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On Generalized Cluster Sets

The paper consists of three parts. In the first we consider σ-ideals of subsets of the plane adjoint with some σ-ideal $\mathcal{J}$ of subsets of the real line. The second part contains some theorems concerning σ-algebras of the form $(\mathcal{B} \Delta \mathcal{J})^2$, where $\mathcal{B}$ is a σ-algebra of Borel sets. In the third part the facts from the two earlier parts are used to study generalized limit numbers of real function defined in the upper half-plane.

1. Let H denote the open upper half-plane above the real line R, S - a σ-algebra of subsets of R and S^2 - the smallest σ-algebra generated by sets $A \times B$, where $A \in S$ and $B \in S$. $L(x,\theta)$ is the halfline beginning at $x \in R$ in the direction θ, $L(x,\theta,r)$ - the segment beginning at $x \in R$ in the direction θ having length r. For $x \in R$ let h_x be the real function defined in H such that $h_x(p) = r$ for $p \in H$, where r is the distance of p from x.

For any σ-ideal $\mathcal{J} \subset S$ and direction $\theta \in (0,\pi)$ we shall define the σ-ideal $\mathcal{J}^2(\theta)$ adjoint with $\mathcal{J}$ in the direction θ:

$$\mathcal{J}^2(\theta) = \{M \in S^2 : \text{there is a set } U \in \mathcal{J} \text{ such that } h_x(L(x,\theta) \cap M) \in \mathcal{J} \text{ for each } x \in R - U\}$$

The σ-ideal $\mathcal{J}^2(\pi/2)$ was defined by R. Ger in his work [3].

Let us notice that for some σ-ideals $\mathcal{J}$ we obtain:

(1) $\mathcal{J}^2(\theta_1) = \mathcal{J}^2(\theta_2)$ for $\theta_1, \theta_2 \in (0,\pi)$, $\theta_1 \neq \theta_2$,

namely for $\mathcal{J} = \{\emptyset\}$, the σ-ideal of measure zero sets and the σ-ideal of first category sets which follow from the Fubini theorem and the Kuratowski-Ulam theorem (see [5], Chapter XIV, XV). However it is possible to give an example of a o-ideal for which (1) is not valid. For instance, let S be the σ-algebra of all subsets of R and $\mathcal{J}$ - the σ-ideal of countable sets. Let A be the set of $x \in [0,1]$ the ternary expansion of which has the form $x = 0,a_1a_2a_3 \ldots$, where

$$a_i = \begin{cases} 0 \text{ or } 2 & \text{for } i \text{ odd} \\ 0 & \text{for } i \text{ even} \end{cases},$$

and B be the set of $y \in [0,1]$ the ternary expansion of which has the form $y = 0,b_1b_2b_3 \ldots$, where

$$b_i = \begin{cases} 0 & \text{for } i \text{ odd} \\ 0 \text{ or } 2 & \text{for } i \text{ even} \end{cases}.$$

Then the set $M = A \times B \in S^2$ and $M \in \mathcal{J}^2(3\pi/4)$, but $M \notin \mathcal{J}^2(\pi/2)$.

Similar to the σ-ideal $\mathcal{J}^2(\theta)$ we shall define a σ-ideal $\mathcal{J}^2(x)$ adjoint with $\mathcal{J}$ at the point x for an arbitrary σ-ideal $\mathcal{J} \subset S$ and for every point $x \in R$.

$$\mathcal{J}^2(x) = \{M \in S^2 : \text{there is a set } \Theta \in \mathcal{J} \text{ such that}$$
$$h_x(L(x,\theta) \cap M) \in \mathcal{J} \text{ for each } \theta \in (0,\pi) - \Theta\} .$$

Let us notice that for such σ-ideals as $\mathcal{J} = \{\emptyset\}$, the family of sets of measure zero and for the family of sets of first category we have

$$(2) \qquad \mathcal{J}^2(x_1) = \mathcal{J}^2(x_2) \quad \text{for } x_1, x_2 \in R, \quad x_1 \neq x_2 .$$

It can be shown that for the σ-ideal of countable sets the equality (2) does not hold. It suffices to transform homeomorfically the unit square Q_0 onto a tetragon in such a way that the points of the halflines $L(x,\pi/2)$ for $x \in [0,1]$ will be transformed into those of the halflines $L(0,\theta)$ for $\theta \in [\pi/4, \text{arc tg } 2]$ and the points of the halflines $L(x,3\pi/4)$ for $x \in [0,2]$ will be transformed into those of the halflines $L(4,\varphi)$ for $\varphi \in [3\pi/4, \pi - \text{arc tg } \frac{1}{2}]$. Then the image E of the set M from the above example, obtained by means of the homeomorphism, belongs to S^2 (assuming that the continuum hypothesis is true, see [6]), and $E \in \mathcal{J}^2(4)$, but $E \notin \mathcal{J}^2(0)$.

It follows from those above considerations that we can define the σ-ideal $\mathcal{J}^2$ adjoint to a given σ-ideal $\mathcal{J} \subset S$ in all directions:

$$\mathcal{J}^2 = \bigcap_{\theta \in (0,\pi)} \mathcal{J}^2(\theta) ,$$

and the σ-ideal $\mathcal{J}^2_*$ adjoint to a given σ-ideal $\mathcal{J} \subset S$ in all points:

$$\mathcal{J}^2_* = \bigcap_{x \in R} \mathcal{J}^2(x) \quad .$$

For the σ-ideal $\mathcal{J}$ of countable sets one can prove that if $A, B \subset (0,\pi)$, $A \cap B = \emptyset$, $A \cup B = (0,\pi)$, $A \neq \emptyset$. $B \neq \emptyset$, then there exists a set $M \subset H$ such that

$$M \notin \mathcal{J}^2(\theta) \quad \text{for each} \quad \theta \in A$$

and

$$M \in \mathcal{J}^2(\theta) \quad \text{for each} \quad \theta \in B \ .$$

(We assume that the continuum hypothesis is true).
It follows from that, that $\mathcal{J}^2_* \not\subset \mathcal{J}^2$. Similarly $\mathcal{J}^2 \not\subset \mathcal{J}^2_*$.

2. We are now going to study the σ-algebra $\mathcal{B} \Delta \mathcal{J} = \{B \Delta U : B \in \mathcal{B}, U \in \mathcal{J}\}$; where $\mathcal{B}$ denotes the σ-algebra of Borel subsets of R and $\mathcal{J}$ will be a σ-ideal in R. Let $\mathcal{G}$ be the family of open sets. We give now some main properties of the σ-algebra $\mathcal{B} \Delta \mathcal{J}$.

<u>Theorem 1</u>. If $\mathcal{J}_0$ is a σ-ideal such that $\mathcal{B} \Delta \mathcal{J}_0 = \mathcal{G} \Delta \mathcal{J}_0$, then for any σ-ideal $\mathcal{J}_1 \supset \mathcal{J}_0$ we have the equality $\mathcal{B} \Delta \mathcal{J}_1 = \mathcal{G} \Delta \mathcal{J}_1$.

It is known that for the σ-ideal $\mathcal{J}$ of sets of the first category we have the equality $\mathcal{B} \Delta \mathcal{J} = \mathcal{G} \Delta \mathcal{J}$. We can construct a σ-ideal, which is proper, movable and essentially larger than either the σ-ideal of measure zero sets or the σ-ideal of first category sets.

Really, let $\mathcal{J}_0$ be the σ-ideal of measure zero sets or the σ-ideal of first category sets and $E_0 \subset R$ be the

Sierpiński set; that is, a nonmeasurable set such that

$$\mathrm{card}((E_0 + x) \Delta E_0) \leq \aleph_0 \quad \text{for every} \quad x \in R ,$$

(see [7], p. 135, C_{70}). The set E_0 does not have the Baire property either.

$$\mathcal{J}_1 = \{C \cup D : C \in \mathcal{J}_0, \quad D \subset E_0\}$$

and

$$\mathcal{J}_2 = \{ \bigcup_{n=1} (E_n + x_n) : E_n \in \mathcal{J}_1 \text{ and } x_n \in R \text{ for every } n\} .$$

Then $\mathcal{J}_2$ is a proper σ-ideal with the required properties.

Theorem 2. If $\mathcal{J}$ is a σ-ideal such that $\mathcal{B} \Delta \mathcal{J} = \mathcal{L} \Delta \mathcal{J}$, then $(\mathcal{B} \Delta \mathcal{J})^2 \subset \mathcal{J}^2(\pi/2)$.

It is known that there exists a set E of the second category in the plane, no three points of which are on a line (see [5], th. 15.5). So it does not have the Baire property. Hence we have that there exists a σ-ideal $\mathcal{J}$ such that $\mathcal{B} \Delta \mathcal{J} = \mathcal{L} \Delta \mathcal{J}$ and $\mathcal{L}^2 \wedge \mathcal{J}^2(\pi/2) \not\subset (\mathcal{B} \Delta \mathcal{J})^2$.

3. We shall consider real functions f defined in the open upper half-plane H and we shall introduce the concepts of directional limit numbers of f with respect to a σ-ideal $\mathcal{J}$ and limit numbers of f with respect to the σ-ideals $\mathcal{J}^2$ and $\mathcal{J}^2_*$. Those concepts are natural generalizations of qualitative limit numbers discussed in the papers [8], [2] and [4].

Let $x \in R$. A real number y is called a limit number of f at x in the direction $\theta \in (0,\pi)$ with respect to a σ-ideal $\mathcal{J}$, if there exists $\varepsilon > 0$ such that

$$h_x(L(x,\theta,r) \cap f^{-1}((y-\varepsilon,\ y+\varepsilon))) \notin \mathcal{J} \quad \text{for each} \quad r > 0 \ .$$

Moreover the number $+\infty$, $(-\infty)$ is called a limit number of f at x in the direction θ with respect to the σ-ideal $\mathcal{J}$, if there exists $a \in R$ such that

$$h_x(L(x,\theta,r) \cap f^{-1}((a,+\infty))) \notin \mathcal{J},\ (h_x(L(x,\theta,r) \cap f^{-1}((-\infty,a))) \notin \mathcal{J})$$

for each $r > 0$. The set of such limit numbers we shall denote by $C_{\mathcal{J}}(f,x,\theta)$.

A real number y is called a limit number of f at x with respect to the σ-ideal $\mathcal{J}^2$, $(\mathcal{J}^2_*)$, if for each $\varepsilon > 0$ and $r > 0$

$$K(x,r) \cap f^{-1}((y-\varepsilon,y+\varepsilon)) \notin \mathcal{J}^2,\ (K(x,r) \cap f^{-1}((y-\varepsilon,y+\varepsilon)) \notin \mathcal{J}^2_*) \ ,$$

where $K(x,r)$ denotes the circle with the center x and radius r. Moreover $+\infty$ is called a limit number of f at x with respect to the σ-ideal $\mathcal{J}^2$, $(\mathcal{J}^2_*)$, if for each $a \in R$ and $r > 0$

$$K(x,r) \cap f^{-1}((a,+\infty)) \notin \mathcal{J}^2,\ (K(x,r) \cap f^{-1}((a,+\infty)) \notin \mathcal{J}^2_*) \ .$$

Similarly we define $-\infty$ as a limit number of f at x with respect to the σ-ideal $\mathcal{J}^2$, $(\mathcal{J}^2_*)$. The set of such limit numbers we shall denote by $C_{\mathcal{J}^2}(f,x)$, $(C_{\mathcal{J}^2_*}(f,x))$.

The following theorem is similar to the theorem concerning essential limit numbers proved in the paper [1].

<u>Theorem 3</u>. Let $\mathcal{J}$ be a σ-ideal which does not include nonempty open sets and satisfies the conditions:

(3) $\mathcal{B} \Delta \mathcal{J} = \mathcal{S} \Delta \mathcal{J}$

and

(4) $\mathcal{J}^2(\theta_1) = \mathcal{J}^2(\theta_2)$ for any two directions

$$\theta_1, \theta_2 \in (0,\pi), \quad \theta_1 \neq \theta_2 .$$

If $f : H \to R$ is a measurable function with respect to the σ-algebra $(\mathcal{B} \Delta \mathcal{J})^2$, then for any direction $\theta \in (0,\pi)$ $\sup C_{\mathcal{J}}(f,x,\theta)$ is a measurable function of x with respect to the σ-algebra $\mathcal{B} \Delta \mathcal{J}$.

By similar assumption, we can obtain generalizations of many theorems concerning the qualitative limit numbers. Those theorems can be found in the works [8], [2], [4].

<u>Theorem 4</u>. Let $\mathcal{J}$ be a σ-ideal which does not include nonempty open sets and satisfies conditions (3) and (4). If $f : H \to R$ is a measurable function with respect to the σ-algebra $(\mathcal{B} \Delta \mathcal{J})^2$, then for any two directions $\theta_1, \theta_2 \in (0,\pi)$, $\theta_1 \neq \theta_2$ we have

$$\{x \in R : \sup C_{\mathcal{J}}(f,x,\theta_1) < \sup C_{\mathcal{J}}(f,x,\theta_2)\} \in \mathcal{J} .$$

<u>Theorem 5</u>. If $\mathcal{J}$ is a σ-ideal which does not include nonempty open sets and if $f : H \to R$ is a continuous function, then for any two directions $\theta_1, \theta_2 \in (0,\pi)$, $\theta_1 \neq \theta_2$ the set

$$\{x \in R : \sup C_{\mathcal{J}}(f,x,\theta_1) < \inf C_{\mathcal{J}}(f,x,\theta_2)\}$$

is at most denumerable.

Theorem 6. If $\mathcal{J}$ is a σ-ideal which does not include nonempty open sets and satisfies conditions (3) and (4). If $f : H \to R$ is a measurable function with respect to the σ-algebra $(\mathcal{B} \Delta \mathcal{J})^2$, then for any two directions $\theta_1, \theta_2 \in (0,\pi)$, $\theta_1 \neq \theta_2$ we have

$$\{x \in R : C_{\mathcal{J}}(f,x,\theta_1) - C_{\mathcal{J}}(f,x,\theta_2) \neq \emptyset\} \in \mathcal{J} .$$

Theorem 7. Let $\mathcal{J}$ be a σ-ideal which does not include nonempty open sets and satisfies conditions (3) and (4). If $f : H \to R$ is a measurable function with respect to the σ-algebra $(\mathcal{B} \Delta \mathcal{J})^2$ and $\{\theta_n\}$ is an arbitrary sequence of directions from the interval $(0,\pi)$, then

$$\{x \in R : \bigcap_{n=1}^{\infty} C_{\mathcal{J}}(f,x,\theta_n) = \emptyset\} \in \mathcal{J} .$$

Theorem 8. If a σ-ideal $\mathcal{J}$ does not include nonempty open sets and if $f : H \to R$ is a continuous function, then for every direction $\theta \in (0,\pi)$ and for every $x \in R$

$$C_{\mathcal{J}}(f,x,\theta) \subset C_{\mathcal{J}^2}(f,x) .$$

Example. There exists a σ-ideal $\mathcal{J}$ and a continuous function $f : H \to R$ such that for some direction $\theta \in (0,\pi)$
$\theta \in (0,\pi)$

$$\{x \in R : C_{\mathcal{J}^2}(f,x) \not\subset C_{\mathcal{J}}(f,x,\theta)\} \notin \mathcal{J} .$$

It suffices to take $\mathcal{J} = \{\emptyset\}$ and

$$f(z) = \max((\operatorname{Arg} z - \pi/2),\, 0) \quad \text{for} \quad z \in H .$$

It is easy to prove that for $\theta \in (0,\pi/2)$ and $x_0 = 0$ we have $C_{\mathcal{J}^2}(f,x_0) \not\subset C_{\mathcal{J}}(f,x_0,\theta)$.

<u>Theorem 9</u>. If $f : H \to R$ is an arbitrary function, then for every $x \in R$

$$\{\theta \in (0,\pi) : C_{\mathcal{J}}(f,x,\theta) \subset C_{\mathcal{J}^2_*}(f,x)\} = (0,\pi) - A ,$$

where $A \in \mathcal{J}$.

The directional limit numbers of f with respect to a σ-ideal $\mathcal{J}$ are related to the limit of the function f at x with respect to the σ-ideal $\mathcal{J}$.

Let $f : H \to R$, $x \in R$, $\theta \in (0,\pi)$. The real number y is called the upper limit of f at x in the direction θ with respect to σ-ideal $\mathcal{J}$ if

1^0 for every $\varepsilon > 0$ there is $r > 0$ such that
$$h_x(L(x,\theta,r) \cap f^{-1}([y+\varepsilon, +\infty))) \in \mathcal{J} ,$$

2^0 for every $\varepsilon > 0$ and for every $r > 0$
$$h_x(L(x,\theta,r) \cap f^{-1}((y-\varepsilon, y]))) \notin \mathcal{J}$$

and it will be denoted by $\mathcal{J} - \limsup_{p \to x,\theta} f(p)$.

The real number y is called the lower limit of f at x in the direction θ with respect to the σ-ideal $\mathcal{J}$ if

1° for every $\varepsilon > 0$ there is $r > 0$ such that

$$h_x(L(x,\theta,r) \cap f^{-1}((-\infty, y-\varepsilon])) \in \mathcal{J} ,$$

2° for every $\varepsilon > 0$ and for every $r > 0$

$$h_x(L(x,\theta,r) \cap f^{-1}([y, y+\varepsilon))) \notin \mathcal{J}$$

and it will be denoted by $\mathcal{J} - \liminf\limits_{p \to x, \theta} f(p)$.

The real number y is called the limit of f at x in the direction θ with respect to the σ-ideal $\mathcal{J}$ if

$$\mathcal{J} - \limsup_{p \to x, \theta} f(p) = \mathcal{J} - \liminf_{p \to x, \theta} f(p) = y$$

<u>Theorem 10</u>. Let $f : H \to R$, $\theta \in (0,\pi)$, $x, y_0 \in R$. If the σ-ideal $\mathcal{J}$ does not include nonempty open sets, then the following conditions are equivalent:

1° $y_0 = \mathcal{J} - \lim\limits_{p \to x, \theta} f(p)$,

2° for every $\varepsilon > 0$ there is $r > 0$ such that

$$h_x(L(x,\theta,r) - f^{-1}((y_0 - \varepsilon, y_0 + \varepsilon))) \in \mathcal{J} .$$

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