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ON THE GROUP GENERATED BY QUASI CONTINUOUS FUNCTIONS

Summary. Let X be a separable metrisable Baire space without isolated points. It is proven that every cliquish function $f : X \to R$ is the sum of four quasi continuous functions.

The notion of quasi continuous function was considered in may papers (see, for example [5]). Some algebraic structures of the family of quasi continuous functions were studied by Z. Grande ([1],[2]) and E. Stronska ([6]). In this paper, I show that every cliquish function $f : X \to R$, where X is separable metrisable Baire space without isolated points, is the sum of four quasi continuous functions.

Let X be a topological space and let R be a real line. A function $f : X \to R$ is said cliquish (quasi continuous) at a point $x_0 \in X$ if for every $\varepsilon > 0$ and for any neighborhood U of the point x_0 there exists an open nonempty set $V \subset U$ such that $\operatorname{osc}_V f \leq \varepsilon$ ($|f(x_0) - f(x)|, \varepsilon$ for every $x \in V$). A function $f : X \to R$ is cliquish (quasi continuous) iff it is cliquish (quasi continuous) at every point $x \in X$. Let $Cq(X)$ denotes the family of all cliquish functions $f : X \to R$ and let $Q(X)$ be the family of all quasi continuous functions $f : X \to R$.

We have obviously:

Remark 1. *If $f, g \in Cq(X)$ and $c \in R$ then $cf \in Cq(X)$, $f + g \in Cq(X)$ and $fg \in Cq(X)$.*

Remark 2. *If $f_n \in Q(X)$ $(n = 1, 2, \ldots)$ and $f_n \overset{unif.}{\to} f$ then $f \in Q(X)$.*

Remark 3. ([4]). *If X is a Baire space then for every function $f \in Cq(X)$ the set of all its continuity points is dense in X.*

Remark 4. ([1]). *There exists a topological space X such that all functions $f \in Q(X)$ are constant and there are functions $f \in Cq(X)$ which are not constant.*

The following lemma is a modification of Natkaniec's Lemma from [3].

From now on we shall assume that X is a separable metrisable space without isolated points. Let $\bar{H}$ denote a closure of the set H.

Lemma 1. *If A is a nowhere dense nonempty set in X and $U \subset X$ is an open set such that $\bar{A} \subset \bar{U}$ then there exists a family $(K_{n,m})_{\substack{n=1\\ m\le n}}^{\infty}$ of nonempty open sets satisfying the following conditions:*

(1) $\bar{K}_{n,m} \subset U \setminus \bar{A}$ *for* $n = 1, 2, \ldots, m \le n$;

(2) $\bar{K}_{r,s} \cap \bar{K}_{i,j} = \phi$ *whenever* $(r, s) \ne (i, j)$ $(r, i = 1, 2, \ldots, s \le r$ *and* $j \le i)$;

(3) *for each $x \in \bar{A}$, each neighborhood V of x and an arbitrary m there exists an $n \ge m$ such that $\bar{K}_{n,m} \subset V$;*

(4) *for each $x \in X \setminus \bar{A}$ there exists a neighborhood V of x such that the set $\{(n, m),\ V \cap \bar{K}_{n,m} \ne \phi\}$ has at most one element.*

Proof. Let $(B_n)_{n=1}^{\infty}$ be a countable basis of open sets. Let $(W_n)_{n=1}^{\infty}$ be a sequence of open sets such that

$$W_{n+1} \subset W_n (n = 1, 2, \ldots) \text{ and } \bigcap_{n=1}^{\infty} W_n = \bar{A};$$

we may assume that

$$W_n \supset \bar{W}_{n+1} \text{ for } n = 1, 2, \ldots.$$

Let $(G_n)_{n=1}^{\infty}$ be the sequence of all sets in the basis $(B_n)_{n=1}^{\infty}$ such that for every $n = 1, 2, \ldots,\ U \cap G_n \cap \bar{A} \ne \phi$.

By induction, for every $n = 1, 2, \ldots$, we choose a nonempty open set K_n such that

$$\bar{K}_n \subset U \cap G_n \cap W_n \setminus (\bar{A} \cup \bigcup_{i<n} \bar{K}_i).$$

All sets of the family $(K_n)_{n=1}^{\infty}$ have the following properties:

(i) $\bar{K}_n \subset U \setminus \bar{A}$ for $n = 1, 2, \ldots$;

(ii) $\bar{K}_n \cap \bar{K}_m = \phi$ for $n \ne m$, $n, m = 1, 2, \ldots$;

(iii) for each $x \in \bar{A}$ and each neighborhood V of x the set $\{n;\ \bar{K}_n \subset V\}$ is infinite;

(iv) for each $x \notin \bar{A}$ there exists a neighborhood V of x such that the set $\{n;\ V \cap \bar{K}_n = \phi\}$ has at most one element.

The properties (i) - (iii) are obvious. We shall show that (iv) is also true. Suppose that $x \notin \bar{A}$. Then there exist an n_0 and a neighborhood W of x such that $W \cap W_{n_0} = \phi$. We have obviously

$$\max\{n;\ W \cap \bar{K}_n \neq \phi\} < n_0.$$

If $x \in \bar{K}_m$, for any $m < n_0$, then the set $V = W \setminus \bigcup_{\substack{n<n_0 \\ n \neq m}} \bar{K}_n$ is the required neighborhood some of the point x. If $x \notin \bar{K}_n$ for every $n < n_0$, then $V = W \setminus \bigcup_{n<n_0} \bar{K}_n$ is the required neighborhood of x.

Now, for every n choose in the set K_n a family $(K_{n,m})_{m \leq n}$ of nonempty open subsets such that:

(v) $\bar{K}_{n,m} \subset K_n$ for $1 \leq m \leq n$

(vi) $\bar{K}_{n,m} \cap \bar{K}_{n,t} = \phi$ for $m \neq t$ and $m, t \leq n$.

The construction of $(K_{n,m})_{m \leq n}$ (for every $n = 1, 2, \ldots$) is following. Fix n and a point $x_0 \in K_n$. Let $(D_m)_{m=1}^{\infty}$ be a basis of the space X in the point x_0. By induction we choose $x_m \in K_n$ and open sets V_m, $(K_{n,m})_{m \leq n}$ such that

$$- \ x_1 \in K_n \backslash \{x_0\},\ x_0 \in V_1 \subset \bar{V}_1 \subset K_n \cap D_1 \backslash \{x_1\},$$

$$x_1 \in K_{n,1} \subset \bar{K}_{n,1} \subset K_n \backslash \bar{V}_1;$$

- for $1 < m \leq n$ we have

$$x_m \in V_{m-1} \backslash \{x_0\},\ x_0 \in V_m \subset \bar{V}_m \subset (V_{m-1} \cap D_m) \backslash \{x_m\},$$

$$x_m \in K_{n,m} \subset \bar{K}_{n,m} \subset V_{m-1} \backslash \bar{V}_m.$$

For every $n = 1, 2, \ldots$ the family $(K_{n,m})_{m \leq n}$ fulfills the properties (v), (vi) and the proof is concluded.

Let $C(f)$ be the set of all continuity points of function f.

Now, we assume that X is a separable metrisable Baire space without isolated points.

Theorem 1. *If $f \in Cq(X)$ then $f = g + h$, where $g \in Q(X)$ and $h \in Cq(X)$ satisfies:*

(1) *for every $x \in X$ there exists a sequence $(x_n)_{n=1}^{\infty}$ of points of $C(h)$ convergent to x and such that the limit $\lim_{n \to \infty} h(x_n)$ exists and is finite.*

Proof. Let A be the set of all points $x \in X$ such that for every sequence $(x_n)_{n=1}^{\infty}$ of points of $C(f)$ convergent to x, if the limit $\lim_{n \to \infty} f(x_n)$ exists, then it is equal to $+\infty$ or $-\infty$. We can assume that $A \neq \phi$. Of course $A \subset \{x \in X;\ \operatorname{osc} f(x) \geq 1\}$.

Since $f \in Cq$, the set A is nowhere dense in X. Let $(K_{n,m})_{\substack{n=1 \\ m \leq n}}^{\infty}$ be a family of open sets satisfying the conditions of Lemma 1, for the nowhere dense set A and $U = X$. Since X is a Baire space, there exists a sequence $(x_{n,m})_{\substack{n=1 \\ m \leq n}}^{\infty}$ of points of $C(f)$ such that

$$x_{n,m} \in K_{n,m} \text{ for every } n = 1, 2, \ldots \text{ and } m \leq n.$$

Let

$$g(x) = \begin{cases} f(x_{n,1}) & \text{for } x \in \bar{K}_{n,1},\ n = 1, 2, \ldots \\ 0 & \text{for } x \in X \setminus \bigcup_{n=1}^{\infty} \bar{K}_{n,1}, \end{cases}$$

and let $h = f - g$. Then $g \in Q(X)$, $h \in Cq(X)$, h satisfies condition (1) and the proof of our theorem is concluded.

Theorem 2. *Let $h \in Cq(X)$ satisfies the following condition:*

(1) *for every $x \in X$ there exists a sequence $(x_n)_{n=1}^{\infty}$ of points of $C(h)$ convergent to x such that the limit $\lim_{n\to\infty} h(x_n)$ exists and is finite.*

Then $h = u + w$, where $u \in Q(X)$, $w \in Cq(X)$ and for the function w the inclusion $w^{-1}(0) \supset C(w)$ holds.

Proof. From the assumption it follows that for every $x \in X$ there exists a sequence $(x_n)_{n=1}^{\infty}$ of points of $C(h)$ convergent to x, such that there exists a finite limit $\lim_{n\to\infty} h(x_n) = \alpha(x) \in R$. Obviously, for any $x \in X$ there can exist many sequences (x_n) and the corresponding numbers $\alpha(x)$. Let us now choose for each $x \in X$ only one $\alpha(x)$.

Let

$$u(x) = \begin{cases} h(x) & \text{if } x \in C(h) \\ \alpha(x) & \text{if } x \notin C(h). \end{cases}$$

and $w = h - u$.

Then $u \in Q(X)$, $w \in Cq(X)$ and $w^{-1}(0) \supset C(w)$; thus the proof of our theorem is concluded.

Theorem 3. *Let $w \in Cq(X)$ be such that $w^{-1}(0)$ is dense in X. Then there exist functions $s, t \in Q(X)$ such that $w = s + t$.*

Proof. If w is a continuous function, then the proof is obvious. In the opposite case observe that

$$X \setminus w^{-1}(0) \subset \bigcup_{k=1}^{\infty} A_k,$$

where

$$A_k = \{x \in X;\ \text{osc}\, w(x) \geq 2^{-k}\} \text{ for } k = 1, 2, \ldots.$$

The set A_k $(k = 1, 2, \ldots)$ is closed and nowhere dense in X.

In the first step, from Lemma 1 where $A = A_1$, $U = X$, we obtain a family of nonempty open sets $(K_{n,m})_{\substack{n=1\\ m\leq n}}^{\infty}$ such that:

- $\bar{K}_{n,m} \subset X \backslash A_1$ $(n = 1, 2, \ldots$ and $m \leq n)$;
- $\bar{K}_{n,m} \cap \bar{K}_{r,s} = \phi$ whenever $(n, m) \neq (r, s)$, $(n, r = 1, 2 \ldots$ and $m \leq n$, $s \leq r)$;
- for each $x \in A$, each neighborhood V of x and an arbitrary m there exists an $n \geq m$ such that $\bar{K}_{n,m} \subset V$;
- for each $x \notin A$ there exists a neighborhood V of x such that the set $\{(n, m);$ $V \cap \bar{K}_{n,m} \neq \phi\}$ has at most one element.

From the above conditions we conclude that if $x \notin A_1$ and $x \notin \bigcup_{n=1}^{\infty} \bigcup_{m\leq n} \bar{K}_{n,m}$, then there exists an open set W such that $x \in W$ and $W \cap \bar{K}_{n,m} = \phi$ for $n = 1, 2, \ldots, m \leq n$. Arrange all rational numbers in a sequence $(w_{1,1}, \ldots, w_{1,n}, \ldots)$ such that $w_{1,i} \neq w_{1,j}$ for $i \neq j$ $(i, j = 1, 2, \ldots)$ and define

$$g_1(x) = \begin{cases} w(x) & \text{for } x \in A_1 \\ w_{1,m} & \text{for } x \in \bar{K}_{n,m},\ m \leq n \text{ and } n = 1, 2, \ldots \\ 0 & \text{at the remaining points of } X \end{cases}$$

and

$$h_1(x) = \begin{cases} -w_{1,m} & \text{for } x \in \bar{K}_{n,m},\ m \leq n \text{ and } n = 1, 2, \ldots \\ 0 & \text{at the remaining points of } X. \end{cases}$$

In the second step, arrange all sets $K_{n,m}$ $(m \leq n$ and $n = 1, 2, \ldots)$ from the first step and the set

$$X \backslash \bigcup_{n=1}^{\infty} \bigcup_{m\leq n} \bar{K}_{n,m} \backslash A_2$$

in a sequence $Z_{1,1}, Z_{1,2}, \ldots, Z_{1,n}, \ldots$. For every $Z_{1,k}$ $(k = 1, 2, \ldots)$ with $\bar{Z}_{1,k} \cap A_2 \neq \phi$ from Lemma 1, where $A = A_2 \cap \bar{Z}_{1,k}$ and $U = Z_{1,k}$, we obtain a family of nonempty open sets $(K_{1,k,n,m})_{\substack{n=1\\ m\leq n}}^{\infty}$ such that:

- $\bar{K}_{1,k,n,m} \subset Z_{1,k} \backslash A_2$ $(n = 1, 2, \ldots$ and $m \leq n)$;

- if $\bar{K}_{1,k,n,m} \cap \bar{K}_{1,k,r,s} \neq \phi$, then $(n,m) = (r,s)$, where $n,r = 1,2,\ldots$ and $m \leq n$, $s \leq r$;

- for each $x \in A_2 \cap \bar{Z}_{1,k}$, each neighborhood V of x and an arbitrary m there exists an $n \geq m$ such that $\bar{K}_{1,k,m,n} \subset V$;

- for each $x \notin A_2$ there exists a neighborhood V of x such that the set $\{(n,m);\ \bar{K}_{1,k,n,m} \cap V = \phi\}$ has at most one element.

As in the first step, we see that if

$$x \notin A_2 \cup \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{n \leq m} \bar{K}_{1,k,n,m},$$

then there exists an open set W such that $x \in W$ and $W \cap \bar{K}_{1,k,n,m} = \phi$ for every k,n,m and $n \geq m$. Arrange all rational numbers from the interval $[-2^{-1}, 2^{-1}]$ in a sequence $(w_{2,1}, \ldots, w_{2,n}, \ldots)$ with $w_{2,i} \neq w_{2,j}$ for $i \neq j$, $i,j = 1,2,\ldots$. If $\bar{Z}_{1,k} \cap A_2 \neq \phi$ $(k = 1,2,\ldots)$, then let

$$g_{1,k}(x) = \begin{cases} w(x) & \text{for } x \in (A_2 \backslash A_1) \cap \bar{Z}_{1,k} \\ w_{2,m-1} & \text{for } x \in \bar{K}_{1,k,n,m},\ n = 1,2,\ldots,1 < m \leq n \\ 0 & \text{at the remaining points of } \bar{Z}_{1,k} \end{cases}$$

and

$$h_{1,k}(x) = \begin{cases} -w_{2,m-1} & \text{for } x \in \bar{K}_{1,k,n,m},\ n = 2,3,\ldots,1 < m \leq n \\ 0 & \text{at the remaining points of } \bar{Z}_{1,k}. \end{cases}$$

If $\bar{Z}_{1,k} \cap A_2 = \phi$ $(k = 1,2,\ldots)$, then let

$$g_{1,k}(x) = 0 \text{ and } h_{1,k}(x) = 0 \text{ for } x \in \bar{Z}_{1,k}.$$

Finally, in the second step, we define the functions g_2, h_2 as follows:

$$g_2(x) = \begin{cases} g_{1,k}(x) & \text{for } x \in \bar{Z}_{1,k}\ (k = 1,2,\ldots) \\ 0 & \text{at the remaining points of } X \end{cases}$$

and

$$h_2(x) = \begin{cases} h_{1,k}(x) & \text{for } x \in \bar{K}_{1,k},\ (k = 1,2,\ldots) \\ 0 & \text{at the remaining points of } X. \end{cases}$$

In general, in the $(n+1)$-st step $(n > 1)$, we arrange all open sets of the form

$$Z_{1,k} \backslash \bigcup_{m=1}^{\infty} \bar{Z}_{2,m} \backslash A_{n+1},\ Z_{2,k} \backslash \bigcup_{m=1}^{\infty} \bar{Z}_{3,m} \backslash A_{n+1}, \ldots,$$

$$Z_{n-2,k}\setminus\bigcup_{m=1}^{\infty}\bar{Z}_{n-1,m}\setminus A_{n+1},\quad Z_{n-1,k}\setminus\bigcup_{i=1}^{\infty}\bigcup_{j\le i}\bar{K}_{n-1,k,i,j}\setminus A_{n+1}$$

$$\text{and}\quad K_{n-1,k,i,j}\ (k,i=1,2,\ldots\ \text{and}\ j\le i)$$

in a sequence $Z_{n,1}, Z_{n,2}, \ldots, Z_{n,m}, \ldots$.

If $\bar{K}_{n,k}\cap A_{n+1}\ne\phi$, then by Lemma 1, where $U=Z_{n,k}$ and $A=A_{n+1}\cap\bar{Z}_{n,k}$, there exists a family of open nonempty sets $(K_{n,k,i,j})_{\substack{i=1\\ j\le i}}^{\infty}$ such that

- $\bar{K}_{n,k,i,j}\subset Z_{n,k}\setminus A_{n+1}$ $(i=1,2,\ldots$ and $j\le i)$;
- if $\bar{K}_{n,k,i,j}\cap\bar{K}_{n,k,r,s}\ne\phi$, then $(i,j)=(r,s)$, where $i,r=1,2,\ldots$ and $j\le i,\ s\le r$;
- for each $x\in A_{n+1}\cap\bar{Z}_{n,k}$, each neighborhood V of x and an arbitrary j there exists an $i\ge j$ such that $\bar{K}_{n,k,i,j}\subset V$;
- for each $x\notin A_{n+1}$ there exists a neighborhood V of x such that the set $\{(i,j),\bar{K}_{n,k,i,j}\cap V\ne\phi\}$ has at most one element.

Remark that

(*) if $x\notin A_{n+1}\cup\bigcup_{k=1}^{\infty}\bigcup_{i=1}^{\infty}\bigcup_{j\le i}\bar{K}_{n,k,i,j}$, then there exists an open set W such that $x\in W$ and $W\cap\bar{K}_{n,k,i,j}=\phi$ for every k,i,j, where $j\le i$.

Arrange all rational numbers from the interval $[-2^{-n},2^{-n}]$ in a sequence $w_{n+1,1}, \ldots, w_{n+1,m},\ldots)$ with $w_{n+1,i}\ne w_{n+1,j}$ for $i\ne j$, $i,j=1,2,\ldots$. If $\bar{Z}_{n,k}\cap A_{n+1}\ne\phi$ $(k=1,2,\ldots)$, then let

$$g_{n,k}(x)=\begin{cases} w(x) & \text{for } x\in(A_{n+1}\setminus A_n)\cap\bar{Z}_{n,k}\\ w_{n+1,j-1} & \text{for } x\in\bar{K}_{n,k,i,j},\ i=1,2,\ldots \text{ and } 1<j\le i\\ 0 & \text{at the remaining points of } \bar{Z}_{n,k},\end{cases}$$

and

$$h_{n,k}(x)=\begin{cases} -w_{n+1,j-1} & \text{for } x\in\bar{K}_{n,k,i,j},\ i=1,2,\ldots \text{ and } 1<j\le i\\ 0 & \text{at the remaining points of } \bar{Z}_{n,k}.\end{cases}$$

If there exist $k=1,2,\ldots$ such that $\bar{Z}_{n,k}\cap A_{n+1}=\phi$, then let

$$g_{n,k}(x)=h_{n,k}(x)=0\ \text{ for }\ x\in\bar{Z}_{n,k}.$$

Furthermore define functions g_{n+1} and h_{n+1} as follows:

$$g_{n+1}(x) = \begin{cases} g_{n,k}(x) & \text{for } x \in \bar{Z}_{n,k} \ (k = 1, 2, \ldots) \\ 0 & \text{at the remaining points of } X \end{cases}$$

and

$$h_{n+1}(x) = \begin{cases} h_{n,k}(x) & \text{for } x \in \bar{Z}_{n,k} \ (k = 1, 2, \ldots) \\ 0 & \text{at the remaining points of } X \end{cases}$$

Finally, we put

$$s(x) = \sum_{n=1}^{\infty} g_n(x) \text{ and } t(x) = \sum_{n=1}^{\infty} h_n(x) \text{ for } x \in X.$$

Observe that series are uniformly convergent. Since for every $n = 1, 2, \ldots$, each of the functions g_{n+1} and h_{n+1} is continuous at each point

$$x \in \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcup_{j \le i} K_{n,k,i,j} \cup (X \setminus \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcup_{j \le i} \bar{K}_{n,k,i,j} \setminus A_{n+1}),$$

the functions $s - g_1$ and $t - h_1$ are continuous at each point

$$x \in \bigcap_{n=1}^{\infty} \{ \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcup_{j \le i} K_{n,k,i,j} \cup (X \setminus \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcup_{j \le i} \bar{K}_{n,k,i,j} \setminus A_{n+1}) \} = M.$$

The functions g_1, h_1 are quasi continuous at every $x \in X$. Consequently, the functions s and t are quasi continuous at each point $x \in M$, because they are sums of everywhere quasi continuous functions g_1, h_1 and the functions $s - g_1, t - h_1$ are continuous at this point.

Now let $x \in X \setminus M$. Then

$$x \in \bar{K}_{n,k,i,j} \setminus K_{n,k,i,j} \setminus \bigcup_{n=1}^{\infty} A_n \text{ for some } k, i, j \ (j \le i),$$

or

$$x \in \bigcup_{n=2}^{\infty} A_n.$$

Let $x \in \bar{K}_{n,k,i,j} \setminus K_{n,k,i,j} \setminus \bigcup_{n=1}^{\infty} A_n$, let $\varepsilon > 0$ be a number and let V be a neighborhood of x. Since all functions $g_1, h_1, \ldots, g_n, h_n$ are continuous in the point x, there exists an open nonempty neighborhood $U \subset V$ of the point x such that

$$|\sum_{k \le n} g_k(x) - \sum_{k \le n} g_k(u)| < \frac{\varepsilon}{4} \text{ and } |\sum_{k \le n} h_k(x) - \sum_{k \le n} h_k(u)| < \frac{\varepsilon}{4}$$

for all points $u \in U$. Observe too, that the functions g_{n+1}, h_{n+1} are constant on the set $\bar{K}_{n,k,i,j}$, and that

$$U \cap K_{n,k,i,j} \neq \phi, \;\; x \in \overline{U \cap K}_{n,k,i,j}.$$

Since series $\sum_{n=1}^{\infty} g_n$, $\sum_{n=1}^{\infty} h_n$ are uniformly convergent, there exists a natural index $N > n+1$ such that

$$|\sum_{i=N+1}^{\infty} g_i(u)| < \frac{\varepsilon}{4} \text{ and } |\sum_{i=N+1}^{\infty} h_i(u)| < \frac{\varepsilon}{4}$$

for every $u \in X$. Since $x \in \bar{K}_{n,k,i,j} \backslash K_{n,k,i,j}$, we have $x \notin K_{m,l,r,s}$ for $m > n$, $l, r = 1, 2, \ldots$ and $s \le r$.

From the condition (*) it follows that there exists an open set W containing x such that for $m = n+1, \ldots, N$

$$W \cap \bar{K}_{m,l,r,s} = \phi \;\; (l, r = 1, 2, \ldots \text{ and } s \le r).$$

Finally, for $u \in W \cap U \cap \bar{K}_{n,k,i,j}$, we have

$$|s(x) - s(u)| = |\sum_{l=1}^{\infty} g_l(x) - \sum_{l=1}^{\infty} g_l(u)| \le$$

$$\le \sum_{l \le n} g_l(x) - \sum_{l \le n} g_l(u)| + |g_{n+1}(x) - g_{n+1}(u)| +$$

$$+ |\sum_{l=n+2}^{N} g_l(x) - \sum_{l=n+2}^{N} g_l(u)| + |\sum_{l=N+1}^{\infty} g_l(x) - \sum_{l=N+1}^{\infty} g_l(u)| <$$

$$< \frac{\varepsilon}{4} + 0 + 0 + \frac{\varepsilon}{4} < \varepsilon,$$

which shows that s is quasi continuous at x. It may be shown similarly that the function t is quasi continuous at $x \in \bar{K}_{n,k,i,j} \backslash K_{n,k,i,j} \backslash \bigcup_{n=1}^{\infty} A_n$. In an analogous way one shows that functions s and t are quasi continuous at all points

$$x \in \bar{K}_{n,m} \backslash K_{n,m} \;\; (n = 1, 2, \ldots \text{ and } m \le n).$$

Now let $x \in A_n \backslash A_{n-1}$ for any natural number $n > 1$. Fix an $\varepsilon > 0$ and an open neighborhood V of the point x. Observe, that for $1 < n$ all functions, except maybe one which we denote by g_{l_1}, are continuous at x. Moreover, if there exists $l_1 < n$ such that the function g_{l_1} is discontinuous at x, then $x \in \bar{K}_{l_2,k,r,s} \backslash K_{l_2,k,r,s}$, where $l_2 = l_1 - 1$. Consequently there exists an open neighborhood $U \subset V$ of x such that

$$|\sum_{\substack{l<n \\ l \neq l_1}} g_l(x) - \sum_{\substack{l<n \\ l \neq l_1}} g_l(u)| < \frac{\varepsilon}{4} \text{ for every } u \in U.$$

Since the series $\sum_{l=1}^{\infty} g_l$ is uniformly convergent, there exists a natural index $N > n$ such that

$$|\sum_{l=N+1}^{\infty} g_l(u)| < \frac{\varepsilon}{4} \text{ for every } u \in X.$$

Observe, moreover, that $g_n(x) = w(x)$ and osc $w(x) < 2^{-n+1}$ for $n \geq 2$. There exists a rational $w_{n,m} \in [-2^{-n+1}, 2^{-n+1}]$ such that $|g_n(x) - w_{n,m}| < \frac{\varepsilon}{4}$. Since $x \in A_n \cap \bar{K}_{l_2,k,r,s}$, there exists an open set $K_{n-1,k_2,r_2,m} \subset K_{l_2,k,r,s} \cap U$ $(m > 1)$. Of course,

$$|g_n(x) - g_n(u)| = |w(x) - w_{n,m}| < \frac{\varepsilon}{4}$$

for $u \in K_{n-1,k_2,r_2,m}$. If $\bar{K}_{n-1,k_2,r_2,m} \cap A_{n+1} \neq \phi$ then there exists an open set $K_{n,k_n,r_n,1} \subset K_{n-1,k_2,r_2,m}$, and if $\bar{K}_{i,k_i,r_i,1} \cap A_{i+2} \neq \phi$ for $i = n,\ n+1, \ldots, N-1$, then we choose successively open sets $K_{i,k_i,r_i,1}$ $(i = n+1, \ldots, N)$ such that

$$K_{i,k_i,r_i,1} \subset K_{i-1,k_{i-1},r_{i-1},1} \text{ for } i = n+1, \ldots, N.$$

Of course $\bar{K}_{n,k_n,r_n,1} \subset U \cap K_{l_1,k,r,s} \subset V$ (if $\bar{K}_{i,k_i,r_i,1} \cap A_{i+2} = \phi$ for any $i = n, n+1, \ldots, N-1$, then $g_{i+1} \equiv 0$ on the set $\bar{K}_{i,k_i,r_i,1}$).

For each $u \in K_{N,k_N,r_N,1}$ we have

$$\begin{aligned}
|s(x) - s(u)| &= |\sum_{l=1}^{\infty} g_l(x) - \sum_{l=1}^{\infty} g_l(u)| \leq |\sum_{\substack{l<n \\ l\neq l_1}} g_l(x) - \sum_{\substack{l<n \\ l\neq l_1}} g_l(u)| + \\
&+ |g_{l_1}(x) - g_{l_1}(u)| + |g_n(x) - g_n(u)| + \\
&+ |\sum_{l=n+1}^{N} g_l(x) - \sum_{l=n+1}^{N} g_l(u)| + |\sum_{l=N+1}^{\infty} g_l(x) - \sum_{l=N+1}^{\infty} g_l(u)| < \\
&< \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4} < \varepsilon,
\end{aligned}$$

so that the function s is quasi continuous at x.

Now let $x \in A_1$. Fix an $\varepsilon > 0$ and an open neighborhood V of the point x. There exists a rational number $w_{1,m} \in R$ such that $|w_{1,m} - w(x)| < \frac{\varepsilon}{4}$. There exists also an open set $K_{n_1,m} \subset V$ such that

$$|g_1(x) - g_1(u)| = |w_{1,m} - w(x)| < \frac{\varepsilon}{4}$$

for every $u \in K_{n_1,m}$. The series $\sum_{l=1}^{\infty} g_l$ is uniformly convergent, so that there exists a natural index $N > 1$ such that

$$|\sum_{l=N+1}^{\infty} g_l(u)| < \frac{\varepsilon}{4} \text{ for each } u \in X.$$

If $\bar{K}_{n_1,m} \cap A_2 \neq \phi$ then, similarly as before, there exists an open set $K_{1,k_1,r_1,1}$ such that $\bar{K}_{1,k_1,r_1,1} \subset K_{n_1,m}$ and if, for $i = 1,2,\dots,N-1$, we have $K_{i,k_1,r_1,1} \cap A_{i+1} \neq \phi$, then we choose successively open sets $K_{i,k_i,r_i,1}$ $(i = 2,3,\dots,N)$ such that

$$K_{n_1,m} \supset K_{1,k_1,r_1,1} \supset K_{2,k_2,r_2,1} \supset \cdots \supset K_{N,k_N,r_N,1}$$

.

Then, for $u \in K_{N,k_N,r_N,1}$, we have

$$\begin{aligned}
|s(x) - s(u)| &= |\sum_{l=1}^{\infty} g_l(x) - \sum_{l=1}^{\infty} g_l(u)| \leq |g_1(x) - g_1(u)| + \\
&+ |\sum_{l=2}^{N} g_l(x) - \sum_{l=2}^{N} g_l(u)| + |\sum_{l=N+1}^{\infty} g_l(x) - \sum_{l=N+1}^{\infty} g_l(u)| < \\
&< \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4} < \varepsilon.
\end{aligned}$$

Consequently, s is quasi continuous at each $x \in \bigcup_{n=1}^{\infty} A_n$. Similarly, the function t is quasi continuous at $x \in \bigcup_{n=1}^{\infty} A_n$.

Now observe, that from the definition of the functions g_l and h_l it follows that for $l = 1,2,\dots$ we have

$$h_l(x) + g_l(x) = w(x) \text{ for every } x \in A_l \backslash A_{l-1}$$

and

$$h_l(x) + g_l(x) = 0 \text{ for every } x \in X \backslash (A_l \backslash A_{l-1}).$$

Finally, for $l = 1,2,\dots$ we have

$$\sum_{k=1}^{l} [g_k(x) + h_k(x)] = \begin{cases} w(x) & \text{if } x \in A_l \\ 0 & \text{if } x \in X \backslash A_l \end{cases}$$

and

$$\begin{aligned}
s(x) + t(x) &= \sum_{l=1}^{\infty} g_l(x) + \sum_{l=1}^{\infty} h_l(x) = \sum_{l=1}^{\infty} [g_l(x) + h_l(x)] = \\
&= \lim_{l \to \infty} \sum_{k=1}^{l} [g_k(x) + h_k(x)] = w(x),
\end{aligned}$$

for each $x \in X$; thus the proof is concluded.

From Theorems 1, 2 and 3 we obtain:

Theorem 4. *If $f \in Cq(X)$, then $f = g + m + s + t$, where $g, m, s, t \in Q(X)$.*

Remark 5. *If $f \in Cq(X)$ is a locally bounded function then $f = g + h + t$, where $g, h, t \in Q(X)$.*

Let $B_\alpha(1 \leq \alpha < \omega_1$, where ω_1 is the first uncountable ordinal number) be the set of all functions $f : X \to R$ of the Baire class α.

Remark 6. *If $f \in Cq(X)$ then f is the sum of functions $g, u, s, t \in Q(X)$, with $g, u, t \in B_1$.*

Proof. From the proof of Lemma 1 it follows that $g \in B_1$. Since in the proof of Theorem 3 $t = \sum_{l=1}^{\infty} h_l$, where $h_l \in B_1$ for each $l = 1, 2, \dots,$ $t \in B_1$.

Finally, observe that if h from Theorem 2 is the function h from the proof of Theorem 1, then in the proof of Theorem 2, u can be defined by the formula

$$u(x) = \begin{cases} h(x) & \text{if } x \in C(h) \\ \limsup\limits_{\substack{t \to x \\ t \in C(h)}} h(t) & \text{if } x \notin C(h) \textit{ and } \\ & \quad x \in \{t \in X;\ \operatorname{osc} h(t) < 1\} \\ 0 & \text{at the remaining points } x \in X, \end{cases}$$

so that $u \in B_1$, and thus the proof is concluded.

Let $M(X)$ be the family of all functions $f : X \to R$ which are measurable relative to a σ-ring containing all Borel sets in the space X.

Corollary 1. *If $f \in Cq(X) \cap B_\alpha$ (or $f \in Cq(X) \cap M(X)$) then f is the sum of four functions $g, u, s, t \in Q(X) \cap B_\alpha$ (f is the sum of four functions $g, u, s, t \in Q(X) \cap M(X)$). Moreover, if $f \in Cq(X) \cap B_\alpha$ (or $f \in Cq(X) \cap M(X)$) and f is a locally bounded then f is the sum of three functions $g, h, t \in Q(X) \cap B_\alpha$ ($g, h, t \in Q(X) \cap M(X)$).*

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