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THE FRÉCHET BOUNDS REVISITED

What have become known as the Fréchet bounds,

$$\max(F(x) + G(y) - 1, 0) \leq H(x, y) \leq \min(F(x), G(y)),$$

were published by Fréchet [2] in 1951 and even earlier by Hoeffding [3] in 1940. Here H is the joint distribution function of a pair X, Y of random variables whose one-dimensional distribution functions are F and G, respectively. It is well-known that $H(x, y)$ is identically equal to its Fréchet upper (lower) bound if and only if the mass of H is concentrated on a nondecreasing (nonincreasing) curve. Fréchet (1951) discussed this result in both the discrete case and the continuous case, and went on to say that things worked in essentially the same way in the general case. Hoeffding (1940) discussed the continuous case and said that his discussion of the discontinuous case would appear elsewhere. Motivated to some extent, perhaps, by the relative inaccessibility of these papers but also, undoubtedly, by a desire for a "better" proof, a number of others, including Dall'Aglio [1], Kimeldorf and Sampson [4], and Wolff [5], have since given proofs.

To our knowledge, each proof in the literature is either limited to the discrete or continuous case or else is quite sketchy. Perhaps it is fair to say that the literature even lacks a clear formulation of the result. Our purpose, in this paper, is to give a clear formulation of the result accompanied by a simple proof which makes no such assumptions about the nature of the marginals, F and G.

We begin with a definition. A subset S of $\mathbb{R}^2$ is *nondecreasing* if and only if, for all $(x, y), (u, v)$ in S,

$$x < u \text{ implies } y \leq v.$$

Lemma 1 *Let $S \subset \mathbb{R}^2$ be nondecreasing. Let (x, y) be an arbitrary element of $\mathbb{R}^2$. Either*

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(1) For all $(u,v) \in S$, $u < x$ *implies* $v < y$

or

(2) For all $(u,v) \in S$, $v < y$ *implies* $u < x$.

Proof: If (1) and (2) are false, points $(a,b),(c,d) \in S$ exist with $a < x, b \geq y, d < y$, and $c \geq x$ yielding $a < c$ and $d < b$, a contradiction because S is nondecreasing. □

A picture gives worthwhile intuition. In Figure 1, condition (1) is satisfied; in Figure 2, condition (2) is satisfied. Loosely speaking, Lemma 1 tells us that at (x,y), either "quadrant II" or "quadrant IV" has an empty intersection with S.

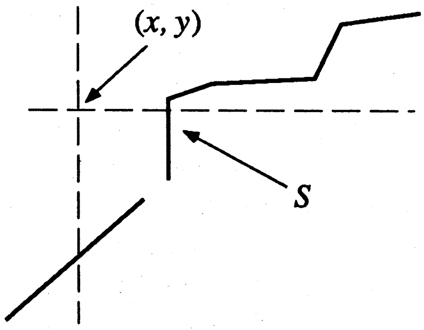

Figure 1

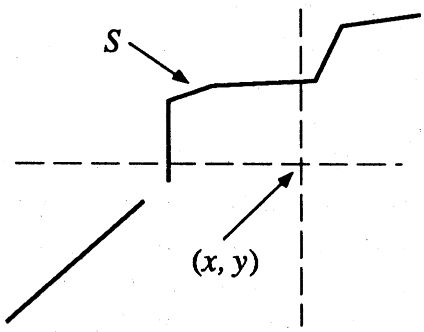

Figure 2

Proposition 2 *Let* $S \subset \mathbb{R}^2$ *be nondecreasing. Let* X *and* Y *be random variables such that* (X,Y) *is almost surely in* S. *Then* $H(x,y)$ *is identically equal to its Fréchet upper bound.*

Proof: Let $(x,y) \in \mathbb{R}^2$. Either (1) and (2) of Lemma 1 is satisfied. Suppose (1) is satisfied. Then,

$$\begin{aligned} F(x) &= P[X < x] \\ &= P[X < x, Y < y] + P[X < x, Y \geq y] \\ &= P[X < x, Y < y] + 0 \qquad \text{(because (1) is satisfied)} \\ &\leq P[X < x, Y < y] + P[X \geq x, Y < y] \\ &= P[Y < y] = G(y) \end{aligned}$$

Thus,

$$H(x,y) = P[X < x, Y < y] = \min(F(x), G(y)).$$

Similarly, because of the symmetry, if (2) is satisfied, then

$$G(y) = P[X < x, Y < y] \leq F(x).$$

Hence, we again have

$$H(x,y) = \min(F(x), G(y)). \quad \square$$

Lemma 3 *Suppose μ is a Lebesgue-Stieltjes measure on $\mathbb{R}^2$ and A is a subset of $\mathbb{R}^2$ having positive μ-measure. Then there is a point $p \in A$ such that*

$$\mu(B(p,\varepsilon)) > 0$$

for every $\varepsilon > 0$ where $B(p,\varepsilon)$ denotes the open ball about p with radius ε.

Proof: If there is no such p, we can produce $\mathfrak{B}$, a countable open cover for A by sets of μ-measure zero, so that $0 \leq \mu(A) \leq \mu(\cup\mathfrak{B}) = 0$, which is a contradiction. $\square$

Lemma 4 *Let μ be a Lebesgue-Stieltjes probability measure on $\mathbb{R}^2$. Then $\mu(D) = 1$, where*

$$D = \{(x,y) \in \mathbb{R}^2 : \textit{ for every } \varepsilon > 0,\ \mu(B((x,y),\varepsilon)) > 0\}.$$

Proof: Lemma 3 implies that $\mu(\mathbb{R}^2 - D) = 0$. Hence

$$\mu(D) = 1 - \mu(\mathbb{R}^2 - D) = 1 \quad \square$$

.

Lemma 5 *Let $H(x,y)$ be identically equal to its Fréchet upper bound. Then, for every $(x,y) \in \mathbb{R}^2$, either $P[X \geq x, Y < y] = 0$ or $P[X < x, Y \geq y] = 0$.*

Proof: Notice that

$$\begin{aligned} F(x) &= P[X < x] = P[X < x, Y < y] + P[X < x, Y \geq y] \\ &= H(x,y) + P[X < x, Y \geq y] \end{aligned}$$

and

$$\begin{aligned} G(y) &= P[Y < y] = P[X < x, Y < y] + P[X \geq x, Y < y] \\ &= H(x,y) + P[X \geq x, Y < y]. \end{aligned}$$

Thus,

$$\begin{aligned} H(x,y) &= \min(F(x), G(y)) \\ &= H(x,y) + \min(P[X < x, Y \geq y], P[X \geq x, Y < y]). \end{aligned}$$

It is now clear that either $P[X < x, Y \geq y] = 0$ or $P[X \geq x, Y < y] = 0$. □

Proposition 6 *Let $H(x,y)$ be identically equal to its Fréchet upper bound. Then there is a nondecreasing subset D of $\mathbb{R}^2$ such that (X,Y) is almost surely in D.*

Proof: Let μ be the Lebesgue-Stieltjes measure on $\mathbb{R}^2$ induced by H. Let

$$D = \{(x,y) \in \mathbb{R}^2 : \text{ for every } \varepsilon > 0, \ \mu(B((x,y),\varepsilon)) > 0\}.$$

We know from Lemma 4 that (X,Y) is almost surely in D so we need only show that D is nondecreasing. To this end, suppose (x_1,y_1) and (x_2,y_2) are both in D. Suppose further that $x_1 < x_2$ and $y_2 < y_1$. Let $x_0 = (x_1+x_2)/2$, let $y_0 = (y_1+y_2)/2$, and let ε be the smaller of $|x_2 - x_1|/2$ and $|y_2 - y_1|/2$. Then,

$$\mu(B((x_1,y_1),\varepsilon)) > 0 \text{ and } \mu(B((x_2,y_2,\varepsilon)) > 0.$$

Moreover, if $(x,y) \in B((x_1,y_1),\varepsilon)$ then $x < x_0$ and $y > y_0$. Hence

$$P[X < x_0, Y \geq y_0] > 0.$$

Likewise, if $(x,y) \in B((x_2,y_2),\varepsilon)$ then $x > x_0$ and $y < y_0$. Hence

$$P[X \geq x_0, Y < y_0] > 0.$$

By Lemma 5, $H(x,y)$ is not identical to its Fréchet upper bound, a contradiction. □

We may now combine Propositions 2 and 6 to obtain

Proposition 7 *$H(x, y)$ is identically equal to its Fréchet upper bound if and only if (X, Y) lies almost surely in a nondecreasing subset of $\mathbb{R}^2$.*

Turning our attention to the Fréchet lower bound, we again start with a definition. A subset S of $\mathbb{R}^2$ is *nonincreasing* if and only if, for every $(x, y), (u, v) \in S$,

$$x < u \text{ implies } y \geq v.$$

There are two ways to get the corresponding result for the Fréchet lower bound. One way is to proceed in a fashion similar to our approach given above for the Fréchet upper bound. The other way is to notice that the set S^*, given by

$$S^* = \{(x, y) : (x, -y) \in S\},$$

is nondecreasing if and only if the set S is nonincreasing. Also, (X, Y) lies a.s. in S if and only if $(X, -Y)$ lies a.s. in S^*. Finally, one makes use of the fact that

$$\max(a + b - 1, 0) = a - \min(a, 1 - b)$$

to prove that the distribution function of $(X, -Y)$ is identically equal to its Fréchet upper bound if and only if the distribution function of (X, Y) is identically equal to its Fréchet lower bound. Either path easily yields the following:

Proposition 8 *$H(x, y)$ is identically equal to its Fréchet lower bound if and only if (X, Y) lies almost surely in a nonincreasing subset of $\mathbb{R}^2$.*

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