Real Analysis Exchange Vol. 17 (1991–92)
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ON THE DARBOUX PROPERTY OF THE SUM OF CLIQUISH FUNCTIONS

Let $\mathbf{R}$ be the set of reals. A function $f : \mathbf{R} \to \mathbf{R}$ is said to be cliquish at a point $x \in \mathbf{R}$ ([1]) if for every $\varepsilon > 0$ and for every open neighborhood U of x there exists a nonempty open set $V \subset U$ such that $\operatorname{osc}_V f \leq \varepsilon$. Observe that $f : \mathbf{R} \to \mathbf{R}$ is cliquish at each point $x \in \mathbf{R}$ iff the set of its continuity points is dense.

In 1987 , H. W. Pu and H. H. Pu established the following theorem (See [2].):

Theorem P.P. *Let A be a finite family of Baire 1 functions. Then there exists a Baire 1 function f such that $f + g$ is a Darboux function for every $g \in A$.*

In this paper I prove that this theorem is true for finite families A of cliquish functions.

Let $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, \infty\}$. For a given function $f : \mathbf{R} \to \overline{\mathbf{R}}$ such that the set $\{x \in \mathbf{R} : f(x) = +\infty \text{ or } -\infty\}$ is nowhere dense, let $C(f)$ be the set of continuity points of f and let $D_n(f) = \{x \in \mathbf{R} : \operatorname{osc} f(x) \geq 2^{-n}\}$ $(n = 1, 2, \ldots)$.

We start with the following lemma:

Lemma 1. *Let $f : \mathbf{R} \to \overline{\mathbf{R}}$ be an upper semicontinuous function (a lower semicontinuous function) such that $f > -\infty$ ($f < \infty$) and $\{x \in \mathbf{R} : f(x) = \infty\}$ ($\{x \in \mathbf{R} : f(x) = -\infty\}$) is nowhere dense. Then for every $c \in \mathbf{R}$ there is an upper semicontinuous (a lower semicontinuous) function $g : \mathbf{R} \to \overline{\mathbf{R}}$ such that $D_n(f) = D_n(g)$ for $n = 1, 2, \ldots, f|C(f) = g|C(g)$, and $c \notin g(\mathbf{R} \setminus C(g))$.*

Proof. Suppose that f is upper semicontinuous. If f is lower semicontinuous, it suffices to consider the function $-f$. Since f and the oscillation of f are upper semicontinuous, all sets $D_n(f)$ $(n = 1, 2, \ldots)$ are closed and nowhere dense. For every $n = 2, 3, \ldots$ there are disjoint finite open intervals I_{nk} with ends belonging to $C(f)$ such that

$$D_n - D_{n-1} = \bigcup_k (D_n \cap I_{nk}).$$

Since every set $D_n \cap I_{nk}$ is compact,

$$2^{-n} \leq d_{nk} = \max\{\operatorname{osc} f(t) : t \in D_n \cap I_{nk}\} < 2^{1-n}.$$

Denote by $c\ell$ the closure operation and let $D = \{x \in \mathbf{R} \setminus C : f(x) = c\}$. Let

$$g(x) = \begin{cases} +\infty & \text{for } x \in Cl\ D \cap D_1(f) \\ c + \min((2^{1-n} - d_{nk})/2, 2^{-n-k}) & \text{for } x \in c\ell\ D \cap I_{nk} \cap D_n \\ & (n = 2, 3, \ldots, k = 1, 2, \ldots) \\ f(x) & \text{otherwise.} \end{cases}$$

Since f is upper semicontinuous, it follows from the definition of g that g is upper semicontinuous, $g|C(g) = f|C(f), D_1(f) = D_1(g)$, and $D_n(f) \setminus D_{n-1}(f) = D_n(g) \setminus D_{n-1}(g)$ for $n = 2, 3, \ldots$. Evidently $c \notin g(\mathbf{R} \setminus C(g))$.

Theorem 1. *Suppose that the functions $g_1, \ldots, g_k : \mathbf{R} \to \overline{\mathbf{R}}$ are Baire 1 and the sets $\{x : g_i(x) = +\infty \text{ or } -\infty\}$ are nowhere dense. Then there is a Baire 1 function $f : \mathbf{R} \to \mathbf{R}$ such that $f + g_i$ is a Darboux function for $i = 1, \ldots, k$.*

Proof. The proof is the same as the proof of Theorem in [2]. Since every g_j $(j = 1, 2, \ldots, k)$ is a Baire 1 function, each $D_i = \bigcup_{j=1}^k D_i(g_j)$ is a closed nowhere dense set and $D = \bigcup_{i=1}^\infty D_i$ is of first category.

The construction involves a sequence of open residual sets $(G_k)_k$. Each G_k has components $((a_{kj}, b_{kj}))_j$ (j runs from 1 to ∞ or to a certain integer depending on k). Let $r_1 = +\infty$ and $r_k = 2^{-(k-2)}$ if $k \geq 2$. We take D as above, $(a, b) = (a_{kj}, b_{kj})$, $l = r_k$. By Lemma in [2], there exist a Darboux Baire 1 function $h_{kj} : (a_{kj}, b_{kj}) \to \mathbf{R}$ and a first category set $P_{kj} \subset (a_{kj}, b_{kj})$ such that

(i) $P_{kj} \cap D = \emptyset$,

(ii) $c\ell\ P_{kj} = P_{kj} \cup \{a_{kj}, b_{kj}\}$,

(iii) $|h_{kj}(x)| < r_k$ for every $x \in (a_{kj}, b_{kj})$,

(iv) $\{x : h_{kj}(x) \neq 0\} \subset P_{kj}$,

(v) $\limsup_{x \to a_{kj}+} h_{kj}(x) = \limsup_{x \to b_{jk}-} h_{kj}(x) = r_k$, and
$\liminf_{x \to a_{kj}+} h_{kj}(x) = \liminf_{x \to b_{kj}-} h_{kj}(x) = -r_k$.

For the case $k = 1$, we require more of each h_{1j}. This will be made clear later.

For each k, we define h_k on $\mathbf{R}$ by

$$h_k(x) = \begin{cases} h_{kj}(x) & \text{if } x \in (a_{kj}, b_{kj}) \text{ for some } j, \\ 0 & \text{if } x \notin G_k, \end{cases}$$

and set $P_k = \bigcup_{i=1}^k \bigcup_j P_{kj}$. Clearly h_k is a Baire 1 function and P_k is a first category set disjoint from D. Moreover, by (ii), (ii+) $c\ell(\bigcup_j P_{kj}) \subset (\mathbf{R} \setminus G_k) \cup \bigcup_J P_{kj}$ for each k.

Also, since each G_k is an open residual set, the sets $\{a_{kj}\}_j$ and $\{b_{kj}\}_j$ are dense in $\mathbf{R} \setminus G_k$. Using (v), we can easily show

(v+) $\limsup_{t \to x+} h_k(t) = \limsup_{t \to x-} h_k(t) = r_k$, and
$\liminf_{t \to x+} h_k(t) = \liminf_{t \to x-} h_k(t) = -r_k$ at each $x_k \in \mathbf{R} \setminus G_k$.

Let $G_1 = \mathbf{R} \setminus D_1$ and a component (a_{1j}, b_{1j}) be fixed. Let the intervals (a_{1j}, b_{1j}), I_{jn}, J_{jn} $(n = 1, 2, \ldots)$ correspond to (a, b), I_n, J_n in Lemma in [2]. For each n, $(I_{jn} \cup J_{jn}) \cap D_1 = \emptyset$, and hence osc $g_i(x) < 1/2$ for every $x \in I_{jn} \cup J_{jn}$ and $i = 1, 2, \ldots, k$. Since each $I_{jn} \cup J_{jn}$ is a compact set, there exists $M_{jn} > 0$ such that $|g_i(x)| < M_{jn}$ $(i = 1, \ldots, k)$ for every $x \in I_{jn} \cup J_{jn}$. With no loss of generality, we assume that $M_{j1} \leq M_{j2} \leq \ldots$. Let $r_1 = +\infty$, $r_{jn} = 2M_{jn} + n$ correspond to l and l_n in Lemma in [2]. Then h_{1j} can be chosen to satisfy the conditions (i) - (v) (for $k = 1$) and

(vi) $\sup h_{1j}(I_{jn}) = \sup h_{1j}(J_{jn}) = r_{jn}$ if n is even ,
$\inf h_{1j}(I_{jn}) = \inf h_{1j}(J_{jn}) = -r_{jn}$ if n is odd .

We now proceed with the induction step. Assume that for some $k \geq 1$, we have constructed an open residual set G_k, the associated functions h_{kj} (j runs through the enumeration of the components of G_k) and h_k, the associated first category set P_{kj} and P_k such that $D_k \cup P_k$ is closed. Clearly $D_{k+1} \cup P_k$ is a closed first category set. We take $G_{k+1} = \mathbf{R} \setminus (D_{k+1} \cup P_k)$. The associated functions and sets are described above. To complete the induction, we need to show that $D_{k+1} \cup P_{k+1}$ is closed. By (ii+) and the choice of G_{k+1},

$$c\ell\,(\bigcup_j P_{k+1,j}) \subset (\bigcup_j P_{k+1,j}) \cup (D_{k+1} \cup P_k) = D_{k+1} \cup P_{k+1}.$$

Since $D_{k+1} \cup P_k$ is closed, $D_{k+1} \cup P_k = c\ell\,(D_{k+1} \cup P_k) = D_{k+1} \cup c\ell\ P_k$. Consequently,

$$D_{k+1} \cup P_{k+1} \supset D_{k+1} \cup c\ell\ P_k \cup c\ell\,(\bigcup_j P_{k+1,j}) = D_{k+1} \cup c\ell\ P_{k+1}.$$

This implies that $D_{k+1} \cup P_{k+1}$ is closed. Thus we have constructed the sequence $(h_k)_k$ by induction. Note that the series $\sum_{k=1}^{\infty} h_k$ converges uniformly on $\mathbf{R}$. Therefore we can define a function f on $\mathbf{R}$ by letting $f = \sum_{k=1}^{\infty} h_k$ and comclude that f is a Baire 1 function.

As in the proof of Theorem P.P. in [2] we may show that f is a Darboux function on $\mathbf{R}$ and $f + g_1$ $(i = 1, \ldots, k)$ have the Darboux property on each interval $[a, b]$

such that $(f+g_i)([a,b]) \subset \mathbf{R}$. Suppose that $[a,b]$ is a closed interval such that $(f+g_i)([a,b]) \not\subset \mathbf{R}$ and $f(a)+g_i(a) \neq f(b)+g_i(b)$ for some $i \leq k$. Let

$$c \in (\min(f(a)+g_i(a), f(b)+g_i(b)), \max(f(a)+g_i(a), f(b)+g_i(b))).$$

Since $(f+g_i)([a,b]) \not\subset \mathbf{R}$, it follows from the construction of f that there exists an interval $[a_1,b_1] \subset (a,b)$ such that $(f+g_i)([a_1,b_1]) \subset \mathbf{R}$, and

$$\min(f(a_1)+g_i(a_1), f(b_1)+g_i(b_1)) < c < \max(f(a_1)+g_i(a_1), f(b_1)+g_i(b_1)).$$

Since $f+g_i$ has the Darboux property on the interval $[a_1,b_1]$, there is a point $d \in (a_1,b_1)$ with $f(d)+g_i(d)=c$.

This completes the proof.

Remark 1. In the above construction, the sets P_{kj} can be chosen to have Lebesgue measure zero. Then the function f equals zero except on a first category set of Lebesgue measure zero.

Remark 2. Preserve all hypothesis and notations of Theorem 1 and its proof. If $f_1,\dots,f_k : \mathbf{R} \to \overline{\mathbf{R}}$ are Baire 1 functions such that

$$f_i|(\mathbf{R}\setminus D) = g_i|(\mathbf{R}\setminus D) \text{ for } i=1,\dots,k, \text{ and}$$

$$D_j = \bigcup_{i=1}^{k} D_j(f_i) \text{ for } j=1,2,\dots ,$$

then every function $f+f_i$ $(i=1,\dots,k)$ has the Darboux property. Of course, it suffices to observe that in the proof of Theorem 1 the construction of the function f for the system $(f_1,\dots,f_k)$ can be the same as for the system $(g_1,\dots,g_k)$.

Remark 3. Preserve all assumptions and notation of Theorem 1 and its proof. Moreover, suppose that the functions g_i, $i=1,\dots,k$, are upper semicontinuous everywhere or lower semicontinuous everywhere. If $z \in \mathbf{R}\setminus C(g_i)$ for some $i \leq k$, $g_i(z) = c \in \mathbf{R}$, and $[u,v]$ is a closed interval containing z, then there exists a point $w \in (u,v) \cap \bigcap_{j=1}^{k} C(g_j)$ such that $f(w)+g_i(w)=c$.

Proof. By Lemma 1 there exists a function $g : \mathbf{R} \to \overline{\mathbf{R}}$ such that $c \notin g(\mathbf{R}\setminus C(g))$, $g|C(g) = g_i|C(g_i)$, $D_j(g) = D_j(g_i)$ for $j=1,2,\dots$, and g is upper (lower) semicontinuous everywhere whenever g_i is the same. For every $n=1,2,\dots$ there are disjoint finite open intervals K_{nm}, $m=1,2,\dots$, with ends belonging to $\bigcap_{j=1}^{k} C(g_j)$ such that

$$D_1 \setminus D_1(g_i) = \bigcup_m (D_1 \cap K_{1m}), \text{ and}$$

$$D_n \setminus D_{n-1} \setminus D_n(g_i) = \bigcup_m (D_n \cap K_{nm}) \text{ for } n = 2, 3, \ldots .$$

Let $E = \{x \in D : g_i(x) = c\}$. Set

$$\bar{g}(x) = \begin{cases} c + 2^{-n} & \text{for } x \in c\ell\, E \cap K_{nm} \cap D_n \quad (n, m = 1, 2, \ldots) \\ g(x) & \text{otherwise} \end{cases}$$

whenever g_i is upper semicontinuous, or

$$\bar{g}(x) = \begin{cases} c - 2^{-n} & \text{for } x \in c\ell\, E \cap K_{nm} \cap D_n \quad (n, m = 1, 2, \ldots) \\ g(x) & \text{otherwise,} \end{cases}$$

whenever g_i is lower semicontinuous.

Note that $D_n = \bigcup_{\substack{j=1 \\ j \neq i}}^{k} D_n(g_j) \cup D_n(\bar{g})$, $n = 1, 2, \ldots$, and $c \notin \bar{g}(D)$. Moreover $\bar{g}$ is upper (lower) semicontinuous everywhere. Since $z \in D$, it follows from (v+) and from the construction of f that there are points $u_0, v_0 \in (u, v) \cap (\mathbf{R} \setminus D)$ such that

$$f(u_0) + g_i(u_0) < c \text{ and } f(v_0) + g_i(v_0) > c.$$

With no loss of generality, we may assume that $u_0 < v_0$. If

$$\{x \in (u_0, v_0) : f(x) + g_i(x) = c\} \cap (\mathbf{R} \setminus D) = \emptyset \text{ then}$$
$$\{x \in (u_0, v_0) : f(x) + \bar{g}(x) = c\} = \emptyset,$$

contrary to Remark 2.

Theorem 2. *Let $f_1, \ldots, f_k : \mathbf{R} \to \mathbf{R}$ be cliquish functions. There is a Baire 1 function $f : \mathbf{R} \to \mathbf{R}$ such that $\{x \in \mathbf{R} : f(x) \neq 0\}$ is of Lebesgue measure zero and all sums $f + f_i$, $i = 1, \ldots, k$, are Darboux functions.*

Proof: For $i = 1, \ldots, k$ let

$$g_i(x) = \lim_{r \to 0+} \inf\{f_i(t) : |t - x| < r\}, \text{ and}$$
$$h_i(x) = \lim_{r \to 0+} \sup\{f_i(t) : |t - x| < r\}$$

for $x \in \mathbf{R}$.

Evidently, g_i (h_i), $i = 1, \ldots, k$, are lower (upper) semicontinuous, $g_i \leq f_i \leq h_i$, $g_i(x) = f_i(x) = h_i(x)$ for $x \in C(f_i)$, and the sets $\{x : g_i(x) = -\infty\}$ and

$\{x : h_i(x) = \infty\}$ are nowhere dense. By Theorem 1, there exists a Darboux Baire 1 function $f : \mathbf{R} \to \mathbf{R}$ such that $\{x : f(x) \neq 0\}$ is of measure zero and all sums $f + g_i$, $f + h_i$, $i = 1, \ldots, k$, are Darboux functions. Fix $i \leq k$. Let $[a, b]$ be a closed interval such that $f(a) + f_i(a) \neq f(b) + f_i(b)$, for example $f(a) + f_i(a) < f(b) + f_i(b)$. Let c be a number such that $f(a) + f_i(a) < c < f(b) + f_i(b)$. If $\min(f(a) + h_i(a), f(b) + h_i(b)) < c < \max(f(a) + h_i(a), f(b) + h_i(b))$, then there is a point $u \in (a, b)$ such that $f(u) + h_i(u) = c$. If $u \in C(f_i)$, then $h_i(u) = f_i(u)$ and $c = f(u) + f_i(u)$. If $u \in (a, b) \setminus C(f_i)$ then, by Remark 3, there is a point $v \in (a, b) \cap C(g_i) \cap C(h_i) = (a, b) \cap C(f_i)$ such that $f(v) + f_i(v) = f(v) + h_i(v) = f(u) + h_i(u) = c$. In the case where $c \leq \min(f(a) + h_i(a), f(b) + h_i(b))$ we remark that $f(a) + g_i(a) < c$. If $b \in C(f_i)$, then $f(b) + g_i(b) = f(b) + f_i(b) > c$ and there is a point $u \in (a, b)$ such that $f(u) + g_i(u) = c$. If $u \in C(f_i)$, then $f(u) + f_i(u) = f(u) + g_i(u) = c$. If $u \in (a, b) \setminus C(f_i)$, then by Remark 3, there is a point $v \in (a, b) \cap C(f_i)$ such that $f(v) + f_i(v) = f(v) + g_i(v) = c$. In the case where $c \leq \min(f(a) + h_i(a), f(b) + h_i(b))$ and $b \notin C(f_i)$, Remark 3 implies that there is a point $w \in (a, b) \cap C(f_i)$ with $f(w) + f_i(w) = f(w) + g_i(w) > c$. Consequently, as above, there is a point $u \in (a, w)$ such that $f(u) + f_i(u) = c$.

Remark 4. Theorem 2 is false for an infinite family A of cliquish functions. (See [2], Example in 3.)

References

[1] W. W. Bledsoe, *Neighbourly functions*, Proc. Amer. Math. Soc. 3 (1972), 114-115.

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Received February 27, 1991