g})$. For each $v \in V(D)$, choose a suitable function ρ_v on D_v with $\mathrm{supp}(\rho_v) \subset \mathrm{Int}(D_v)$ and deform the metric $\bar{g}$ on D_v so that $\int_{D_v} h dV(M,g) = f(v)$. By setting $\rho(x) = \rho_v(x)$ for $x \in D_v$, $v \in V(D)$, we have the desired Riemannian metric of M. □

PROOF OF THEOREM. Assume that f is admissible. We shall show that there is a labelling g_D of D so that $f = \delta_{g_D} 1$. By Lemma 2, there is a Riemannian metric g of M such that $\Gamma_{dV}(h) = f$, where dV is the volume element of the Riemannian manifold (M, g) and $-h$ is the mean curvature function of $\mathcal{F}$ with respect to the Riemannian metric g. Let N be the unit vector field on M orthogonal to $\mathcal{F}$ such that the orientation coincides with the transverse orientation of $\mathcal{F}$. Then it is well-known that $\mathrm{div}_g(N) = h$.

Now, define a labelling $g_D = (g_V, g_A)$ of D by

$$g_V(v) = 1 \quad \text{and} \quad g_A(e) = \mathrm{Area}(L_e, g|L_e)\,,$$

where L_e, $e = (u, v) \in A(D)$, is the unique leaf in $D_u \cap D_v$ intersecting with $\psi(e)$ (cf. Theorem O1), and $\mathrm{Area}(L_e, g|L_e)$ is the volume of the Riemannian manifold $(L_e, g|L_e)$. Set $\partial^+ D_v = \{$compact leaves $L \subset \partial D_v$ with N pointing outwards on $L\}$ and $\partial^- D_v = \{$compact leaves $L \subset \partial D_v$ with N pointing inwards on $L\}$. Then, for each $v \in V(D)$, we have

$$\int_{D_v} h = \int_{D_v} \mathrm{div}_g(N) = \sum_{L\in\partial^+ D_v} \mathrm{Area}(L) - \sum_{L'\in\partial^- D_v} \mathrm{Area}(L')\,.$$

This implies, for $v \in V(D)$, that

$$\begin{aligned} f(v) = \Gamma_{dV}(h) &= \int_{D_v} h \\ &= \sum_{L\in\partial^+ D_v} \mathrm{Area}(L) - \sum_{L'\in\partial^- D_v} \mathrm{Area}(L') \\ &= \sum_{e\in\Gamma^+(v)} g_A(e) - \sum_{e'\in\Gamma^-(v)} g_A(e') \\ &= \delta_{g_D} 1\,. \end{aligned}$$

Thus, g_D is the desired one.

Conversely, Assume that $f = \delta_{g_D} 1$ for some labelling g_D. Then, by Proposition and $\partial D = \emptyset$, we have

$$\int_D f = \int_D \delta_{g_D} 1 = \int_{\partial D} 1 = 0\,.$$

This implies that $f \equiv 0$ or $f(u) > 0$ and $f(v) < 0$ for some $u, v \in V(D)$. If D is strongly connected, then, as D has no $(\pm)$-subdigraphs, f is, by definition, admissible. Assume D is not strongly connected and H is a minimal $(+)$-subdigraph. As H is a $(+)$-subdigraph, we have $\Gamma^+(V(H)) \neq \emptyset$ and $\Gamma^-(V(H)) = \emptyset$. It follows that

$$\int_H f = \int_H \delta_{g_D} 1 = \int_{\partial H} 1 = \int_{\Gamma^+(V(H))} 1 > 0\,,$$

which means that $f(v) > 0$ for some $v \in V(H)$. For a minimal $(-)$-subdigraph K, by the same argument, we have $\int_K f < 0$, which means that $f(w) < 0$ for some $w \in V(K)$. Thus, f is, by definition, admissible. This completes the proof. □

REMARK. From the construction of the labelling in the proof, we get the following result corresponding to Theorem O2 for a labeled digraph (D, g_D):

For any labeled digraph (D, g_D), there are a foliated manifold $(M, \mathcal{F})$ and a Riemannian metric g of M such that $D = \Gamma(M, \mathcal{F})$ and the labelling g_D is given as in the proof of the theorem from the Riemannian metric g.

In [10], the author studied a relation of Cheeger constant and strong connectivity of finite digraphs. As a corollary to the theorem, we give a labeled digraph version of this result. To this end, define Cheeger constant $\mathrm{ch}(D, g)$ for a labeled finite digraph (D, g) by

$$\begin{aligned} &\mathrm{ch}(D, g) \\ &\quad = \min\left\{ \frac{\sum_{e\in\partial H} g_A(e)}{\sum_{v\in V(H)} g_V(v)} \,\middle|\, H \text{ is a subdigraph of } D \text{ with } \sum_{v\in V(H)} g_V(v) \le \frac{1}{2} \sum_{v\in V(D)} g_V(v) \right\}. \end{aligned}$$

Recall that a digraph is weakly connected if the underlying graph is connected.

COROLLARY. *Let D be a weakly connected finite digraph. Then, D is strongly connected if and only if there is a labelling g_D of D such that $\mathrm{ch}(D, g_D) > \max_{v\in V(D)} |\delta_{g_D} 1(v)|$.*

PROOF. If D is strongly connected, then the function $f \equiv 0$ is admissible, and, by the theorem, there is a labelling g_D of D such that $f = \delta_{g_D} 1 \equiv 0$. As $\mathrm{ch}(D, g_D) > 0$, it follows that $\mathrm{ch}(D, g_D) > \max_{v\in V(D)} |\delta_{g_D} 1(v)| = 0$.

Conversely, assume that there is a labelling g_D of D such that $\mathrm{ch}(D, g_D) > \max_{v\in V(D)} |\delta_{g_D} 1(v)|$. If D is not strongly connected, then, by Lemma 1, there is a $(+)$-subdigraph H. We may assume that $\sum_{v\in V(H)} g_V(v) \le 1/2 \cdot \sum_{v\in V(D)} g_V(v)$ (if not, consider the $(-)$-subdigraph $D \setminus H$). As $\partial H = \Gamma^+(H)$, by Proposition, it follows that

$$\sum_{e\in\partial H} g_A(e) = \int_{\partial H} 1 = \int_H \delta_{g_D} 1 < \mathrm{ch}(D, g_D) \cdot \int_H 1\,.$$

Thus, we have

$$\mathrm{ch}(D, g_D) \le \frac{\displaystyle\sum_{e\in\partial H} g_A(e)}{\displaystyle\sum_{v\in V(H)} g_V(v)} = \frac{\displaystyle\int_H \delta_{g_D} 1}{\displaystyle\int_H 1} < \frac{\mathrm{ch}(D, g_D) \cdot \displaystyle\int_H 1}{\displaystyle\int_H 1} = \mathrm{ch}(D, g_D)\,,$$

which is a contradiction, and this completes the proof. □

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