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Variation of f on E and Lebesgue Outer Measure of fE

Let f be a real-valued function on a cell $K = [a, b]$. By "cell" we mean a closed, bounded, nondegenerate interval in $\mathbf{R}$. The total variation of f is given by a Kurzweil-Henstock integral $\int_K |df| \leq \infty$ defined as the gauge-filtered limit of approximating sums over cell divisions with endpoint tags. For a development of this type of integral and its associated definition of differential, see [3,4,5]. We hope the reader will be impressed with the utility of our differential formulations based on an "honest" defintion of differential. We define the variation of f on a subset E of K to be the upper integral $\overline{\int}_K 1_E|df| \leq \infty$ where 1_E is the indicator of E. We call E df-null if this integral is zero, that is, if the differential $1_E\, df = 0$ [3,4]. Before the advent of the Kurzweil-Henstock integral df-null sets E were treated indirectly by using the condition that the image fE be Lebesgue-null. Indeed, as we shall show in Theorem 2, fE is Lebesgue-null if E is df-null. This result enables us to avoid the usual tedious proofs that an image fE is Lebesgue-null by resorting to a concise proof of the inherently stronger condition that E is df-null. Theorem 11 gives a converse to Theorem 2 for f a continuous function of bounded variation. For such f a set E is df-null if and only if fE is Lebesgue-null. So for continuous f of bounded variation Lusin's condition (N) that f map Lebesgue-null sets into Lebesgue-null sets is obviously just the absolute continuity conditon that every Lebesgue-null set is df-null. Let m be Lebesgue measure and m^* be Lebesgue outer measure.

Theorem 1. Let E be a subset of K such that at each point of E f is either left or right continuous. Then

$$m^*(fE) \leq 2\overline{\int}_K 1_E|df|. \tag{1}$$

Proof: Let D be the set of those t in E for which there exist cells J containing t with diam $fJ = 0$, that is, with f constant on J. Clearly fD is countable, so $m(fD) = 0$. Given a gauge δ on K and $\varepsilon > 0$ each t in $E \setminus D$ is an endpoint of some cell J in K such that (J, t) is δ-fine and $0 <$ diam $FJ < \varepsilon$. Given $c > 1$ choose s in J such that

$$(2) \qquad 0 < \sup_{r \in J} |f(r) - f(t)| \leq c|f(s) - f(t)|.$$

Let I be the cell with endpoints s, t. Then by (2) we have

$$(3) \qquad \operatorname{diam} fI \leq \operatorname{diam} fJ \leq 2c|\Delta f(I)|.$$

Thus

$$(4) \qquad (I, t) \quad \text{is a } \delta\text{-fine tagged cell with } t \text{ in } E/D \text{ and } 0 < \operatorname{diam} fI \leq 2c|\Delta f(I)|.$$

Moreover,

$$(5) \qquad \operatorname{diam} fI < \varepsilon.$$

Let $\mathcal{H}$ be the set of all cells H of the form $[\inf fI, \sup fI]$ for some (I, t) satisfying (4). By (5) $\mathcal{H}$ is a Vitali covering of $f(E \setminus D)$, hence of m-all of fE. So m-all of fE is covered by a countable set $\{H_i\}$ of disjoint members H_i of $\mathcal{H}$. For each H_i choose (I_i, t_i) satisfying (4) with H_i the convex closure of fI_i. The I_i's are disjoint since $fI_i \subseteq H_i$. Thus by (4)

$$(6) \qquad \begin{aligned} m^*(fE) &\leq \Sigma_i\, m(H_i) = \Sigma_i \operatorname{diam} fI_i \leq 2c\, \Sigma_i |\Delta f(I_i)| \\ &\leq 2c\, \bar{\Sigma}(1_E |\Delta f|, \delta) \end{aligned}$$

where the upper sum is the supremum of all approximating sums over δ-divisions of K, [3,4]. Since δ is an arbitrary gauge (6) gives

$$(7) \qquad m^*(fE) \leq 2c \overline{\int_K} 1_E |df| \quad \text{for all } c > 1.$$

Clearly (7) implies (1). $\square$

Theorem 2. If A is df-null then $m(fA) = 0$.

Proof: Since $1_A\, df = 0$, $1_p\, df = 0$ for every point p in A. That is, f is continuous at every p in A. So Theorem 1 gives $m(fA) = 0$. $\square$

Under certain conditions on f the inequality (1) in Theorem 1 can be sharpened by halving the coefficient 2 on the right side in (1). Such is our next result.

Theorem 3. Let E be a subset of K and A a df-null subset of E such that for each t in $E \setminus A$ either f is left continuous and $(df/|df|)_-(t)$ exists, or f is right continuous and $(df/|df|)_+(t)$ exists. Then

$$m^*(fE) \leq \overline{\int}_K 1_E|df|. \tag{8}$$

Proof: By Theorem 2 we may assume A is empty. Existence (in the narrow sense [5]) of $(df/|df|)_-(t) = \lim_{I \to t-} \Delta f(I)/|\Delta f(I)|$ means that for all sufficiently small I with right endpoint t sgn $\Delta f(I) = (df/|df|)_-(t) \neq 0$. If this left derivative equals 1 then $f(t) = \text{Max } fI$. If it equals -1 then $f(t) = \text{Min } fI$. Analogous statements hold for the right derivative. In either case the radius of fI about $f(t)$ equals diam fI. So the coefficient 2 in (3) can be replaced by 1 inducing the replacement of $2c$ by c in (4), (6), (7). The modified proof of Theorem 1 then gives (8) in place of (1). □

A classical special case of Theorem 3 is the following.

Theorem 4. If $f'(t)$ exists and is finite for df-all t in E then (8) holds.

Proof: If $f'(t)$ exists and is not 0 then so does $df/|df|(t) =$ sgn $f'(t) = \pm 1$. Let B be the set of all t where $f'(t)$ exists and is finite. Then $1_B\ df = 1_B\ f'\ dt$. So the set of points where $f' = 0$ is df-null. By hypothesis the subset of E where f' fails to exist is df-null. Hence, the set A of all t in E where either $f'(t) = 0$ or $f'(t)$ fails to exist is df-null. At each t in $E \setminus A$ f has a nonzero derivative. So f is continuous at t and $df/|df|\ (t)$ exists. Hence, Theorem 3 gives (8). □

Our next result gives (8) in particular for continuous functions of bounded variation. df is <u>dampable</u> if there exists an everywhere positive function u on K such that both $u\ df$ and $u|df|$ are integrable [3,4].

Theorem 5. Let f be continuous with df dampable on K. Then (8) holds for every subset E of K.

Proof: By Prop. 21 of [5] there is a df-null subset A of K such that $df/|df|\ (t)$ exists in the narrow sense at every t in $K \setminus A$. So Theorem 3 applies and gives (8). □

Our next result generalizes an exercise in [6] whose utility was pointedly noted by Varberg [7].

Theorem 6. Let $|f'(t)| \leq c$ at df-all t in E. Then $m^*(fE) \leq c\, m^*(E)$.

Proof: By hypothesis there is a df-null subset A of E such that $|f'(t)| \leq c$ for all t in $E \backslash A$. Thus $1_E|df| = 1_{E\backslash A}|df| = 1_{E\backslash A}|f'|dt \leq c\ 1_E\ dt$. By Theorem 4 (8) holds. So $m^*(fE) \leq \overline{\int}_K 1_E|df| \leq c \overline{\int}_K 1_E\ dt \leq c\, m^*(E)$. □

In the results that follow we shall apply Theorem 2 to get Lebesgue nullity of image sets in Theorems 7, 8, 9, 11 and 12.

Theorem 7. Let f be continuous with df dampable on K. Then the set A of all points where f has either a left or right derivative equal to zero is df-null. So $m(fA) = 0$.

Proof: Let B be the set of points where f has a right derivative equal to 0. We need only show B is df-null since a similar proof applies for the left derivative. Let $P(I,t) = 1$ if t is the left endpoint of I, 0 if t is the right endpoint. Then $P\ 1_B\ \ \Delta f = o(\Delta t)$ so $P\ 1_B\ df = 0$. By Theorem 16 of [4] $P\ 1_B$ is tag-null df-everywhere. That is, for df-all t in B the indicator $P(I,t) = 0$ ultimately as $I \to t$. But this can only occur at b since for $t < b$ we have $P(I,t) = 1$ as $I \to t$. But this can only occur at b since for $t < b$ we have $P(I,t) = 1$ as $I \to t+$. Clearly b cannot belong to B. So the empty set is df-all of B. That is, B is df-null. $m(fB) = 0$ by Theorem 2. □

Theorem 8. Let A be a Lebesgue-null subset of K such that all four Dini derivates of f are finite at each point in A. Then A is df-null, so $m(fA) = 0$.

Proof: By hypothesis there exist a gauge δ and a function g on K such that $|\Delta f|/\Delta t\ (I) \leq g(t)$ for all δ-fine (I,t) with t in A. That is, $1_A|\Delta f| \leq 1_A g\ \Delta t$ at each δ-fine tagged cell. So $1_A|df| \leq 1_A g\ dt = 0$ since $1_A\ dt = 0$ for Lebesgue-null A. Hence, $1_A\ df = 0$. $m(fA) = 0$ by Theorem 2. □

Theorem 2 easily gives Theorem 1 of [2] which we formulate in terms of differentials as our next result.

Theorem 9. If $df = g\ dt$ on K and A is a Lebesgue-null subset of K then fA is Lebesgue-null.

Proof: Since $1_A\ dt = 0$, $1_A\ df = g\ 1_A\ dt = 0$. So $m(fA) = 0$ by Theorem 2. □

Our next result is an extension of Banach's indicatrix theorem [1].

Theorem 10. Let f be a continuous function of bounded variation on K. Let $N(t)$ be the number of points s in K such that $f(s) = t$. Then for every Borel set E in $\mathbf{R}$

$$\int_K 1_{f^{-1}E}\,|df| = \int_{\mathbf{R}} 1_E\,N\,dt. \tag{9}$$

Proof: (Since $N = 0$ outside fK the integral on the right in (9) is effectively over fK rather than over $\mathbf{R}$.) For f of bounded variation $\int_K 1_B|df|$ defines a measure on the Borel sets B in K [4]. Since f is Borel measurable $f^{-1}E$ is a Borel set for E a Borel set. So $\alpha(E)$ defined by the left side of (9) gives a Borel measure α on $\mathbf{R}$, indeed on fK. By Banach's indicatrix theorem [1,6] $N < \infty$ dt-everywhere and (9) holds for $E = \mathbf{R}$. That is,

$$\int_K |df| = \int_K N\,dt. \tag{10}$$

Since $N\,dt$ is integrable and nonnegative it defines a Borel measure β on $\mathbf{R}$ with $\beta(E)$ given by the right side of (9) for each Borel set E in $\mathbf{R}$. We need only show $\alpha = \beta$. Given D open in K apply (10) to $K_i = \overline{I}_i$ for each component I_i of D. Since f is continuous, the integral of $|df|$ over K_i equals the integral of $1_{I_i}|df|$ over K. So summation of (10) over K_i gives

$$\int_K 1_D|df| = \int_{\mathbf{R}} N_D\,dt \tag{11}$$

where $N_D(t)$ is the number of points s in D such that $f(s) = t$. Given an open subset B of $\mathbf{R}$ apply (11) to the open subset $D = f^{-1}B$ of K, noting that $N_D = 1_B\,N$, to conclude that $\alpha(B) = \beta(B)$. So $\alpha = \beta$ since Borel measures on $\mathbf{R}$ are regular. □

Theorem 10 gives a converse to Theorem 2 for continuous functions of bounded variation. This is our next result. The conclusion that $m(gA) = 0$ is Theorem 18 in [7].

Theorem 11. Let f be a continuous function of bounded variation on K. Let A be a subset of K such that $m(fA) = 0$. Then A is df-null and $m(gA) = 0$ for $dg = |df|$.

Proof: The Lebesgue-null fA is contained in a Lebesgue-null Borel set E, $1_E\,dt = 0$. So $f^{-1}E$ is df-null by (9). Thus, since $A \subseteq f^{-1}E$, A is df-null. For $dg = |df|$, A is dg-null. Hence, $m(gA) = 0$ by Theorem 2. □

Our next result characterizes monotoneity in terms of the upper right derivate. A similar result holds for the upper left derivate. The open interval U can be bounded or unbounded.

Theorem 12. Let f be continuous on an open interval U. Let A be the set of all t in U where the upper right derivate $D^+f(t) \leq 0$. Then the following conditions are equivalent:

(i) $f(s) \leq f(t)$ for all $s \leq t$ in U,

(ii) A is df-null on every cell K in U,

(iii) fA is Lebesgue-null,

(iv) fA has no interior points.

Proof: Given (i) all derivates of f are nonnegative. Hence, f has a right derivate equal to 0 at every point in A. So (ii) follows from Theorem 7. (ii) implies (iii) by Theorem 2 and the countable additivity of m. (iii) trivially implies (iv). Given (iv) suppose (i) false. Then $f(a) > f(b)$ for some $a < b$ in U. We contend this implies $(f(b), f(a)) \subseteq fA$ contradicting the hypothesis (iv). Let $K = [a, b]$. Consider any t in the open interval $(f(b), f(a))$. We contend t is in fA. Since f is continuous $K \cap f^{-1}t$ is nonempty by the intermediate value theorem. It is moreover compact. Let q be its last point. Then since $f(b) < t$ the intermediate value theorem implies $f(s) < t$ for all s in $(q, b]$, hence $f(s) - f(q)/s - q < 0$. So q is in A. Hence $t = f(q)$ is in fA. □

We finish with a pair of exercises characterizing monotone and monotone, continuous functions.

Theorem 13. Let f be a function on K such that

$$\operatorname{diam} fK = \int_K |df| < \infty. \tag{12}$$

Then f is monotone.

Proof: We may assume $f(a) \leq f(b)$. (Otherwise consider $-f$.) Given $s \leq t$ in K we contend $f(s) \leq f(t)$. Take $s_n \leq t_n$ in K such that

$$|f(s_n) - f(t_n)| \to \operatorname{diam} fK \text{ as } n \to \infty. \tag{13}$$

Since $a \leq s_n \leq t_n \leq b$, $|f(a) - f(s_n)| + |f(s_n) - f(t_n)| + |f(t_n) - f(b)| \leq \int_K |df|$. So for all $n = 1, 2, \ldots$

$$(14) \qquad |f(a) - f(s_n)| + |f(t_n) - f(b)| \leq \int_K |df| - |f(s_n) - f(t_n)|.$$

By (14), (13), (12) $f(s_n) \to f(a)$ and $f(t_n) \to f(b)$ as $n \to \infty$. So $f(b) - f(a) = \int_K |df|$ by (13), (12). Thus $f(s) - f(a) + |f(t) - f(s)| + f(b) - f(t) \leq f(b) - f(a)$ since $a \leq s \leq t \leq b$. Hence $|f(t) - f(s)| \leq f(t) - f(s)$. That is, $f(s) \leq f(t)$. □

Our final result follows easily from Theorem 13.

Theorem 14. Let f be a function on K such that

$$(15) \qquad m^*(fK) = \int_K |df| < \infty.$$

Then f is monotone and continuous.

Proof: Let $J = [\inf fK, \ \sup fK]$. Then $m^*(fK) \leq m(J) = \ \operatorname{diam} fK \leq \int_K |df|$. By (15) equality holds throughout these inequalities. In particular (12) holds. So f is monotone by Theorem 13. Since $m(fK) = m(J)$, fK is dense in J. So the monotone f cannot have a saltus in K. Hence, f is continuous. □

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