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BOUNDED VARIATION AND POROSITY

A set $E \subset R^2$ is said to have porosity p at a point x if

$$p = p(x) = \overline{\lim_{r \to 0}} \frac{p(x,r)}{r}$$

where $p(x,r)$ is the radius of the largest open ball missing E and contained in the neighborhood of radius r about x. Then x is a <u>point of porosity</u> of E if $p(x) > 0$; E is said to be <u>porous</u> if each point of E is a point of porosity of E; E is said to be <u>σ-porous</u> if E is a countable union of porous sets. In [1] it was shown that a continuous function defined on R or on $[0,1]$ can have a graph which is not σ-porous. The function constructed was wildly oscillatory and it seemed natural to ask whether functions of bounded variation must have σ-porous graphs. While the answer to this question has some interesting consequences, that answer may not be easy to come by. The example given below shows that a continuous function of bounded variation may have a c-dense set of points which are points of non-porosity of the graph. In fact, from the example it follows that such functions are dense in the space of functions of bounded variation with the variation norm. It is not clear whether this set of functions is of first or second category in this space.

In order to work more easily with porosity, it is desirable to be able to determine non-porosity at a point by using a net. In particular, the net

$$A = \left\{ \left[\frac{i}{3^n}, \frac{i+1}{3^n} \right] \times \left[\frac{j}{3^n}, \frac{j+1}{3^n} \right] \right\}$$

will be used. For this purpose the following lemma is needed:

<u>Lemma</u>. <u>If</u> $E \subset R^2$ <u>and</u> $x \in E$, x <u>is a point of non-porosity of</u> E <u>provided that for each natural number</u> K <u>there is a number</u> N <u>so that if</u> $n > N = N(K)$ <u>and</u>

$$x \in \left[\frac{i}{3^n}, \frac{i+1}{3^n} \right] \times \left[\frac{j}{3^n}, \frac{j+1}{3^n} \right] = I_{i,j,n}$$

E <u>contains a point in each square of the form</u>

$$\left[\frac{k}{3^{n+K}}, \frac{k+1}{3^{n+K}} \right] \times \left[\frac{k'}{3^{n+K}}, \frac{k'+1}{3^{n+K}} \right]$$

contained within the boundary of the 11×11 *grid of* 121 *squares of side length* 3^{-n} *whose central square* $I_{i,j,n}$ *contains* x.

Proof. Let x be a point of E satisfying the conditions of the lemma. Suppose that E has porosity $p > 0$ at x. Choose K so that $3^{-K+1} < p$ and $N = N(K)$. If a ball of radius $r < 3^{-N}$ centered at x contains a net element containing x and having side length 3^{-n} and contains no larger net element containing x, then the next larger net element containing x has a point in it outside of the ball. Then $r \leq 3 \cdot 3^{-n} \cdot \sqrt{2} < 5 \cdot 3^{-n}$. It follows that the ball is contained in the 121 squares described by the statement of the lemma. Since E contains a point in each square of side length 3^{-n-K} within the boundaries of these 121 squares, no ball of radius $r' > 3^{-n-K} \cdot \sqrt{2}$ within the ball of radius r misses E. Thus, if a ball of radius r' within the ball of radius r misses E,

$$\frac{r'}{r} < \frac{3^{-n-K}\sqrt{2}}{3^{-n}} = 3^{-K}\sqrt{2} < \frac{p}{2}.$$

This contradicts the assumption that the porosity of E at x

is p and since p is arbitrary, E is not porous at x.

The following example is auxiliary to the construction of a function of bounded variation for which a c-dense set of points are mapped to points of non-porosity on the graph.

Example 1. *There is a function of bounded variation defined on* $[0,1]$ *which has a graph with* c *points of non-porosity. In fact such a function can be absolutely continuous.*

Construction. Let $h(0) = 0 = h(1)$, $h(1/3) = -1$, $h(2/3) = 1$ and let h be linear on $[0, 1/3]$, $[1/3, 2/3]$, and $[2/3, 1]$. The function h will be used to produce non-porosity by adapting it to intervals in the complement of a perfect set. The set is

$$X = \left\{ x : x = \sum_{m=1}^{\infty} \frac{a_m}{3^m} \right\}$$

where $a_m = 0$ or $a_m = 2$ if 3 divides m; otherwise, $a_m = 1$. Note that the set X is the intersection of the sets E_n, $n = 0, 1, \ldots$ consisting of 2^k intervals $k = 0, 1, \ldots$ of the form $[i/3^n, (i+1)/3^n]$ where $3k \leq n \leq 3k+2$. For $n = 3, 4, 5, \ldots$; $k = 1, 2, \ldots$, each such interval I of E_n is to be partitioned into 3^{k+1} equal intervals and five blocks of 3^{k+1}

intervals of the same length as the 3^{k+1} are to be placed at each side of I resulting in $3^{k+1}\cdot 11$ smaller intervals $k = 0, 1, 2, \ldots$. Corresponding to E_n, $n = 3, 4, \ldots$ there will be a total of fewer than $2^{k}\cdot 3^{k+1}\cdot 11$ distinct smaller intervals. When this is done for $k = 1, 2, \ldots,$ then from the interior of each of these small intervals corresponding to E_n, distinct points are to be chosen in the complement of E_{n+k+2} and distinct from previously chosen points. Since for each $n = 3, 4, \ldots,$ the blocks of intervals from E_{n+3} are always contained in E_n, the resulting set of points consists of all isolated points. Thus disjoint intervals contained in the complement of X each containing exactly one of the points can be selected so as to be contained in the small interval from which the point was chosen. If such a selected interval is (a,b), contained in one of the intervals from the block associated with E_n, then $f(x)$ is defined on that interval to be

$$h_{n,a,b}(x) = \frac{6h\left(\frac{x - a}{b - a}\right)}{3^{n+1}} \;;$$

otherwise, $f(x) = 0$. It follows that the variation of f is equal to $\sum \mathrm{Var}\, h_{n,a,b}(x)$ where the sum is taken over all (a,b) and n. Thus,

$$\operatorname{Var} f \le 3 \sum_k 2\cdot 12\cdot\frac{1}{3^{3k}}\cdot 3^{k+1}\cdot 11\cdot 2^k < \infty .$$

Here the sum is carried out in groups of three k's, the variation of each $h_{n,a,b}$ is less than or equal to $2\cdot 12\cdot 1/3^{3k}$ and the number of intervals on which each h_n is used is at most $3^{k+1}\cdot 11\cdot 2^k$. Since $f(x) = \sum h_{n,a,b}(x)$ uniformly, f is continuous. Since $f(x)$ clearly satisfies Lusin's condition (N), f is absolutely continuous (cf.[2], p.227). Clearly the graph of f satisfies the condition of the Lemma and thus each point of X is a point of non-porosity of the graph.

Example 2. *There is a function F of bounded variation such that the set of x for which $(x, F(x))$ is a point of non-porosity of the graph of F is a c-dense subset of $[0,1]$. Again, such a function can be absolutely continuous.*

Construction. In the construction of Example 1, the function $h(x)$ can be replaced with a function $g(x)$ with $g(0) = g(1) = 0$, $g(1/3) = -1$, $g(2/3) = 1$ and such that g is decreasing on $[0, 1/3]$ and $[2/3, 1]$, increasing on $[1/3, 2/3]$, g is absolutely continuous and the intervals of constancy of g are dense in $[0,1]$. The resulting function will have the same properties as that of f in Example 1 but will have a dense set of intervals of constancy. Calling this function F_1, an absolutely continuous function F_2 can be designed so that it agrees with F_1 at each point which is not in an interval of constancy and has variation less than two times that of F_1 and has c points of non-porosity in each interval of constancy of F_1 and has a dense set of intervals of constancy. Continuing in this fashion, functions F_n can be defined inductively such that F_n has variation less than two times that of F_1, a dense set of intervals of constancy none of which is larger than 2^{-n} and F_n has c points of non-porosity in each interval of constancy of F_{n-1}. Furthermore F_n agrees with F_{n-1} at each point which does not belong to an interval of constancy of F_{n-1}. The limit of this sequence of functions is a function F and

$$\operatorname{Var} F = \sum \operatorname{Var} (F_{n+1} - F_n) = \lim \operatorname{Var} F_n.$$

Since F is the limit in variation of a sequence of absolutely continuous functions, F is absolutely continuous.

<u>Note</u>. It is possible to describe the set of points of non-porosity of a continuous function using derivates defined in terms of porosity. For example, let

$$\overline{D}_p^+ f(x) = \sup \lim_{n\to\infty} \frac{f(x + h_n) - f(x)}{h_n}$$

where the sup is taken over all sequences which have 0 as a point of non-porosity on the right and for which the limit exists (finite or infinite). The four derivates $\overline{D}_p^+$, $\overline{D}_p^-$, $\underline{D}_p^+$, and $\underline{D}_p^-$ are thus defined in this way and a point $(x,f(x))$ is a point of non-porosity of the graph of a continuous function f if and only if

$$\overline{D}_p^+ f(x) = \overline{D}_p^- f(x) = \infty \quad \text{and} \quad \underline{D}_p^+ f(x) = \underline{D}_p^- f(x) = -\infty .$$

To see this suppose that, given a natural number K, sequences $h_{n,1}$, $h_{n,2}$, $h_{n,3}$, and $h_{n,4}$ are chosen so that

$$\lim \frac{f(x + h_{n,i}) - f(x)}{h_{n,i}} (-1)^i > 2 \cdot 6 \cdot 3^K (-1)^i$$

when $h_{n,1}$ and $h_{n,2}$ decrease to 0 and are non-porous at 0 on the right and $h_{n,3}$ and $h_{n,4}$ increase to 0 and are non-porous at 0 on the left. Then there is $\varepsilon > 0$ such that for $3^{-N} < \varepsilon$ and $n > N$ the graph of f intersects each of the $11 \cdot 3^K$ subdivisions of the line segments

$$\left[\frac{i - 5}{3^n}, \frac{i + 6}{3^n}\right] \times \left\{f(x) + \frac{6}{3^n}\right\}$$

and

$$\left[\frac{i - 5}{3^n}, \frac{i + 6}{3^n}\right] \times \quad \{f(x) - \frac{6}{3^n}\}$$

when x is in $[i/3^n, (i + 1)/3^n]$. By the mean value theorem for continuous functions, f satisfies the conditions of the Lemma. Thus the graph of f is non-porous at $(x, f(x))$. Since the above proof requires only the mean value theorem, the conclusion is true for Darboux functions. The proof of the necessity of the condition

$$\overline{D}_p^{+} f(x) = \overline{D}_p^{-} f(x) = \infty \quad \text{and} \quad \underline{D}_p^{+} f(x) = \underline{D}_p^{-} f(x) = -\infty \ .$$

is as follows: Suppose, for example, that

$$\overline{D}_p^{+} f(x) < M < \infty \ .$$

Then $A = \{x|\ [f(x+h) - f(x)]/h > M\}$ has 0 as a point of porosity. That is, there is $\varepsilon > 0$ and a sequence of intervals $[a_n, b_n]$ such that $a_n > 0$, $b_n \downarrow 0$, $(b_n - a_n)/b_n > \varepsilon$, and $[a_n, b_n] \quad A = 0$. It follows that $[x + a_n, x + b_n] \times [Mb_n, \infty]$ does not meet the graph of f. These cross products contain circles of radius $\frac{1}{2}(b_n - a_n)$ which are in turn contained in the circle of radius $Mb_n + a_n$ centered at $(x,f(x))$. Thus, the porosity of the graph of f at $(x,f(x))$ is at least

$$\overline{\lim} \frac{\frac{1}{2}(b_n - a_n)}{Mb_n + a_n} > \overline{\lim} \frac{\frac{1}{2}(b_n - a_n)}{2Mb_n} > \frac{\varepsilon}{4M}.$$

Therefore $(x,f(x))$ is a point of porosity of the graph of f.

REFERENCES

1. J. Foran, Continuous functions need not have σ-porous graphs, Real Analysis Exchange, Vol.11, #1 (1985-86) p.194-203.

2. S. Saks, Theory of the Integral, Dover, 1964.

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