

Initial coefficient bounds for a subclass of m -fold symmetric bi-univalent functions

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Abstract

Let Σ denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

belonging to the normalized analytic function class \mathcal{A} in the open unit disk \mathbb{U} , which are bi-univalent in \mathbb{U} , that is, both the function f and its inverse f^{-1} are univalent in \mathbb{U} . The usual method for computation of the coefficients of the inverse function $f^{-1}(z)$ by means of the relation $f^{-1}(f(z)) = z$ is too difficult to apply in the case of m -fold symmetric analytic functions in \mathbb{U} . Here, in our present investigation, we aim at overcoming this difficulty by using a general formula to compute the coefficients of $f^{-1}(z)$ in conjunction with the residue calculus. As an application, we introduce two new subclasses of the bi-univalent function class Σ in which both $f(z)$ and $f^{-1}(z)$ are m -fold symmetric analytic functions with their derivatives in the class \mathcal{P} of analytic functions with positive real part in \mathbb{U} . For functions in each of the subclasses introduced in this paper, we obtain the coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$.

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1 Introduction and definitions

Let \mathcal{A} denote the class of functions of the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

which are *analytic* in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We denote by \mathcal{S} the class of all functions in \mathcal{A} which are *univalent* in \mathbb{U} (see, for details, [2, 4]).

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1} , which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}) \quad (1.2)$$

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and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right). \quad (1.3)$$

In fact, the inverse function f^{-1} may be analytically continued to \mathbb{U} as follows:

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \quad (1.4)$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in \mathbb{U} if both f and f^{-1} are univalent in \mathbb{U} . Let Σ denote the class of all functions $f \in \mathcal{A}$ which are bi-univalent in \mathbb{U} and are given by the equation (1.1).

Lewin [8] investigated the class Σ of bi-univalent functions and obtained a bound given by

$$|a_2| \leq 1.51.$$

Motivated by the work of Lewin [8], Brannan and Clunie [1] conjectured that

$$|a_2| \leq \sqrt{2}.$$

Some examples of bi-univalent functions are given (see also the work of Srivastava *et al.* [13]):

$$\frac{z}{1-z}, \quad \frac{1}{2} \log \left(\frac{1+z}{1-z} \right) \quad \text{and} \quad -\log(1-z).$$

Indeed, as pointed out by Srivastava *et al.* [13], the coefficient estimate problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \quad (n \in \mathbb{N} \setminus \{1, 2\}; \mathbb{N} := \{1, 2, 3, \dots\})$$

is still open.

In recent years, the study of bi-univalent functions has gained momentum mainly due to the work of Srivastava *et al.* [13], which has apparently revived the subject. Motivated by their work [13], many researchers (see, for example, [3, 5, 6, 11, 13, 14, 12, 15, 16, 17, 18]; see also the various closely-related papers on the subject, which are cited in some of these works) have recently investigated several interesting subclasses of the bi-univalent function class Σ and found non-sharp estimates on the first two Taylor-Maclaurin coefficients of functions belonging to these subclasses.

For each function $f \in \mathcal{S}$, the function

$$h(z) = \sqrt[m]{f(z^m)} \quad (z \in \mathbb{U}; m \in \mathbb{N})$$

is univalent and maps the unit disk \mathbb{U} into a region with m -fold symmetry. A function is said to be m -fold symmetric (see [9, 10]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \quad (z \in \mathbb{U}; m \in \mathbb{N}). \quad (1.5)$$

We denote by \mathcal{S}_m the class of m -fold symmetric univalent functions in \mathbb{U} , which are normalized by the series expansion (1.5). In fact, the functions in the class \mathcal{S} are one-fold symmetric.

Analogous to the concept of m -fold symmetric univalent functions, we here introduced the concept of m -fold symmetric bi-univalent functions. Each function $f \in \Sigma$ generates an m -fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of f is given as in (1.5) and the series expansion for f^{-1} is given as follows by using Theorem (1):

$$g(w) = w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1} \right] w^{3m+1} + \dots, \quad (1.6)$$

where $f^{-1} = g$. We denote by Σ_m the class of m -fold symmetric bi-univalent functions in \mathbb{U} . For $m = 1$, the formula (1.6) coincides with the formula (1.4) of the class Σ . Some examples of m -fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m} \right)^{\frac{1}{m}}, \quad \left[\frac{1}{2} \log \left(\frac{1+z^m}{1-z^m} \right) \right]^{\frac{1}{m}} \quad \text{and} \quad [-\log(1-z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions given by

$$\left(\frac{w^m}{1+w^m} \right)^{\frac{1}{m}}, \quad \left(\frac{e^{2w^m} - 1}{e^{2w^m} + 1} \right)^{\frac{1}{m}} \quad \text{and} \quad \left(\frac{e^{w^m} - 1}{e^{w^m}} \right)^{\frac{1}{m}},$$

respectively.

Finally, we denote by \mathcal{P} the Carathéodary class of functions $p(z)$ of the following form:

$$p(z) = 1 + p_1z + p_2z^2 + \dots, \quad (1.7)$$

which are analytic in \mathbb{U} such that

$$\Re \{p(z)\} > 0 \quad (z \in \mathbb{U}). \quad (1.8)$$

The objective of the present paper is to introduce an elegant formula for computing the coefficients of the inverse functions for the class Σ_m of m -fold symmetric functions by means of residue calculus. As an application, we introduce two new subclasses of bi-univalent functions in which both f and f^{-1} are m -fold symmetric analytic functions with their derivative in the class \mathcal{P} and obtain coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses.

In order to derive our main results, we use the following lemma for the Carathéodary class \mathcal{P} .

Lemma. *If $h \in \mathcal{P}$, then*

$$|c_k| \leq 2 \quad (k \in \mathbb{N}),$$

where the Carathéodary class \mathcal{P} is the family of all functions h , analytic in \mathbb{U} , for which

$$\Re \{h(z)\} > 0 \quad (z \in \mathbb{U})$$

and

$$h(z) = 1 + c_1z + c_2z^2 + \dots \quad (z \in \mathbb{U}), \quad (1.9)$$

the extremal function being given by

$$h(z) = \frac{1+z}{1-z} \quad (z \in \mathbb{U}).$$

2 Coefficients of the inverse functions

Coefficients of the inverse f^{-1} of a given function f are obtained generally by virtue of the relation

$$f^{-1}(f(z)) = z.$$

However, this technique is much too difficult to apply in the case of m -fold symmetric functions. To overcome this difficulty, we use a general formula [4] to compute the coefficients of f^{-1} by means of the residue calculus as follows.

Theorem 1. Let $f(z)$ be in the class \mathcal{S} . Then the coefficients γ_n of the inverse function

$$f^{-1}(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \cdots$$

are given by

$$\gamma_n = \frac{1}{n!} \lim_{z \rightarrow 0} \left\{ \frac{d^{n-1}}{dz^{n-1}} \left(\frac{z}{f(z)} \right)^n \right\}. \quad (2.1)$$

Proof. Let $f(z)$, given by (1.4), be in the class \mathcal{S} and suppose that

$$g(w) = f^{-1}(w)$$

is given as in the hypothesis of Theorem 1. Then, from Cauchy's integral formula and by the method described in [7], we have

$$\gamma_n = \frac{1}{2\pi i} \int_{\mathcal{C}_\varepsilon} \left(\frac{g(w)}{w^{n+1}} \right) dw = \frac{1}{2n\pi i} \int_{|z|=r} \frac{dz}{[f(z)]^n} \quad (0 < r < 1), \quad (2.2)$$

where \mathcal{C}_ε denotes a closed positively-oriented circle in the complex w -plane with centre at $w = 0$ and radius ε ($0 < \varepsilon < 1$). Since the function $f(z)$ is univalent and $f(0) = 0$, the integrand $\frac{1}{[f(z)]^n}$ has a pole of order n at $z = 0$. Therefore, we have

$$\gamma_n = \frac{1}{n!} \lim_{z \rightarrow 0} \left\{ \frac{d^{n-1}}{dz^{n-1}} \left(\frac{z}{f(z)} \right)^n \right\}, \quad (2.3)$$

which evidently proves Theorem 1. Q.E.D.

In view of Theorem 1, the first two coefficients γ_{m+1} and γ_{2m+1} of the m -fold symmetric bi-univalent function $f(z)$ in (1.5) are given by

$$\gamma_{m+1} = \frac{1}{(m+1)!} \lim_{z \rightarrow 0} \left\{ \frac{d^m}{dz^m} \left(\frac{z}{f(z)} \right)^{m+1} \right\} \quad (2.4)$$

and

$$\gamma_{2m+1} = \frac{1}{(2m+1)!} \lim_{z \rightarrow 0} \left\{ \frac{d^{2m}}{dz^{2m}} \left(\frac{z}{f(z)} \right)^{2m+1} \right\}. \quad (2.5)$$

3 Coefficient bounds for the function class $\mathcal{H}_{\Sigma,m}^\alpha$

Definition 1. A function $f(z)$, given by (1.5), is said to be in the class $\mathcal{H}_{\Sigma,m}^\alpha$ if the following conditions are satisfied:

$$f \in \Sigma_m \quad \text{and} \quad |\arg \{f'(z)\}| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{U}; 0 < \alpha \leq 1) \quad (3.1)$$

and

$$|\arg \{g'(w)\}| < \frac{\alpha\pi}{2} \quad (w \in \mathbb{U}; 0 < \alpha \leq 1), \quad (3.2)$$

where the function $g(w)$ is given by (1.6).

In this section, we first state and prove the following theorem.

Theorem 2. Let the function $f(z)$, given by (1.5), be in the class $\mathcal{H}_{\Sigma,m}^\alpha$ ($0 < \alpha \leq 1$). Then

$$|a_{m+1}| \leq \frac{2\alpha}{m\sqrt{(m+1)(\alpha m + m + 1)}} \quad (3.3)$$

and

$$|a_{2m+1}| \leq \frac{2\alpha [(2m+1)\alpha + m + 1]}{(m+1)(2m+1)}. \quad (3.4)$$

Proof. From (3.1) and (3.2), we get

$$f'(z) = [p(z)]^\alpha \quad (3.5)$$

and

$$g'(w) = [q(w)]^\alpha, \quad (3.6)$$

where the functions $p(z)$ and $q(w)$ are in the class \mathcal{P} and have the following series representations:

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots \quad (3.7)$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots. \quad (3.8)$$

Now, equating the coefficients in (3.5) and (3.6), we find that

$$(m+1)a_{m+1} = \alpha p_m, \quad (3.9)$$

$$(2m+1)a_{2m+1} = \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2} p_m^2, \quad (3.10)$$

$$-(m+1)a_{m+1} = \alpha q_m \quad (3.11)$$

and

$$(2m+1) [(m+1)a_{m+1}^2 - a_{2m+1}] = \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2} q_m^2. \quad (3.12)$$

From (3.9) and (3.11), we get

$$p_m = -q_m \quad (3.13)$$

and

$$2(m+1)^2 a_{m+1}^2 = \alpha^2 (p_m^2 + q_m^2). \quad (3.14)$$

Also, from (3.10), (3.12) and (3.14), a simple computation shows that

$$\begin{aligned} (2m+1)(m+1)a_{m+1}^2 &= \alpha(p_{2m} + q_{2m}) + \frac{\alpha(\alpha-1)}{2} (p_m^2 + q_m^2) \\ &= \alpha(p_{2m} + q_{2m}) + \frac{\alpha(\alpha-1)}{2} \frac{2(m+1)^2}{\alpha^2} a_{m+1}^2. \end{aligned}$$

Therefore, we have

$$a_{m+1}^2 = \frac{\alpha^2 (p_{2m} + q_{2m})}{(m+1)(\alpha m + m + 1)}.$$

Applying the Lemma of the preceding section for the coefficients p_{2m} and q_{2m} , we obtain

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{(m+1)(\alpha m + m + 1)}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (3.3).

Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (3.12) from (3.10), we get

$$\begin{aligned} 2(2m+1)a_{2m+1} - (m+1)(2m+1)a_{m+1}^2 \\ = \alpha(p_{2m} - q_{2m}) + \frac{\alpha(\alpha-1)}{2} (p_m^2 - q_m^2). \end{aligned} \quad (3.15)$$

By a simple computation, and using (3.13) to (3.15), we get

$$a_{2m+1} = \frac{\alpha^2 (p_m^2 + q_m^2)}{4(m+1)} + \frac{\alpha(p_{2m} - q_{2m})}{2(2m+1)}.$$

Thus, by applying the Lemma of Section 1 again for the coefficients p_m, p_{2m}, q_m and q_{2m} , we find that

$$|a_{2m+1}| \leq \frac{2\alpha [(2m+1)\alpha + m + 1]}{(m+1)(2m+1)}.$$

This completes the proof of Theorem 2.

Q.E.D.

4 Coefficient bounds for the function class $\mathcal{H}_{\Sigma,m}(\beta)$

Definition 2. A function $f(z)$, given by (1.5), is said to be in the class $\mathcal{H}_{\Sigma,m}(\beta)$ if the following conditions are satisfied:

$$f \in \Sigma_m \quad \text{and} \quad \Re \{f'(z)\} > \beta \quad (z \in \mathbb{U}; 0 \leq \beta < 1) \quad (4.1)$$

and

$$\Re \{g'(w)\} > \beta \quad (z \in \mathbb{U}; 0 \leq \beta < 1), \quad (4.2)$$

where the function g is defined by (1.6).

In this section, we state and prove the following theorem.

Theorem 3. Let the function $f(z)$, given by (1.5), be in the class $\mathcal{H}_{\Sigma,m}(\beta)$ ($0 \leq \beta < 1$). Then

$$|a_{m+1}| \leq 2\sqrt{\frac{(1-\beta)}{(m+1)(2m+1)}} \quad (4.3)$$

and

$$|a_{2m+1}| \leq 2(1-\beta) \left(\frac{(1-\beta)(2m+1) + m+1}{(m+1)(2m+1)} \right). \quad (4.4)$$

Proof. First of all, the argument inequalities in (4.1) and (4.2) can be written in the following forms:

$$f'(z) = \beta + (1-\beta)p(z) \quad (4.5)$$

and

$$g'(w) = \beta + (1-\beta)q(w), \quad (4.6)$$

where the functions $p(z)$ and $q(w)$ are in the class \mathcal{P} and have the forms given by (3.7) and (3.8), respectively. Now, as in the proof of Theorem 2, by equating the coefficients in (4.5) and (4.6), we get

$$(m+1)a_{m+1} = (1-\beta)p_m, \quad (4.7)$$

$$(2m+1)a_{2m+1} = (1-\beta)p_{2m}, \quad (4.8)$$

$$-(m+1)a_{m+1} = (1-\beta)q_m \quad (4.9)$$

and

$$(2m+1) [(m+1)a_{m+1}^2 - a_{2m+1}] = (1-\beta)q_{2m}. \quad (4.10)$$

From (4.7) and (4.9), we get

$$p_m = -q_m \quad (4.11)$$

and

$$2(m+1)^2 a_{m+1}^2 = (1-\beta)^2 (p_m^2 + q_m^2). \quad (4.12)$$

Also, from (4.8) and (4.10), we obtain

$$(m+1)(2m+1)a_{m+1}^2 = (1-\beta)(p_{2m} + q_{2m}).$$

Thus, clearly, we have

$$|a_{m+1}|^2 \leq \frac{(1-\beta)(|p_{2m}| + |q_{2m}|)}{(m+1)(2m+1)} \leq \frac{4(1-\beta)}{(m+1)(2m+1)}.$$

This gives the bound on $|a_{m+1}|$ as asserted in (4.3).

Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (4.10) from (4.8), we get

$$2(2m+1)a_{2m+1} - (m+1)(2m+1)a_{m+1}^2 = (1-\beta)(p_{2m} - q_{2m})$$

or, equivalently,

$$a_{2m+1} = \frac{(m+1)}{2}a_{m+1}^2 + \frac{(1-\beta)(p_{2m} - q_{2m})}{2(2m+1)}. \quad (4.13)$$

Upon substituting the value of a_{m+1}^2 from (4.12), we get

$$a_{2m+1} = \frac{(1-\beta)^2(p_m^2 + q_m^2)}{4(m+1)} + \frac{(1-\beta)(p_{2m} - q_{2m})}{2(2m+1)}. \quad (4.14)$$

Applying the Lemma of Section 1 for the coefficients p_m , p_{2m} , q_m and q_{2m} , we find that

$$|a_{2m+1}| \leq 2(1-\beta) \left(\frac{(1-\beta)(2m+1) + m+1}{(m+1)(2m+1)} \right) \quad (4.15)$$

which is the bound on $|a_3|$ as asserted in (4.4).

Q.E.D.

5 Corollaries and consequences

For one-fold symmetric bi-univalent functions, Theorem 2 and Theorem 3 reduce to Corollary 1 and Corollary 2, respectively, which were proven earlier by Srivastava *et al.* [13].

Corollary 1. (see [13]) Let the function $f(z)$, given by (1.1), be in the class $\mathcal{H}_\Sigma^\alpha$ ($0 < \alpha \leq 1$). Then

$$|a_2| \leq \alpha \sqrt{\frac{2}{2+\alpha}} \quad (5.1)$$

and

$$|a_3| \leq \frac{\alpha(3\alpha+2)}{3}. \quad (5.2)$$

Corollary 2. (see [13]) Let the function $f(z)$, given by (1.1), be in the class $\mathcal{H}_\Sigma(\beta)$ ($0 \leq \beta < 1$). Then

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3}} \quad (5.3)$$

and

$$|a_3| \leq \frac{(1-\beta)(5-3\beta)}{3}. \quad (5.4)$$

Numerous other (presumably new) corollaries and consequences of our main results can also be deduced.

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