SYMMETRIC REAL ANALYSIS: A SURVEY

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Section 1: Introduction

This survey concerns what we call, for lack of a better name, "symmetric real analysis." Under the heading of symmetric real analysis, we include such topics as symmetric differentiation, symmetric continuity, symmetric functions, smooth functions and locally symmetric sets. In short, this name is a catch-all phrase for any definition or property which is intrinsically based upon a symmetric difference of some order.

All of the examples of symmetric real analysis mentioned above originate in the study of trigonometric series. For this reason, they play an important role in the classical works of Riemann [Ri], Lebesgue [Le], Fatou [Fa] and others. Since their inception, however, they have been used in such areas as approximation theory and harmonic analysis. This survey will not be concerned with these applications of symmetry. Rather, we will examine the behavior of functions satisfying certain symmetry properties which have proved useful, mostly in the study of the pointwise convergence of trigonometric series. Any reader interested in the application of these ideas to Fourier series may refer to Hobson [Ho, Vol. 2, Ch. 8] and Zygmund [Zy, Vol. 2, Ch. 11].

For a function $f:R \to R$, we define the n'th symmetric difference of f at x to be

$$\Delta^n f(x,h) = \Sigma_{i=0}^{n}(-1)^i\binom{n}{i}f(x + (n - 2i)h). \tag{1}$$

Of particular importance are the first and second symmetric differences of f,

$$\Delta f(x,h) = \Delta^1 f(x,h) = f(x + h) - f(x - h)$$

and

$$\Delta^2 f(x,h) = f(x + h) + f(x - h) - 2f(x).$$

A function f is <u>symmetrically continuous</u> at x iff

$$\lim_{h\to 0}\Delta f(x,h) = 0.$$

It is <u>a-smooth</u> $(a \geq 0)$ iff $\lim_{h\to 0}\Delta^2 f(x,h)/h^a = 0$. In the special cases when $a = 0$ or

$a = 1$, the function is usually said to be symmetric or smooth at x, respectively.

The upper (lower) first symmetric derivative of f at x is defined to be

$$\overline{f}^{(1)}(x) = \lim\sup_{h\to 0}\Delta f(x,h)/2h \ (\underline{f}^{(1)}(x) = \lim\inf_{h\to 0}\Delta f(x,h)/2h).$$

When these two are equal, finite or infinite, their common value is the first symmetric derivative of f at x, denoted $f^{(1)}(x)$. It is easy to see that when $f^{(1)}(x)$ exists, then the ordinary derivative, $f'(x)$, also exists and the two are equal. The converse is not true as can be seen by considering $f(x) = |x|$.

The second symmetric derivative of f is defined similarly as

$$f^{(2)}(x) = \lim_{h\to 0}\Delta^2 f(x,h)/(2h)^2.$$

The second symmetric derivates of f are defined analogously.

Various authors use different names for the first and second symmetric derivatives. The most common alternate names are the Schwarz derivative and the Riemann derivative. (Also rarely used are the Lebesgue derivative [We] and de la Vallée-Poisson derivative [We].) This confusion is further enhanced by the fact that there are two non-equivalent, common ways to extend the symmetric derivative to higher orders.

The first way, usually called the Riemann derivative, is the most natural extension of the definitions given above. It is defined as

$$f^{(n)}(x) = \lim_{h\to 0}\Delta^n f(x,h)/(2h)^n.$$

The corresponding upper and lower derivates are defined as usual.

The second method, due to de la Vallée-Poisson, is reminiscent of the Peano derivative and is usually called the n'th symmetric derivative. To define it, we say that f has a symmetric derivative of order n at x_0 iff there exists a polynomial P(t) of degree n such that

$$[f(x_0 + t) + (-1)^n f(x_0 - t)]/2 = P(t) + o(t^n)$$

as $t\to 0$. This derivative is a_n if $a_n/n!$ is the leading coefficient of P(t).

The ambiguity created by having two definitions will cause little difficulty because both definitions agree for $n = 1$ and $n = 2$, and because we shall rarely need symmetric derivatives of orders higher than 2.

If A is an arbitrary subset of R then its complement is denoted A^c. The inner (outer) measure of A is written $|A|_i$ ($|A|_o$). If A is measurable, then its measure is $|A|$.

For any real valued function f defined on a subset of R, C(f) is the set of points at which f is continuous. D(f) is the set of points at which the ordinary derivative of f, denoted f', exists and is finite. SC(f) is the set of points where f is symmetrically continuous. $\bar{D}^+f$, $\bar{D}^-f$, $\underline{D}^+f$ and $\underline{D}^-f$ are the Dini derivates of f.

The remaining sections are concerned with the following topics:

Section 2 deals with the relationship between ordinary continuity and symmetric continuity.

Section 3 is concerned with the relationship between ordinary and symmetric differentiability.

Section 4 examines the relationship between ordinary continuity and symmetric differentiation.

Section 5 presents symmetric analogues to the standard theorems of ordinary differentiation such as monotonicity theorems.

Section 6 deals with generalizations of the symmetric derivative such as approximate, L_p and qualitative versions.

Section 7 is concerned with symmetric, smooth and a-smooth functions.

Finally, we would like to draw your attention to the bibliography, which we believe is the most complete ever assembled on this subject.

Section 2: Symmetric Continuity

The study of symmetric continuity by itself, and not in the context of symmetric differentiability, apparently began in 1935, when F. Hausdorff [Ha] asked whether the set of points where a symmetrically continuous function is discontinuous can be of the second category. This question was answered in the negative two years later by H. Fried [Fr], who proved the following theorem.

<u>THEOREM 2.1</u>. If SC(f) is residual in an interval I, then C(f) is residual in I.

Of course, the following corollary is immediate from this theorem.

<u>COROLLARY 2.2</u>. If f is symmetrically continuous everywhere on an interval I, then C(f) is residual in I.

It is known that, in a sense, these are the strongest category statements which can be made about the relationship between C(f) and SC(f). P. Erdős [Er] has shown that if the continuum hypothesis is assumed, then there exists an additive subgroup G of R such that $|G| = 0$ and $G \cap I$ is a second category set for every interval, I. If f is the characteristic function of G, then f is symmetrically continuous at each point of G, but continuous nowhere.

A comparison in the measure sense between C(f) and the set of points where f is symmetrically continuous is supplied by the following theorem.

<u>THEOREM 2.3</u>. If f is symmetrically continuous everywhere on an interval I, then f is continuous almost everywhere on I.

This theorem was first proved in 1960 for measurable functions by E. M. Stein and A. Zygmund [SZ, Lemma 9]. It was later extended by D. Preiss [Pr] to the general case given above. In the same paper, Preiss also gives an example of a symmetrically continuous function f such that $C(f)^c$ is uncountable.

C. Belna [Be] recently proved the following stronger version of Theorem 2.3.

<u>THEOREM 2.4</u>. If f is an arbitrary function, then

$$|SC(f) \cap C(f)^c|_1 = 0.$$

As Belna points out, if we assume the continuum hypothesis, then this is the strongest measure theoretic statement concerning the relationship between $SC(f)$ and $C(f)$ which can be made. P. Erdös [Er] has shown the existence of an additive subgroup G of R that is of the first category and has full outer measure. If f is the characteristic function of such a group, then

$$|(SC(f) \cap C(f)^c) \cap I|_o = |I|$$

for every interval I.

It is unknown whether symmetrically continuous functions are in any Baire class.

Section 3: Symmetric and Ordinary Differentiability

Any discussion of the relationship between symmetric and ordinary differentiability must begin with the following theorem which was proved in 1927 by A. Khintchine [Kh].

THEOREM 3.1. If f is a measurable function, then f has a finite ordinary derivative almost everywhere on the set $\{x: \bar{f}^{(1)}(x) < \infty\}$.

This theorem has several immediate consequences.

COROLLARY 3.2. If f is a measurable function which is symmetrically differentiable everywhere, then $D(f)$ has full measure.

Using Theorem 2.1, we have the following.

COROLLARY 3.3. If f is an arbitrary function such that $f^{(1)}$ exists and is finite everywhere, then f is measurable and $D(f)$ has full measure.

In 1973 N. Kundu [Ku4] pointed out that Theorem 3.1 almost immediately implies a Denjoy-Young-Saks theorem for symmetric derivates. (See also C. Ezzell and J. Nyman [NM].)

COROLLARY 3.4. If f is measurable, then the symmetric derivates of f must satisfy one of the following two conditions at almost every point:

$$\text{(i) } \bar{f}^{(1)}(x) = +\infty \text{ and } \underline{f}^{(1)}(x) = -\infty; \text{ or,}$$

$$\text{(ii) } x \in D(f) \text{ and consequently, } \bar{f}^{(1)}(x) = \underline{f}^{(1)}(x) = f^{(1)}(x) \text{ is finite.}$$

Various refinements to Theorem 3.1 have appeared in recent years, the most important of which was contributed by J. Uher [Uh] in 1982. He defines f to be upper (lower) symmetrically semicontinuous at x iff

$$\limsup_{h\to 0^+} \Delta f(x,h) \leq 0 \;\; (\liminf_{h\to 0^+} \Delta f(x,h) \geq 0).$$

If f is either upper or lower symmetrically semicontinuous at x, then it is just called symmetrically semicontinuous at x.

THEOREM 3.5. (Uher [Uh]) For an arbitrary function f, define

$$A = \{x: \overline{f}^{(1)}(x) < \infty \text{ or } \underline{f}^{(1)}(x) > -\infty\},$$

and

$$B = \{x: f \text{ is not symmetrically semicontinuous at } x\}.$$

Then f' exists almost everywhere on $A - B$.

This theorem immediately implies Theorem 3.1. It also has the following two important corollaries.

COROLLARY 3.6. If a function f is symmetrically differentiable almost everywhere on a measurable set E, then $|E - D(f)| = 0$.

COROLLARY 3.7. If f is symmetrically differentiable almost everywhere (finite or infinite), then f is measurable.

Another direction one might take in attempting to improve upon Theorem 3.1 is to strengthen the "almost everywhere" part of the theorem. One such change would be to prove a category analogue. But, the exceptional set of Theorem 3.1, although of measure zero, need not be small in the category sense. To see this, let G be an additive subgroup of R which is of the second category in every interval. (For the existence of such an object, see P. Erdős [Er].) As in Section 2, let f be the characteristic function of G. Then $f^{(1)}(x) = 0$ for each $x \in G$, but f' exists nowhere. But, the situation can get even worse than this. It is pointed out by Belna, Evans and Humke [BEH1], that given any bounded, second category set Z of measure zero, there exists a function f with bounded symmetric derivates such that f' exists nowhere on Z. (In fact, f may be chosen to be Lipschitz.) From these examples, it is clear that further conditions must be placed upon f in order to prove a category analogue of Theorem 3.1. The following theorem was proved by Belna, Evans and Humke [BEH1].

THEOREM 3.8. If f is a function such that $C(f)$ is dense, then for all but a σ-porous set of points, both of the following inequalities hold:

$$\text{(i) } \underline{f}^{(1)}(x) = \min\{\underline{D}^+f(x), \underline{D}^-f(x)\},$$

$$\text{(ii) } \overline{f}^{(1)}(x) = \max\{\overline{D}^+f(x), \overline{D}^-f(x)\}.$$

Note that since σ-porous sets are both of measure zero and of the first category, the conclusion of this theorem is stronger than that of Theorem 3.1

<u>COROLLARY 3.9.</u> ([BEH1]) If C(f) is dense, then f′ exists at all but a σ-porous set of points where $f^{(1)}$ exists.

Using Corollary 3.9 along with Corollary 3.2, we arrive at yet another corollary.

<u>COROLLARY 3.10.</u> ([BEH1]) If f is symmetrically differentiable on R (finite or infinite), then $f^{(1)}$ exists for all but a σ-porous set of points.

Corollary 3.10 was originally proved with the additional assumption that f is measurable. However, according to Corollary 3.7, this measurability assumption is redundant.

It is not known whether the above results are sharp. In relation to this, S. N. Mukhopadhyay [Mu3] constructed a continuous function with a finite symmetric derivative everywhere, but lacking an ordinary derivative on a countable dense set. J. Foran [Fo2] constructed a continuous, finitely symmetrically differentiable function which is not differentiable on a nonempty perfect set.

Theorem 3.1 was partially extended to higher order derivatives by J. Marcinkiewicz and A. Zygmund [MZ].

<u>THEOREM 3.10.</u> If f is a measurable function such that $f^{(n)}$ exists (in the de la Vallée-Poisson sense) everywhere on a set E, then f has an n'th Peano derivative a.e. in E.

For more information about the relationship between n'th symmetric derivatives and Peano derivatives, see the survey by M. Evans and C. Weil [EW] or the two papers, [As1] and [As2], by J. M. Ash.

Section 4: Continuity and Symmetric Differentiability

As we shall see, the relationship between symmetric differentiability and continuity is far better understood than the relationship between symmetric and ordinary differentiability. From Theorem 3.1, we arrive at the following.

THEOREM 4.1. If f is a measurable function, then f is continuous almost everywhere on the set

$$\{x: \underline{f}^{(1)}(x) > -\infty \text{ or } \overline{f}^{(1)}(x) < \infty\}.$$

COROLLARY 4.2. If f is symmetrically differentiable (finite or infinite), then $C(f)$ is residual and of full measure.

Z. Charzynski [Ch] proved a stronger result in the case when $f^{(1)}$ is finite.

THEOREM 4.3. If f is a function such that $f^{(1)}$ exists everywhere and is finite, then $C(f)^c$ has no subset which is dense in itself.

Any set which has the property that no subset of it is dense in itself has been called a "clairsemé" set [Go] or a "scattered" set [Ha2]. It is known that such sets are countable and cannot be dense in any perfect set.

A companion theorem was proved right on the heels of this one by E. Szpilrajn [Sz].

THEOREM 4.4. If A is any set which has no dense in itself subset, then there exists a function f such that $f^{(1)}$ exists everywhere and

$$A = C(f)^c.$$

Thus, $C(f)$ is completely characterized for finitely symmetrically differentiable functions. No such characterization is known, if we allow $f^{(1)}$ to assume infinite values.

A related topic which is best discussed at this point is that of locally symmetric sets. A set S is said to be locally symmetric iff its characteristic function has symmetric derivative 0 everywhere. It follows easily from Theorem 4.3 that every locally symmetric set has countable closure. (See R. Davies [Da] or M. Foran [Fo1].)

It is well-known that a continuous function need not be differentiable anywhere.

That the same is true of the symmetric derivative was shown by L. Filipczak [Fi4], [Fi5]. In fact, the following theorems were proved by F. Filipczak [Fi3]. (See also [Fi1] and [Fi2].)

THEOREM 4.5. Let f be an approximately continuous function and S be the set of all points where $f^{(1)}$ exists (finite or infinite). Then

$$S^c = A \cup B,$$

where A is a G_δ set and B is a $G_{\delta\sigma}$ set of measure zero. Furthermore, the same decomposition holds if S is the set where $f^{(1)}$ is finite.

THEOREM 4.6. If A is a G_δ set and B is a $G_{\delta\sigma}$ set of measure zero, then there exists a bounded continuous function f such that $(A \cup B)^c = D(f)$ and $f^{(1)}$ exists nowhere on $A \cup B$.

We end this section with the following theorem of P. Kostyrko [Ko].

THEOREM 4.7. Let C be the Banach space of all continuous functions on $[0,1]$ with the L_∞ norm and let M be the set of all $f \in C$ such that $\underline{f}^{(1)}(x) = -\infty$ and $\overline{f}^{(1)}(x) = \infty$ for each $x \in (0,1)$. Then M is residual in C.

Section 5: Symmetric Analogues to Ordinary Differentiation Theorems

Much of the research on symmetric differentiation has been concerned with extending the theorems of ordinary differentiation to the more general case of the symmetric derivative. There are two obstacles which make this task more difficult than it seems at first glance: Direct one-to-one extensions of ordinary differentiation theorems are usually not true; and, the symmetric derivative is much harder to handle than the ordinary derivative. Nevertheless, much progress has been made, and it has been shown that, when certain natural conditions are satisfied, monotonicity theorems, mean value theorems, Baire classifications and Zahorski properties can all be shown to hold for the symmetric derivative.

One of the first theorems of this type was proved by A. Khintchine [Kh] in 1928. He showed that a continuous function with a nonnegative symmetric derivative is nondecreasing. There was a lapse of almost forty years before this was improved upon.

THEOREM 5.1. (S. Mukhopadhyay [Mu1]) If f is a function satisfying

(i) $\limsup_{a \to x-} f(a) \leq f(x) \leq \limsup_{a \to x+} f(a)$ for all x,

(ii) $\underline{f}^{(1)} \geq 0$ a.e., and

(iii) $\underline{f}^{(1)} > -\infty$ except for a countable set,

then f is nondecreasing.

(Actually, Mukhopadhyay's proof has a slight oversight in it, which was elegantly corrected by H. W. Pu and H. H. Pu [PP3].)

The following improvement of Theorem 5.1 was proved by N. Kundu [Ku6].

THEOREM 5.2. Let f satisfy conditions (i) and (iii) of Theorem 5.1 and replace condition (ii) by the following:

(ii') $\overline{f}^{(1)} \geq 0$ a.e..

Then f is nondecreasing.

By weakening condition (i) and strengthening condition (iii), M. Evans [Ev3] proved the following. (See also C. Weil [We1].)

THEOREM 5.3 If f is a measurable function satisfying

(i) $\liminf_{a\to x} f(a) \leq f(x) \leq \limsup_{a\to x} f(a)$

(ii) $\underline{f}^{(1)} \geq 0$ a.e., and

(iii) $\underline{f}^{(1)} > -\infty$ everywhere,

then f is nondecreasing.

It should be noted that requiring the function to be measurable in Theorem 5.3 is not a new restriction, because measurability is implicit in condition (i) of the previous theorems. In fact, condition (i) implies that f is actually upper semicontinuous off of a countable set.

Another strengthening of Theorem 5.1 was given in 1973 by H. W. Pu and H. H. Pu [PP3], who proved:

THEOREM 5.4. If f is a function satisfying (i) of theorem 5.1 such that $f(E)$ contains no non-degenerate interval, where

$$E = \{x: \underline{f}^{(1)}(x) \leq 0\},$$

then f is nondecreasing.

They went on to show that Theorem 5.4 implies Theorem 5.1. (For other results similar to Theorem 5.4, see N. Kundu [Ku1].)

Another direction was taken by L. Larson [La1], who proved the following. (Although it was not stated with quite this generality.)

THEOREM 5.5. If f is a measurable function such that $\underline{f}^{(1)} \geq 0$ a.e. and $\underline{f}^{(1)} > -\infty$ everywhere, then $f|_{C(f)}$ is nondecreasing.

This theorem may be proved through a clever covering argument developed by B. Thomson [Th].

Theorem 5.5 has the following interesting corollary.

COROLLARY 5.6. (L. Larson [La3]) If f is a measurable, symmetrically differentiable function (allowing infinite values) such that $f^{(1)} \geq 0$ a.e. and $f^{(1)}$ has the Darboux property, then f is nondecreasing.

It is easy to see that no direct analogues for the ordinary mean value theorem or the Darboux property hold with the symmetric derivative. A function as simple as $f(x) = |x|$ provides counterexamples for both. But, the monotonicity theorems mentioned above do allow some generalized versions, which reduce to the usual theorems in the case of ordinary differentiability. The first of these so-called "quasi-mean value" theorems was apparently due to C. Aull [A1], who used Khintchine's monotonicity theorem to prove the following.

THEOREM 5.7. If f is a continuous function such that $f^{(1)}$ exists and is finite everywhere and $a < b$, then there are numbers $c,d \in (a,b)$ such that

$$f^{(1)}(c) \leq [f(b) - f(a)]/(b - a) \leq f^{(1)}(d).$$

It is easy to see that this reduces to the usual mean value theorem if $f^{(1)}$ has the Darboux property.

Using Theorem 5.2, N. Kundu [Ku6] proved

THEOREM 5.8. Let f be a function satisfying

(i) $\liminf_{a \to x+} f(a) \leq f(x) \leq \limsup_{a \to x+} f(a)$ everywhere,

(ii) $\liminf_{a \to x-} f(a) = f(x)$ everywhere, and

(iii) $\underline{f}^{(1)}$ and $\overline{f}^{(1)}$ are both finite off of a countable set.

If $a < b$, then there are two sets of positive measure, C and D, contained in (a,b) such that

$$\overline{f}^{(1)}(c) \leq [f(b) - f(a)]/(b - a) \leq \underline{f}^{(1)}(d) \tag{1}$$

for all $c \in C$ and $d \in D$.

Another example, using Theorem 5.5, was proved in [La1].

THEOREM 5.9. Let f be a such that $f^{(1)}$ exists and is finite everywhere. If $a,b \in C(f)$ such that $a < b$, then there are two G_δ sets of positive measure, C and D, such that (1) holds for every $c \in C$ and $d \in D$.

An application of Theorem 5.9 is the following quasi-Darboux property for symmetric derivatives [La1].

COROLLARY 5.10. If f is function such that $f^{(1)}$ exists and is finite everywhere, then for all $x \in R$,

$$\liminf_{h\to 0}[f^{(1)}(x+h)+f^{(1)}(x-h)]/2 \le f^{(1)}(x) \le$$

$$\le \limsup_{h\to 0}[f^{(1)}(x+h)+f^{(1)}(x-h)]/2.$$

For other quasi-mean value theorems and applications of these theorems see [Al], [Ev3], [Ku1], [La1] and [Mu1].

Another area where progress has been made recently is in the Baire classification of symmetric derivatives. The following was proved by L. Larson [La1].

THEOREM 5.11. If f is a function such that $f^{(1)}$ exists everywhere (finite or infinite), then $f^{(1)}$ is in Baire class one.

F. M. Filipczak [Fi3], proved the existence of a measurable function f such that $\underline{f}^{(1)}$ and $\overline{f}^{(1)}$ are both nonmeasurable and the set where $f^{(1)}$ exists is not Borel measurable. By putting restrictions on f, however, he was able to prove the following theorem.

THEOREM 5.12. ([Fi3]) If f is an approximately continuous function, then $\underline{f}^{(1)}$ and $\overline{f}^{(1)}$ are in Baire class two.

Various other analogous properties have been explored. For example: Analogues to Dini's theorem have been proved [Ku1], [PP2]; an analogue to the Denjoy-Young-Saks theorem has been proved [EN], [Ku4]; the Zahorski properties have been extended [Ku7], [La3].

We end this section by noting a few results which have been proved about the second symmetric derivative.

It has long been known that if f is a continuous function such that $f^{(2)} \ge 0$ everywhere, then f is convex [Zy, Vol. 1, p. 22]. C. Weil [W1] proved the following generalization to this theorem in 1976.

THEOREM 5.13. If f is a function in Baire class one with the Darboux property such that $\underline{f}^{(2)} \ge 0$ everywhere, then f is convex.

A recent variation on this thoerem is that if f is measurable and $0 \le f^{(2)} < \infty$ everywhere, then f is convex [La4]. It is not known whether measurability can be substituted for the Baire class one condition in Theorem 5.13.

Section 6: Generalized Symmetric Derivatives

(A) Approximate symmetric derivatives. We denote the n'th approximate symmetric derivative of a function f by $f_{ap}^{(n)}$ and the corresponding derivates by $\underline{f}_{ap}^{(n)}$ and $\overline{f}_{ap}^{(n)}$. For the definitions, see H. H. Pu and H. W. Pu [PP5].

When one considers what is known about the ordinary symmetric derivative, it is surprising that so little is known about the approximate symmetric derivative. In the last ten years, however, some progress has been made. We begin by mentioning the following theorem, which we hope will prove useful in future investigations.

THEOREM 6.1. (L. Larson [La1],[La4]) If f is a measurable function, then $\underline{f}_{ap}^{(1)}$ and $\overline{f}_{ap}^{(1)}$ are both in Baire class three. Further, if $f_{ap}^{(1)}$ exists everywhere (finite or infinite), then $f_{ap}^{(1)}$ is in Baire class one.

Several authors have presented monotonicity theorems for the approximate symmetric derivative. Unfortunately, some of the proofs appear flawed [Kb],[Mu6],[Ku5]. (See also [MR].) The one we will state is due to H.H. Pu and H.W. Pu [PP5], and depends upon the following definitions.

For an arbitrary function f, define

$$E(f) = \{x: f \text{ assumes an approximate maximum at } x\},$$

$$E_a(f) = E(f) \cap f^{-1}(a),$$

$$F = \{x: \underline{f}_{ap}^{(1)}(x) \geq 0\}, \text{ and}$$

$$K(f) = \{a: E_a(f) \text{ is uncountable}\}.$$

THEOREM 6.2. If f is a Baire one function satisfying

(i) $f(F) \cup K$ contains no non-degenerate interval, and

(ii) $\text{app lim sup}_{h\to 0^+} f(x-h) \leq f(x) \leq \text{app lim sup}_{h\to 0^+} f(x+h)$ for every x,

then f is nondecreasing.

Using this theorem, they were able to show that the extreme approximate derivates and $[f(x) - f(y)]/(x - y)$ have the same bounds in every interval when f is approximately continuous and satisfies the following condition (which they call T_3):

> If $T(f,c,a) = \{x: f(x) = a - cx\}$, then there is a dense set $R' \subset R$ such that whenever $c \in R$
>
> $$|\{a: T(f,c,a) \text{ is uncountable}\}| = 0.$$

They additionally showed that if either of the two approximate symmetric derivates of this f is continuous at a point, then f' exists a.e. in some neighborhood of that point. Viewed in conjunction with Theorem 6.1, this immediately implies the following theorem.

<u>THEOREM 6.3.</u> If f is approximately continuous, approximately symmetrically differentiable and satisfies condition T_3, then f' exists a.e. on a dense open set.

It appears that no full approximate analogue to Khintchine's theorem (Section 3) has yet been proved. A proof presented by G. Russo and S. Valenti [RV] seems to be incorrect.

Several authors have investigated the relationship between the Dini derivates of a continuous function and the extreme approximate symmetric derivates. (See N. Kundu [Ku3] and H.H. Pu and H. W. Pu [PP4].)

In relation to this, M. Evans [Ev1] has shown that the set of functions which have both approximate symmetric derivates infinite at every point is residual in the space of continuous functions.

(B) <u>L_p-symmetric Differentiation</u>. The L_p-symmetric derivative was introduced by M. Weiss [We]. In that paper, a partial analogue to Khintchine's theorem (Section 3) is proved.

<u>THEOREM 6.4.</u> If f is a locally L_p function ($p \geq 1$) which has an n'th L_p-symmetric derivative at each point of a set E, then f has an ordinary L_p-derivative a.e. in E.

M. Evans [Ev2] proved the following theorem.

<u>THEOREM 6.5.</u> If f is an aproximately continuous function which has a nonnegative L_p-symmetric derivative ($p \geq 1$) everywhere, then f is nondecreasing.

(C) <u>Qualitative symmetric differentiation</u>. M. Evans and L. Larson [EL1] have extensively studied qualitative symmetric differentiation. They established qualitative analogues to most of the theorems in Sections 2-5.

(D) Parametric differentiation. In recent years there has been some investigation into the more general derivative which results when the variable h in the symmetric difference quotient is replaced by a more general function of h, called a parameter function. The resultant derivative, which is defined in the natural way, is called a parametric derivative. We mention these derivatives here because, so far, all of the results and proofs concerning them closely parallel well-known results and proofs for the symmetric derivative.

M. Evans and P. Humke [EH1] have proved monotonicity theorems for such derivatives. L. Larson [La4] has proved that if the parameter functions satisfy rather general conditions, then the approximate parametric derivative of a measurable function is in Baire class one. A particular example of this type derivative is the k-pseudo-symmetric derivative introduced by S. Valenti [Va1],[Va2],[RV],[Gi]. (See the comments following Theorem 6.3.)

Section 7: Symmetric and Smooth Functions

The idea of a smooth function was introduced by Riemann [Ri] in his classical study of trigonometric series. Since then, it has found wide application in areas such as approximation theory and partial differential equations. (For example, see [CZ] or [SZ2].) Because of this usefulness, symmetric and smooth functions have been widely studied, and are probably the best understood of all the function classes we have considered here.

In this section we will have one standing hypothesis: Unless specifically mentioned otherwise, all symmetric and smooth functions are assumed to be measurable. The reason for this is that any solution of the functional equation $f(x) + f(y) = f(x + y)$ is a-smooth for all a, and it is well-known that this equation has nonmeasurable solutions.

We start with the following theorem.

THEOREM 7.1. (L. Larson [La4]) If f is an approximately symmetric function, then f is in Baire class one.

(For related results, see C. Neugebauer [Ne1], [Ne2], where it is proved that symmetric functions and L_p-symmetric functions ($p \geq 1$) are in Baire class one.)

Theorem 7.1 implies that $C(f)$ is residual in the domain of f. This is the best result possible because, according to Zahorski [Za], there exists a bounded approximately continuous function which is discontinuous almost everywhere.. E. Stein and A. Zygmund [SZ2] have shown that this is also the case when f is L_p-symmetric, $p < \infty$. However, in the case $p = \infty$, the result is different.

THEOREM 7.2. (Neugebauer [Ne1]) If f is a symmetric function, then $C(f)$ has full measure.

Of course, the existence of continuous nowhere differentiable functions shows that a symmetric function need not be differentiable anywhere. However, the following two theorems have been proved concerning the extreme derivates of such a function.

THEOREM 7.3. (C. Neugebauer [Ne1]) If f is a symmetric function, then

$$\{x: \bar{D}^+f(x) = \bar{D}^-f(x) \text{ and } \underline{D}^+f(x) = \underline{D}^-f(x)\}$$

is a residual set.

THEOREM 7.4. (H. H. Pu, H. W. Pu [PP1]) If f is a symmetric function, then

$$\{x: \bar{D}^-_{ap}f(x) < \bar{D}^+f(x)\}$$

and

$$\{x: \bar{D}^+_{ap}f(x) < \bar{D}^-f(x)\}$$

are both first category sets.

If f is smooth, it has long been known that $D(f)$ must be uncountable in every interval. (A. Zygmund attributes this result to Z. Zalcwasser [Zy2], but gives no reference.) However, $D(f)$ may have measure zero [SZ1]. (C. Neugebauer [Ne2] has extended these results to the case of L_p-smoothness.) But, if f is a-smooth for any $a > 1$, then $D(f)$ has full measure. This is a consequence of the following theorem.

<u>THEOREM 7.5.</u> (E. Stein, A. Zygmund [SZ1]) Suppose that f is defined on an interval I and that at each point of a set $E \subset I$ we have

$$\Delta^2 f(x,h) = O(h\ V(h)),$$

where $V(h)$ is a function defined in a right-hand neighborhood of $h = 0$, decreases monotonically to 0 and $V^2(h)/h$ is integrable near zero. Then $D(f)$ exists a.e. in E.

A smooth function behaves rather strangely on the set where it is differentiable. It was shown by A. Zygmund [Zy2] that if f is continuous and smooth, then f has the Darboux property when restricted to $D(f)$. If f is not continuous, this need not hold [Ne2]. But, if $D(f)$ is thin in the sense that $|D(f) \cap I| < |I|$ for every interval I, then it is once again true [Ne2].

As for the continuity of smooth functions, C. Neugebauer [Ne2] showed in 1964 that if f is L_p-smooth ($p \geq 1$), then $C(f)$ contains an open dense set. Recently, M. Evans and P. Humke [EH4] constructed an L_p smooth ($p \geq 1$) function with an uncountable number of points at which it is approximately discontinuous. M. Evans and L. Larson [EL2] have characterized the set of points of discontinuity for any smooth function as exactly those sets which have no subset which is dense in itself (clairséme). Using this result, they were able to show that if f is smooth and $\bar{D}^+f(x) \geq 0$ everywhere, then f is nondecreasing [EL3]. This monotonicity theorem is false for symmetric functions, and no such characterization of $C(f)$ is known for them.

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