Real Analysis Exchange Vol.16 (1990–91)
Boguslaw Kaczmarski, Centre of Training Teachers, Warsaw, Poland.

THE SETS WHERE A FUNCTION HAS INFINITE ONE-SIDED DERIVATIVES

In paper [1] Codyks proved following theorem:

Theorem. Let E_1 and E_2 be disjoint subsets of the set of all real numbers. There exists a function f defined on the set of all real numbers such that $E_1 = \{x : f'(x) = +\infty\}$ and $E_2 = \{x : f'(x) = -\infty\}$ if and only if

(i) E_1 and E_2 are of type $F_{\sigma\delta}$ and of measure zero, and

(ii) There exists disjoint sets F_1 and F_2 of type F_σ such that $E_1 \subset F_1$ and $E_2 \subset F_2$.

In the present paper we consider the problem: Is analogus theorem for left (right) - hand derivative of any finite real function true? It turns out that it is not so. We prove that, for any disjoint sets E_1 E_2 of measure zero there exists a function f such that $E_1 = \{x : f'_-(x) = +\infty\}$ and $E_2 = \{x : f'_-(x) = -\infty\}$. Therefore, exists a function for which the sets $\{x : f'_-(x) = +\infty\}$ and $\{x : f'_-(x) = -\infty\}$ are not-Borel.

We shall apply the following notations:

R - the set of all real numbers;

$R \setminus A$ - the complement of the set A;

A^-, A^+ - the set of all accumulation points of the set A from the left, from the right;

$\overline{f}^-(x), \underline{f}^-(x)$ - the upper left-hand, lower left-hand Dini derivatives of a function f at point x;

$m(A)$ - the Lebesgue measure of a set A;

χ_A - the characteristic function of the set A;

$f'_-(x)$ - the left-hand derivatives of the function f at the point x.

Theorem 1. Let $A_1 \subset R$, $A_2 \subset R$, $A_1 \cap A_2 = \emptyset$ and $m(A_1) = m(A_2) = 0$. Then there exists function $f : R \to R$ such that $A_1 = \{x : f'_-(x) = +\infty\} A_2 = \{x : f'_-(x) = -\infty\}$, $(A_1^- \cup A_2^-) \setminus (A_1 \cup A_2) = \{x : f'_-(x)$ does not exist, finite or infinite$\}$, $R \setminus (A_1^- \cup A_1 \cup A_2^- \cup A_2) = \{x : f'_-(x) \neq \pm\infty\}$.

Proof. A_1^- and A_2^- are closed from the left. Therefore $A_1^- \in G_\delta$ and $A_2^- \in G_\delta$. Denote by G_1^*, G_2^* the sets of type G_δ such that $m(G_1^*) = m(G_2^*) = 0$ and $A_1 \subset G_1^*$, $A_2 \subset G_2^*$. Let $G_1 = A_1^- \cap G_1^*$, $G_2 = A_2^- \cap G_2^*$. Then $m(G_1) = m(G_2) = 0$, $G_1 \in G_\delta$, $G_2 \in G_\delta$ and $A_1 \cap A_1^- \subset G_1 \subset A_1^-$, $A_2 \cap A_2^- \subset G_2 \subset A_2^-$.

Let us denote by h_1, h_2 the continuous, non-decreasing, and bounded functions such that $h_1 : R \to R$, $h_1'(x) = +\infty$ for $x \in G_1$, $h_1'(x) \neq +\infty$ for $x \in R \setminus G_1$, $h_1(x) > 0$ for all x and $h_2 : R \to R$, $h_2'(x) = +\infty$ for $x \in G_2$, $h_2'(x) \neq +\infty$ for $x \in R \setminus G_2$, $h_2(x) > 0$ for all x (see Zahorski [3]).

Let $f_1 : R \to R$ and $f_2 : R \to R$ be defined by

$$f_1(x) = \begin{cases} h_1(x) + \chi_{A_1}(x) & \text{for} \quad x \in R \setminus A_2 \\ 0 & \text{for} \quad x \in A_2 \end{cases}$$

and

$$f_2(x) = \begin{cases} h_2(x) + \chi_{A_2}(x) & \text{for} \quad x \in R \setminus A_1 \\ 0 & \text{for} \quad x \in A_1 \end{cases}$$

We have $f_1(x) > 0$ on $R \setminus A_2$ and $f_2(x) > 0$ on $R \setminus A_1$.

We show that $f'_{1-}(x) = +\infty$ for $x \in A_1$. Let $x_0 \in A_1$ and $x < x_0$. Then $f_1(x_0) = h_1(x_0) + 1$ (since $x_0 \notin A_2$) and $f_1(x) \in [0, h_1(x) + 1]$. Hence

$$\frac{f_1(x_0) - f_1(x)}{x_0 - x} \geq \frac{h_1(x_0) - h_1(x)}{x_0 - x}.$$

Thus $f'_{1-}(x_0) = +\infty$ for $x_0 \in A_1 \cap A_1^-$ (since $A_1 \cap A_1^- \subset G_1$). If $x_0 \notin A_1 \cap A_1^-$, then there exists a $\delta > 0$ such that $(x_0 - \delta, x_0) \cap A_1 = \emptyset$. Let $x \in (x_0 - \delta, x_0)$. Then $f_1(x) \in [0, h_1(x)]$. Hence

$$\frac{f_1(x_0) - f_1(x)}{x_0 - x} \geq \frac{h_1(x_0) + 1 - h_1(x)}{x_0 - x} \geq \frac{1}{x_0 - x}.$$

Thus $f'_{1-}(x_0) = +\infty_-$ for $x \in A_1$.

We prove that $\overline{f_1^-}(x) \leq 0$ for $x \in A_2$. Let $x_0 \in A_2$. Then $f_1(x_0) = 0$ and $f_1(x) \geq 0$ for each $x \in R$. So $\overline{f_1^-}(x_0) \leq 0$. Analogously we obtain that $f'_{2-}(x) = +\infty$ for $x \in A_2$ and $\overline{f_2^-}(x) \leq 0$ for $x \in A_1$.

Let $f = f_1 - f_2$. We show that $f'_-(x) = +\infty$ for $x \in A_1$ and $f'_-(x) = -\infty$ for $x \in A_2$. Let $x_0 \in A_1$. Then we have $\underline{f}^-(x_0) = f'_{1-}(x_0) - \overline{f_2}^-(x_0) = +\infty$. Let $x_0 \in A_2$. Then we have $\overline{f}^-(x_0) = \overline{f_1}^-(x_0) - f'_{2-}(x_0) = -\infty$.

We prove that $(A_1^- \cup A_2^-) \setminus (A_1 \cup A_2) = \{x : f'_-(x)$ does not exist, finite or infinite$\}$. First $(A_1^- \cup A_2^-) \setminus (A_1 \cup A_2) = B_1 \cup B_2 \cup B_3$ where $B_1 = (A_1^- \setminus A_2^-) \setminus A_1^- \cup A_2)$, $B_2 = (A_2^- \setminus A_1^-) \setminus (A_1 \cup A_2)$ and $B_3 = (A_1^- \cap A_2^-) \setminus (A_1 \cup A_2)$. Let $x_0 \in B_1$. Then $x_0 \notin A_1 \cup A_2$, $x_0 \in A_1^-$ and $x_0 \notin A_2^-$. Since h_1 is continuous and increasing, it follows that there exists a $\delta > 0$ such that $h_1(x) > h_1(x_0) - \frac{1}{2}$ for $x \in (x_0 - \delta, x_0)$ and $(x_0 - \delta, x_0) \cap A_2 = \emptyset$. Clearly $(x_0 - \delta, x_0) \cap A_1 \neq \emptyset$. Let $y \in (x_0 - \delta, x_0) \cap A_1$. Then $f_1(x_0) = h_1(x_0)$, $f_1(y) = h_1(y) + 1$, $f_2(x_0) = h_2(x_0)$ and $f_2(y) = 0$. Hence

$$\frac{f(x_0) - f(y)}{x_0 - x} = \frac{h_1(x_0) - h_1(y) - \frac{1}{2} - \frac{1}{2} - h_2(x_0)}{x_0 - x} \to -\infty \text{ if } y \nearrow x_0.$$

Let $t \in (x_0 - \delta, x_0) \setminus A_1$. (This is possible since $m(A_1) = 0$.) Since $t \notin A_2$, we have $f_1(t) = h_1(t)$. Then

$$\begin{aligned} \frac{f(x_0) - f(t)}{x_0 - t} &= \frac{h_1(x_0) - h_1(t)}{x_0 - t} - \frac{h_2(x_0) - h_2(t)}{x_0 - t} \to h_1'(x_0) - h_2'(x_0) \\ &\neq -\infty \text{ if } t \nearrow x_0. \end{aligned}$$

(Since $x_0 \notin A_2$, $x_0 \notin G_2$ and hence $h_2'(x_0) \neq +\infty$.) Analogously we obtain that $f'_-(x)$ does not exist, finite or infinite on B_2. Let $x_0 \in B_3$. Since h_1 and h_2 are continuous and increasing, there exists a $\delta > 0$ such that $(x_0 - \delta, x_0) \cap A_1 \neq \emptyset$, $(x_0 - \delta, x_0) \cap A_2 \neq \emptyset$, $x_0 \notin A_1 \cup A_2$ and $h_1(x) > h_1(x_0) - \frac{1}{2}$, $h_2(x) > h_2(x_0) - \frac{1}{2}$ for $x \in (x_0 - \delta, x_0)$. Let $y \in (x_0 - \delta, x_0) \cap A_1$ and $t \in (x_0 - \delta, x_0) \cap A_2$. Then $f_1(x_0) = h_1(x_0)$, $f_2(x_0) = h_2(x_0)$, $f_1(y) = h_1(y) + 1$, $f_2(y) = 0$, $f_1(t) = 0$ and $f_2(t) = h_2(t) + 1$. Clearly

$$\frac{f(x_0) - f(y)}{x_0 - y} \to -\infty \text{ if } y \nearrow x_0 \text{ and}$$

$$\frac{f(x_0) - f(t)}{x_0 - t} = \frac{h_1(x_0) - h_2(x_0) + h_2(t) + \frac{1}{2} + \frac{1}{2}}{x_0 - t} \to +\infty \text{ if } y \nearrow x_0.$$

Let $x_0 \in R \setminus (A_1^- \cup A_1 \cup A_2^- \cup A_2)$. Then there exists a $\delta > 0$ such that $(x_0 - \delta, x_0) \cap A_1 = \emptyset$ and $(x_0 - \delta, x_0) \cap A_2 = \emptyset$. Let $x \in (x_0 - \delta, x_0)$. Then $f_1(x_0) = h_1(x_0)$, $f_2(x_0) = h_2(x_0)$, $f_1(x) = h_1(x)$ and $f_2(x) = h_2(x)$. Hence $f'_-(x_0) = h'(x_0) - h_2'(x_0) \neq \pm\infty$. This ends the proof of Theorem 1.

Modifying the construction from Theorem 1, we obtain analogous results for the sets $\{x : f'_+(x) = +\infty\}$ and $\{x : f'_+(x) = -\infty\}$. Theorem 1 implies the following theorems:

Theorem 2. Let E_1 and E_2 be disjoint subsets of R. There exists a function $f : R \to R$ such that $E_1 = \{x : f'_-(x) = +\infty\}$ and $E_2 = \{x : f'_-(x) = -\infty\}$ if and only if $m(E_1) = m(E_2) = 0$.

The sufficiency of the condition follows from Theorem 1, and its necessity – from the Denjoy-Young-Saks theorem.

Theorem 3. There exists a function $f : R \to R$ for which sets $\{x : f'_-(x) = +\infty\}$ and $\{x : f'_-(x) = -\infty\}$ are non-Borel. We obtain analogous theorems for the right-hand derivatives.

Exists a function $f : R \to R$ for which sets $\{x : f'_-(x) = +\infty\}$ and $\{x : f'_-(x) = -\infty\}$ also $\{x : f'_+(x) = +\infty\}$ and $\{x : f'_+(x) = -\infty\}$ are non-Borel but the union of this sets is Borel.

In paper [2] I prove that: for any set A of measure zero, there exists a function $f : R \to R$ such that $A = \{x : f'_-(x) = +\infty\} \cup \{x : f'_-(x) = -\infty\} \cup \{x : f'_+(x) = +\infty\} \cup \{x : f'_+(x) = -\infty\}$, $\overline{A} \setminus A = \{x : f'_-(x)$ and $f'_+(x)$ does not exist, finite or infinite$\}$ and $R \setminus \overline{A} = \{x : f'(x)$ exist finite$\}$.

References

[1] M. Codyks: On the sets of points at which the derivative is suitable equal $+\infty$ and $-\infty$ (in Russian), Mat. Sb. t.43 (85) (1957), 429-450.

[2] B. Kaczmarski: On the measure and Borel type of the set of points of one-sided non-differentiability, Demonstratio Math. 22 (1989), 441-460.

[3] Z. Zahorski: Ober die Menge der Punkte in welchen die Ableitung unendlich ist, Tohoku Math. 48 (1941) 321-330.

Received July 12, 1990