# Some sequence spaces of Invariant means and lacunary defined by a Musielak-Orlicz function over $n$-normed spaces

Sunil K. Sharma[1], Kuldip Raj[2] and Ajay K. Sharma[3]

[1]Department of Mathematics, Model Institute of Engineering & Technology, Kot Bhalwal 181122, J& K, India
[2]Department of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, J&K, India
[3] Department of Mathematics, Central University of Jammu, Rahya-Suchani (Bagla), Samba-181143, J&K, India
E-mail: sunilksharma42@gmail.com[1], kuldeepraj68@rediffmail.com[2], aksju_76@yahoo.com[3]

**Abstract**

In the present paper we introduce some sequence spaces combining lacunary sequence, invariant means over $n$-normed spaces defined by Musielak-Orlicz function $\mathcal{M} = (M_k)$. We study some topological properties and also prove some inclusion results between these spaces.

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## 1 Introduction and preliminaries

The concept of 2-normed spaces was initially developed by Gähler [4] in the mid of 1960's, while that of $n$-normed spaces one can see in Misiak [14]. Since then, many others have studied this concept and obtained various results, see Gunawan ([5],[6]) and Gunawan and Mashadi [7]. Let $n \in \mathbb{N}$ and $X$ be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension $d$, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on $X^n$ satisfying the following four conditions:ĉ

1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in $X$;

2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;

3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and

4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a $n$-norm on $X$, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a $n$-normed space over the field $\mathbb{K}$. For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean $n$-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the $n$-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be a $n$-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in $X$. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on $X^{n-1}$ defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

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defines an $(n-1)$-norm on $X$ with respect to $\{a_1, a_2, \cdots, a_n\}$.
A sequence $(x_k)$ in a $n$-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence $(x_k)$ in a $n$-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every cauchy sequence in $X$ converges to some $L \in X$, then $X$ is said to be complete with respect to the $n$-norm. Any complete $n$-normed space is said to be $n$-Banach space.
The notion of difference sequence spaces was introduced by Kızmaz [9], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et. and Çolak [2] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let $w$ be the space of all complex or real sequences $x = (x_k)$ and let $m$, $v$ be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta_v^m) = \{x = (x_k) \in w : (\Delta_v^m x_k) \in Z\},$$

where $\Delta_v^m x = (\Delta_v^m x_k) = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_v^m x_k = \sum_{s=0}^{m} (-1)^s \begin{pmatrix} m \\ s \end{pmatrix} x_{k+vs}.$$

Taking $v = 1$, we get the spaces which were introduced and studied by Et. and Çolak [2]. Taking $m = v = 1$, we get the spaces which were studied by Kızmaz [9].
Let $\sigma$ be the mapping of the set of positive integers into itself. A continuous linear functional $\varphi$ on $l_\infty$, is said to be an invariant mean or $\sigma$-mean if and only if

1. $\varphi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all $k$,
2. $\varphi(e) = 1$, where $e = (1, 1, 1, \cdots)$ and
3. $\varphi(x_{\sigma(k)}) = \varphi(x)$ for all $x \in l_\infty$.

If $x = (x_n)$, write $Tx = Tx_n = (x_{\sigma(n)})$. It can be shown in [31] that

$$V_\sigma = \Big\{x \in l_\infty : \lim_k t_{kn}(x) = l, \quad \text{uniformly in } n,\ l = \sigma - \lim x\Big\},$$

where

$$t_{kn}(x) = \frac{x_n + x_{\sigma^1 n} + ... + x_{\sigma^k n}}{k+1}.$$

In the case $\sigma$ is the translation mapping $n \to n+1$, $\sigma$-mean is often called a Banach limit and $V_\sigma$, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences see[10].
By a lacunary sequence $\theta = (k_r)$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by $\theta$ will be denoted

by $I_r = (k_{r-1}, k_r]$. We write $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by $q_r$. The space of lacunary strongly convergent sequence was defined by Freedman et al [3].
Let $X$ be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$ for all $x \in X$,
2. $p(-x) = p(x)$ for all $x \in X$,
3. $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$,
4. if $(\lambda_n)$ is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and $(x_n)$ is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm $p$ for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair $(X, p)$ is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [33], Theorem 10.4.2, pp. 183). For more details about sequence spaces (see [1], [8], [12], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [26], [27], [28], [29], [30], [32]) and reference therein.
An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.
Lindenstrauss and Tzafriri [11] used the idea of Orlicz function to define the sequence space,

$$l_M = \Big\{ x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty \Big\}$$

which is called as an Orlicz sequence space. Also $l_M$ is a Banach space with the norm

$$||x|| = \inf \Big\{ \rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1 \Big\}.$$

Also, it was shown in [11] that every Orlicz sequence space $l_M$ contains a subspace isomorphic to $l_p (p \geq 1)$. The $\Delta_2-$ condition is equivalent to $M(Lx) \leq LM(x)$, for all $L$ with $0 < L < 1$. An Orlicz function $M$ can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t) dt$$

where $\eta$ is known as the kernel of $M$, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, $\eta$ is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.
A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function (see [13],[25]). A sequence $\mathcal{N} = (N_k)$ is called a complementary function of a Musielak-Orlicz function $\mathcal{M}$

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\},\ k = 1, 2, ...$$

For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{ x \in w : I_M(cx) < \infty, \quad \text{for some } \ c > 0 \Big\},$$

$$h_{\mathcal{M}} = \Big\{ x \in w : I_{\mathcal{M}}(cx) < \infty, \text{ for all } c > 0 \Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf \Big\{ k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1 \Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf \Big\{ \frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0 \Big\}.$$

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $(X, ||\cdot, \cdots, \cdot||)$ be a $n$-normed space, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. By $S(n-X)$ we denote the space of all sequences defined over $(X, ||\cdot, \cdots, \cdot||)$. In this paper we define the following sequence spaces :
$w_\sigma^0 \Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta (\Delta_v^m) =$

$$\Big\{ x \in S(n-X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} u_k \Big[ M_k \Big( || \frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1} || \Big) \Big]^{p_k} = 0,$$

$$\text{uniformly in } n \text{ for some } \rho > 0 \Big\},$$

$w_\sigma \Big[\mathcal{M}, u, p, ||., .||\Big]_\theta (\Delta_v^m) =$

$$\Big\{ x \in S(n-X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} u_k \Big[ M_k \Big( || \frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1} || \Big) \Big]^{p_k} = 0,$$

$$\text{uniformly in } n \text{ for some } l \text{ and } \rho > 0, \Big\}$$

and
$w_\sigma^\infty \Big[\mathcal{M}, u, p, ||., .||\Big]_\theta (\Delta_v^m) =$

$$\Big\{ x \in S(n-X) : \sup_{r,n} \frac{1}{h_r} \sum_{k \in I_r} u_k \Big[ M_k \Big( || \frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1} || \Big) \Big]^{p_k} < \infty,$$

$$\text{uniformly in } n \text{ for some } \rho > 0 \Big\}.$$

If we take $\mathcal{M}(x) = x$, we get the spaces
$w_\sigma^0\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) =$

$$\Big\{x \in S(n-X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} u_k \Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} = 0,$$

$$\text{uniformly in } n \text{ for some } \rho > 0\Big\},$$

$w_\sigma\Big[u, p, ||., .||\Big]_\theta(\Delta_v^m) =$

$$\Big\{x \in S(n-X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} u_k \Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} = 0,$$

$$\text{uniformly in } n \text{ for some } l \text{ and } \rho > 0\Big\}$$

and
$w_\sigma^\infty\Big[u, p, ||., .||\Big]_\theta(\Delta_v^m) =$

$$\Big\{x \in S(n-X) : \sup_{r,n} \frac{1}{h_r} \sum_{k\in I_r} u_k \Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} < \infty,$$

$$\text{uniformly in } n \text{ for some } \rho > 0\Big\}.$$

If we take $p = (p_k) = 1$, we get the spaces
$w_\sigma^0\Big[\mathcal{M}, u, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) =$

$$\Big\{x \in S(n-X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} u_k \Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] = 0,$$

$$\text{uniformly in } n \text{ for some } \rho > 0\Big\},$$

$w_\sigma\Big[\mathcal{M}, u, ||., .||\Big]_\theta(\Delta_v^m) =$

$$\Big\{x \in S(n-X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} u_k \Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] = 0,$$

$$\text{uniformly in } n \text{ for some } l \text{ and } \rho > 0\Big\}$$

and
$w_\sigma^\infty\Big[\mathcal{M}, u, ||., .||\Big]_\theta(\Delta_v^m) =$

$$\Big\{x \in S(n-X) : \sup_{r,n} \frac{1}{h_r} \sum_{k\in I_r} u_k \Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] < \infty,$$

$$\text{uniformly in } n \text{ for some } \rho > 0\Big\}.$$

The following inequality will be used throughout the paper. If $0 \le p_k \le \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \le K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.1}$$

for all $k$ and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \le \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main aim of the present paper is to study some topological properties and prove some inclusion relations between above defined sequence spaces.

## 2 Main results

**Theorem 2.1** *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. Then the classes of sequences*

$$w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m), \quad w_\sigma\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \quad \text{and} \quad w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$$

*are linear spaces over the field of complex numbers $\mathbb{C}$.*

*Proof.* The proof is obvious, so we omit it. Q.E.D.

**Theorem 2.2** *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. Then $w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$ is a topological linear space paranormed by*

$$g(x) = \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1, r, n = 1, 2, \cdots\Big\},$$

*where $H = \max(1, \sup_k p_k < \infty)$.*

*Proof.* Clearly $g(x) \ge 0$ for $x = (x_k) \in w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. Since $M_k(0) = 0$, we get $g(0) = 0$. Conversely, suppose that $g(x) = 0$, then

$$\inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1, r, n = 1, 2, \cdots\Big\} = 0.$$

This implies that for a given $\varepsilon > 0$, there exists some $\rho_\varepsilon (0 < \rho_\varepsilon < \varepsilon)$ such that

$$\Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho_\varepsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1.$$

Thus

$$\Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\varepsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}$$

$$\le \Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho_\varepsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1,$$

for each $r$ and $n$. Suppose that $x_k \neq 0$ for each $k \in \mathbb{N}$. This implies that $t_{kn}(\Delta_v^m x_k) \neq 0$, for each $k, n \in \mathbb{N}$. Let $\varepsilon \to 0$, then $||\frac{t_{kn}(\Delta_v^m x_k)}{\varepsilon}, z_1, \cdots, z_{n-1}|| \to \infty$. It follows that

$$\Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\varepsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \to \infty,$$

which is a contradiction. Therefore, $t_{kn}(\Delta_v^m x_k) = 0$ for each $k$ and thus $\Delta_v^m x_k = 0$ for each $k \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1$$

and

$$\Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m y_k)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1$$

for each $r$ . Let $\rho = \rho_1 + \rho_2$. Then by using Minkowski's inequality, we have

$$\begin{aligned}
&\Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m (x_k + y_k))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
\leq\ & \Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k) + t_{kn}(\Delta_v^m y_k)}{\rho_1+\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
\leq\ & \Big(\frac{1}{h_r}\sum_{k\in I_r} \Big[\frac{\rho_1}{\rho_1+\rho_2} u_k M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big) \\
+\ & \frac{\rho_2}{\rho_1+\rho_2} u_k M_k\Big(||\frac{t_{kn}(\Delta_v^m y_k)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
\leq\ & \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
+\ & \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m y_k)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
\leq\ & 1.
\end{aligned}$$

Since $\rho$'s are non-negative, so we have
$g(x+y)$

$$\begin{aligned}
&= \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k) + t_{kn}(\Delta_v^m y_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
&\qquad\qquad\qquad \leq 1, \quad r, n = 1, 2, \cdots\Big\} \\
&\leq \inf\Big\{\rho_1^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\ \Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r, n = 1, 2, \cdots\Big\} \\
&+ \inf\Big\{\rho_2^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m y_k)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r, n = 1, 2, \cdots\Big\}.
\end{aligned}$$

Therefore,

$$g(x+y) \leq g(x) + g(y).$$

Finally, we prove that the scalar multiplication is continuous. Let $\lambda$ be any complex number. By definition,

$$g(\lambda x) = \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m \lambda x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r, n = 1, 2, \cdots\Big\}.$$

Then

$$g(\lambda x_k) = \inf\Big\{(|\lambda|t)^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{t}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r, n = 1, 2, \cdots\Big\},$$

where $t = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{p_r} \leq \max(1, |\lambda|^{\sup p_r})$, we have

$$g(\lambda x) \leq \max(1, |\lambda|^{\sup p_r}) \inf\Big\{t^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{t}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r, n = 1, 2, \cdots\Big\}.$$

So, the fact that scalar multiplication is continuous follows from the above inequality.
This completes the proof of the theorem. Q.E.D.

**Theorem 2.3** *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. Then*

$$w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m).$$

*Proof.* The first inclusion is obvious. We will show that

$$w_\sigma\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m).$$

Let $x \in w_\sigma\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. Then there exists some positive number $\rho_1$ such that

$$\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \to 0 \text{ as } r \to \infty \text{ uniformly in } n.$$

Define $\rho = 2\rho_1$. Since $\mathcal{M} = (M_k)$ is non-decreasing, convex and so by using inequality (1.1), we have

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\qquad\leq \frac{K}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\qquad+ \frac{K}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{l}{\rho_1}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\qquad\leq \frac{K}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\qquad+ K\max\Big\{1, u_k\Big[M_k\Big(||\frac{l}{\rho_1}, z_1,\cdots,z_{n-1}||\Big)\Big]^{H}\Big\}.
\end{aligned}$$

Thus $x \in w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot,\cdots,\cdot||\Big]_\theta(\Delta_v^m)$. Q.E.D.

**Theorem 2.4** *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. If $\sup_k\Big[M_k(t)\Big]^{p_k} < \infty$ for all $t > 0$, then*

$$w_\sigma\Big[\mathcal{M}, u, p, ||\cdot,\cdots,\cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot,\cdots,\cdot||\Big]_\theta(\Delta_v^m).$$

*Proof.* Let $x \in w_\sigma\Big[\mathcal{M}, u, p, ||\cdot,\cdots,\cdot||\Big]_\theta(\Delta_v^m)$. By using inequality (1.1), we have

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\qquad\leq \frac{K}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\qquad+ \frac{K}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{l}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}.
\end{aligned}$$

Since $\sup_k\Big[M_k(t)\Big]^{p_k} < \infty$, we can take that $\sup_k\Big[M_k(t)\Big]^{p_k} = T$. Hence we get $x \in w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot,\cdots,\cdot||\Big]_\theta(\Delta_v^m)$. Q.E.D.

**Theorem 2.5** *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function which satisfies $\Delta_2$-condition for all $k$, then*

$$w_\sigma\Big[u, p, ||\cdot,\cdots,\cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma\Big[\mathcal{M}, u, p, ||\cdot,\cdots,\cdot||\Big]_\theta(\Delta_v^m).$$

*Proof.* Let $x \in w_\sigma\Big[u, p, ||\cdot,\cdots,\cdot||\Big]_\theta(\Delta_v^m)$. Then we have

$$\mathcal{T}_r = \frac{1}{h_r}\sum_{k\in I_r} u_k||t_{kn}(\Delta_v^m x_k - l), z_1,\cdots,z_{n-1}||^{p_k} \to \infty \text{ as } r\to\infty \text{ uniformly in } n, \text{ for some } l.$$

Let $\varepsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that $M_k(t) < \varepsilon$ for $0 \le t \le \delta$ for all $k$. So that
$\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$

$$\begin{aligned} &= \frac{1}{h_r}\sum_{k\in I_r, ||t_{kn}(x-l),z||\le\delta}^{1} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\ &+ \frac{1}{h_r}\sum_{k\in I_r, ||t_{kn}(x-l),z||>\delta}^{2} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}. \end{aligned}$$

For the first summation in the right hand side of the above equation, we have $\sum^{1} \le \varepsilon^H$ by using continuity of $M_k$ for all $k$. For the second summation, we write

$$||t_{kn}(\Delta_v^m x_k - l), z_1, \cdots, z_{n-1}|| \le 1 + ||\frac{t_{kn}(\Delta_v^m x_k - l)}{\delta}, z_1, \cdots, z_{n-1}||.$$

Since $M_k$ is non-decreasing and convex for all $k$, it follows that
$u_k\Big[M_k(||t_{kn}(\Delta_v^m x_k - l), z_1, \cdots, z_{n-1}||)\Big]$

$$\begin{aligned} &< u_k\Big[M_k\Big(1 + ||\frac{t_{kn}(\Delta_v^m x_k - l)}{\delta}, z_1, \cdots, z_{n-1}||\Big)\Big] \\ &\le \frac{1}{2}u_k\big(M_k(2)\big) + \frac{1}{2}u_k\Big[M_k\Big((2)||\frac{t_{kn}(\Delta_v^m x_k - l)}{\delta}, z_1, \cdots, z_{n-1}||\Big)\Big]. \end{aligned}$$

Since $M_k$ satisfies $\Delta_2$-condition for all $k$, we can write

$$\begin{aligned} u_k\Big[M_k(||t_{kn}(\Delta_v^m x_k - l), z_1, \cdots, z_{n-1}||) &\le \frac{1}{2}L||\frac{t_{kn}(\Delta_v^m x_k - l)}{\delta}, z_1, \cdots, z_{n-1}||M_k(2) \\ &+ \frac{1}{2}L||\frac{t_{kn}(\Delta_v^m x_k - l)}{\delta}, z_1, \cdots, z_{n-1}||M_k(2) \\ &= L||\frac{t_{kn}(\Delta_v^m x_k - l)}{\delta}, z_1, \cdots, z_{n-1}||M_k(2). \end{aligned}$$

So we write

$$\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \le \varepsilon^H + [\max(1, LM_k(2))\delta]^H \mathcal{T}_r.$$

Letting $r \to \infty$, it follows that $x \in w_\sigma\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. This completes the proof. Q.E.D.

**Theorem 2.6** *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent :*
*(i)* $w_\sigma^\infty\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$,

*(ii)* $w^0_\sigma\Big[u,p,||\cdot,\cdots,\cdot||\Big]_\theta(\Delta^m_v) \subset w^0_\sigma\Big[\mathcal{M},u,p,||\cdot,\cdots,\cdot||\Big]_\theta(\Delta^m_v)$,

*(iii)* $\sup_r \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k(t)\Big]^{p_k} < \infty$ *for all* $t > 0$.

*Proof.* (i) $\Longrightarrow$ (ii) We have only to show that $w^0_\sigma\Big[u,p,||\cdot,\cdots,\cdot||\Big]_\theta(\Delta^m_v) \subset w^\infty_\sigma\Big[u,p,||\cdot,\cdots,\cdot||\Big]_\theta(\Delta^m_v)$. Let $x \in w^0_\sigma\Big[u,p,||\cdot,\cdots,\cdot||\Big]_\theta(\Delta^m_v)$. Then there exists $r \geq r_0$, for $\varepsilon > 0$, such that

$$\frac{1}{h_r}\sum_{k\in I_r} u_k\Big(||\frac{t_{kn}(\Delta^m_v x_k)}{\rho},z_1,\cdots,z_{n-1}||\Big)^{p_k} < \varepsilon.$$

Hence there exists $H > 0$ such that

$$\sup_{r,n}\frac{1}{h_r}\sum_{k\in I_r} u_k\Big(||\frac{t_{kn}(\Delta^m_v x_k)}{\rho},z_1,\cdots,z_{n-1}||\Big)^{p_k} < H$$

for all $n$ and $r$. So we get $x \in w^\infty_\sigma\Big[u,p,||\cdot,\cdots,\cdot||\Big]_\theta(\Delta^m_v)$.

(ii) $\Longrightarrow$ (iii) Suppose that (iii) does not hold. Then for some $t > 0$

$$\sup_r \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k(t)\Big]^{p_k} = \infty$$

and therefore we can find a subinterval $I_{r(m)}$ of the set of interval $I_r$ such that

$$\frac{1}{h_{r(m)}}\sum_{k\in I_{r(m)}} u_k\Big[M_k(\frac{1}{m})\Big]^{p_k} > m, \quad m = 1,2,\cdots. \tag{2.1}$$

Let us define $x = (x_k)$ as follows, $x_k = \frac{1}{m}$ if $k \in I_{r(m)}$ and $x_k = 0$ if $k \notin I_{r(m)}$. Then $x \in w^0_\sigma\Big[u,p,||\cdot,\cdots,\cdot||\Big]_\theta(\Delta^m_v)$ but by eqn.(2.1), $x \notin w^0_\sigma\Big[\mathcal{M},u,p,||\cdot,\cdots,\cdot||\Big]_\theta(\Delta^m_v)$, which contradicts (ii). Hence (iii) must hold.

(iii) $\Longrightarrow$ (i) Suppose (i) not holds, then for $x \in w^\infty_\sigma\Big[u,p,||\cdot,\cdots,\cdot||\Big]_\theta(\Delta^m_v)$, we have

$$\sup_{r,n}\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta^m_v x_k)}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k} = \infty. \tag{2.2}$$

Let $t = ||\frac{t_{kn}(\Delta^m_v x_k)}{\rho},z_1,\cdots,z_{n-1}||$ for each $k$ and fixed $n$, so that eqn. (2.2) becomes

$$\sup_r \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k(t)\Big]^{p_k} = \infty,$$

which contradicts (iii). Hence (i) must hold. Q.E.D.

**Theorem 2.7** *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent:*

*(i)* $w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^0\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$,

*(ii)* $w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^\infty\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$,

*(iii)* $\inf_r \sum_{k\in I_r} u_k\Big[M_k(t)\Big]^{p_k} > 0$ *for all* $t > 0$.

*Proof.* (i) $\Longrightarrow$ (ii) : It is easy to prove.

(ii) $\Longrightarrow$ (iii) Suppose that (iii) does not hold. Then

$$\inf_r \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k(t)\Big]^{p_k} = 0 \quad \text{for some } \ t > 0,$$

and we can find a subinterval $I_{r(m)}$ of the set of interval $I_r$ such that

$$\frac{1}{h_r} \sum_{k\in I_{r(m)}} u_k[M_k(m)]^{p_k} < \frac{1}{m}, \ \ m = 1, 2, \cdots . \tag{2.3}$$

Let us define $x_k = m$ if $k \in I_{r(m)}$ and $x_k = 0$ if $k \notin I_{r(m)}$. Thus by eqn.(2.3), $x \in w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$ but $x \notin w_\sigma^\infty\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$ which contradicts (ii). Hence (iii) must hold.

(iii) $\Longrightarrow$ (i) It is obvious. Q.E.D.

**Theorem 2.8** *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then* $w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^0\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$ *if and only if*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k(t)\Big]^{p_k} = \infty. \tag{2.4}$$

*Proof.* Let $w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^0\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. Suppose that eqn. (2.4) does not hold. Therefore there is a subinterval $I_{r(m)}$ of the set of interval $I_r$ and a number $t_0 > 0$, where $t_0 = ||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots z_{n-1}||$ for all $k$ and $n$, such that

$$\frac{1}{h_{r(m)}} \sum_{k\in I_{r(m)}} u_k[M_k(t_0)]^{p_k} \le M < \infty, \ \ m = 1, 2, \cdots . \tag{2.5}$$

Let us define $x_k = t_0$ if $k \in I_{r(m)}$ and $x_k = 0$ if $k \notin I_{r(m)}$. Then, by eqn. (2.5), $x \in w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. But $x \notin w_\sigma^0\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. Hence eqn. (2.5) must hold.

Conversely, suppose that eqn. (2.5) hold and that $x \in w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. Then for each $r$ and $n$

$$\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq M < \infty. \tag{2.6}$$

Now suppose that $x \notin w_\sigma^0\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. Then for some number $\varepsilon > 0$ and for a subinterval $I_{ri}$ of the set of interval $I_r$, there is $k_0$ such that $||t_{kn}(\Delta_v^m x_k), z_1, \cdots, z_{n-1}||^{p_k} > \varepsilon$ for $k \geq k_0$. From the properties of sequence of Orlicz functions, we obtain

$$u_k\Big[M_k\Big(\frac{\varepsilon}{\rho}\Big)\Big]^{p_k} \leq u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$

which contradicts eqn.(2.5), by using eqn. (2.6). This completes the proof. Q.E.D.

**Theorem 2.9** *Let $m \geq 1$ be a fixed integer. Then*
*(i)* $w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^{m-1}) \subset w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$;
*(ii)* $w_\sigma\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^{m-1}) \subset w_\sigma\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$;
*(iii)* $w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^{m-1}) \subset w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$.

*Proof.* The proof of the inclusions follows from the following inequality

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||)\Big]^{p_k} \\ &\leq \frac{K}{h_r}\sum_{k\in I_r} u_k\Big[M_k||\frac{t_{kn}(\Delta_v^{m-1} x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big]^{p_k} \\ &+ \frac{K}{h_r}\sum_{k\in I_r} u_k\Big[M_k||\frac{t_{kn}(\Delta_v^{m-1} x_{k+1})}{\rho}, z_1, \cdots, z_{n-1}||\Big]^{p_k}.\end{aligned}$$

Q.E.D.

**Theorem 2.10** *Let $\mathcal{M} = (M_k)$ and $\mathcal{M}' = (M_k')$ are Musielak-Orlicz functions. Then*
*(i)* $w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \cap w_\sigma^\infty\Big[\mathcal{M}', u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$

$$\subset w_\sigma^\infty\Big[\mathcal{M}+\mathcal{M}', u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m);$$

*(ii)* $w_\sigma\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \cap w_\sigma\Big[\mathcal{M}', u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$

$$\subset w_\sigma\Big[\mathcal{M}+\mathcal{M}', u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m);$$

*(iii)* $w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \cap w_\sigma^0\Big[\mathcal{M}', u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$

$$\subset w_\sigma^0\Big[\mathcal{M}+\mathcal{M}', u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m).$$

*Proof.* Let $x \in w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \cap w_\sigma^\infty\Big[\mathcal{M}', u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. Then

$$\sup_{r,n} \frac{1}{h_r} \sum_{k \in I_r} u_k \Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} < \infty \text{ uniformly in } n$$

and

$$\sup_{r,n} \frac{1}{h_r} \sum_{k \in I_r} u_k \Big[M'_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} < \infty \text{ uniformly in } n.$$

Thus by using inequality (1.1) we have

$$\begin{aligned} u_k\Big[(M_k + M'_k)\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \quad &\leq \quad K\Big[u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big] \\ &+ \quad K\Big[u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big] \end{aligned}$$

$$\Longrightarrow \frac{1}{h_r} \sum_{k \in I_r} u_k \Big[(M_k + M'_k)\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$

$$\leq \frac{K}{h_r} \sum_{k \in I_r} u_k \Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$

$$+\frac{K}{h_r} \sum_{k \in I_r} u_k \Big[M'_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \text{ uniformly in } n.$$

This completes the proof. Similarly, we can prove (ii) and (iii). Q.E.D.

**Theorem 2.11** *Let $0 < p_k \leq q_k$ for each $k$ and $\left(\frac{q_k}{p_k}\right)$ be bounded. Then*
*(i)* $w_\sigma^\infty\Big[\mathcal{M}, q, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^\infty\Big[\mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$;
*(ii)* $w_\sigma\Big[\mathcal{M}, q, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma\Big[\mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$;
*(iii)* $w_\sigma^0\Big[\mathcal{M}, q, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^0\Big[\mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$.

*Proof.* (i) Let $x \in w_\sigma^\infty\Big[\mathcal{M}, q, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. Then

$$\sup_{r,n} \frac{1}{h_r} \sum_{k \in I_r} u_k \Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{q_k} < \infty \text{ uniformly in } n.$$

Write $\mu_{k,n} = u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{q_k}$ and $\lambda_k = \frac{p_k}{q_k}$. Since $p_k \leq q_k$ therefore $0 < \lambda < \lambda_k \leq 1$. Define $y_{k,n} = \mu_{k,n}$, $y_{k,n} = 0$ if $\mu_{k,n} \geq 1$ and $z_{k,n} = \mu_{k,n}$, $z_{k,n} = 0$ if $\mu_{k,n} \geq 1$. So $\mu_{k,n} = y_{k,n} + z_{k,n}$ and $\mu_{k,n}^{\lambda_k} = y_{k,n}^{\lambda_k} + z_{k,n}^{\lambda_k}$. Now it follows that $y_{k,n}^{\lambda_k} \leq y_{k,n} \leq z_{k,n}$ and $z_{k,n}^{\lambda_k} \leq z_{k,n}^{\lambda}$. Therefore

$$\frac{1}{h_r} \sum_{k \in I_r} \mu_{k,n}^{\lambda_k} = \frac{1}{h_r} \sum_{k \in I_r} \left(y_{k,n}^{\lambda_k} + z_{k,n}^{\lambda_k}\right) \leq \frac{1}{h_r} \sum_{k \in I_r} y_{k,n} + \frac{1}{h_r} \sum_{k \in I_r} z_{k,n}^{\lambda}.$$

Since $\lambda < 1$ so that $\frac{1}{\lambda} > 1$, for each $n$ and by using Holder's inequality, we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} z_{k,n}^{\lambda} &= \sum_{k\in I_r}\Big(\frac{1}{h_r}z_{k,n}\Big)^{\lambda}\Big(\frac{1}{h_r}\Big)^{1-\lambda} \\ &\le \Big(\sum_{k\in I_r}\Big[\big(\frac{1}{h_r}z_{k,n}\big)^{\lambda}\Big]^{\frac{1}{\lambda}}\Big)^{\lambda}\Big(\sum_{k\in I_r}\Big[\big(\frac{1}{h_r}\big)^{1-\lambda}\Big]^{\frac{1}{(1-\lambda)}}\Big)^{1-\lambda} \\ &= \big(\frac{1}{h_r}\sum_{k\in I_r} z_{k,n}\big)^{\lambda}.\end{aligned}$$

Thus, we have

$$\frac{1}{h_r}\sum_{k\in I_r}\mu_{k,n}^{\lambda_k} \le \frac{1}{h_r}\sum_{k\in I_r}\mu_{k,n} + \big[\frac{1}{h_r}\sum_{k\in I_r} z_{k,n}\big]^{\lambda}.$$

Hence $x \in w_{\sigma}^{\infty}\Big[\mathcal{M}, p, ||\cdot,\cdots,\cdot||\Big]_{\theta}(\Delta_v^m)$. This completes the proof of (i). Similarly, we can prove (ii) and (iii). Q.E.D.

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