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A Note on Denjoy Integrable Functions

It is well known that if $f(x)$ is Denjoy integrable in the wide sense (integrable $\mathcal{D}$) on $[0,1]$, then every closed set contains a portion on which $f(x)$ is Lebesgue integrable. Equivalently, $[0,1]$ is the countable union of closed sets E_n such that $f|E_n$ is Lebesgue integrable. Thus, if $\hat{f}(x)$ is a non-negative measurable function and $\hat{f}(x) = |f(x)|$ where $f(x)$ is $\mathcal{D}$ integrable on $[0,1]$ then $[0,1] = \cup E_n$, E_n closed and $\hat{f}(x)|E_n$ is Lebesgue integrable. The following theorem shows that these conditions are necessary and sufficient. In fact, $\mathcal{D}$ integrable can be replaced by $\mathcal{D}^*$ integrable. The notation used is given in [1].

Theorem: *If $\hat{f}(x)$ is a non-negative function defined on $[0,1]$ and if $[0,1] = \bigcup_{n=1}^{\infty} E_n$ with each E_n closed such that $\hat{f}(x)|E_n$ is Lebesgue integrable, then there is a function $g(x)$ which takes on the values -1 and 1 so that $\hat{f}(x) \cdot g(x)$ is $\mathcal{D}^*$ integrable.* (Thus $\hat{f}(x) = |f(x)\ g(x)|$.)

The proof of this theorem requires two simple lemmas.

Lemma 1: *If $F(x) = \Sigma F_n(x)$ then for any interval I, $O(F;I) \le \Sigma\, O(F_n;I)$, where $O(F;I)$ is the oscillation of F on $I = \sup_{x\varepsilon I} F(x) - \inf_{x\varepsilon I} F(x)$.*

Lemma 2: *Let E denote a bounded closed set, I_o a closed interval containing E and J_k the sequence of intervals contiguous* to E in I_o. Then for any function F which is finite on I_o,

$$\mathcal{O}(F;I_o) \le V\,(F;E) + 2\sum_k \mathcal{O}(F;J_k)$$

where $V\,(F;E)$ is the variation of F on E. Lemma 1 is immediate and Lemma 2 occurs in [1] p. 231.

Proof of the Theorem: Suppose $\hat{f}$ is a non negative measurable function. Suppose $[0,1] = \cup E_n$ with each E_n closed and that for each natural number n, $\hat{f}|E_n$ is Lebesgue integrable. Without loss of generality assume that $E_1 = \{0,1\}$. Let $X_n = \bigcup_1^n E_i$ and $A_n = E_{n+1} \setminus X_n$. Denote the intervals contiguous to each set X_n by I_{nk} $k = 0, 1, \ldots$ and let $A_{nk} = A_n \cap I_{nk}$. Fix n and k and let $a_{nk} = \int c_{A_{nk}}(x)\,\hat{f}(x)\,dx$. Choose a natural number m so large that $a_{nk}/2m < 2^{-n-k-1}$. There is a partition $x_0, x_1, \ldots x_m, \ldots x_{2m}$ of I_{nk} so that

$$\int c_{A_{nk} \cap [x_{i-1}\, x_i]}(x) \cdot \hat{f}(x)\,dx = a_{nk}/2m .$$

Let $g_{nk}(x) = \begin{cases} (-1)^i \text{ if } x \in [x_{i-1}, x_i) \cap A_{nk} \\ 0 \text{ otherwise} . \end{cases}$

Since $(0,1) = \bigcup_{n,k} A_{nk}$ and the A_{nk} are pairwise disjoint

$$g_n(x) = \sum_k g_{nk}(x) \quad\text{and}\quad g(x) = \sum_n g_n(x)$$

are well defined on $(0,1)$. For completeness, let $g(0) = g(1) = 1$.

Now let

$$f_n(x) = \hat{f}(x) \cdot g_n(x)$$

$$f(x) = \hat{f}(x) \cdot g(x)$$

$$F_n(x) = \int_o^x f_n(t)\,dt$$

$$F(x) = \sum_{n=1}^{\infty} F_n(x)$$

Since $g_n(x) = 0$ for each $x \notin A_n$ and thus for each $x \notin X_n$, $f_n(x)$ is Lebesgue integrable and thus F_n is well defined. Note that $F_n(x) = 0$ at each $x \in X_n$. Since $|F_n(x)| \le 2^{-n}$, $F(x)$ is well defined and since each $F_n(x)$ is continuous, $F(x)$ is a continuous function. Since $F(x)|_{E_N} = \sum_1^N F_n(x)$, and $f(x)|_{E_N} = \sum_1^N f_n(x)$, it follows that $F'\text{ ap }(x) = f(x)$ almost everywhere in E_n and since $[0,1] = \cup E_n$, $F'\text{ ap }(x) = f(x)$ a.e.

Thus, in order to show that $F(x)$ is the $\mathcal{D}^*$ integral of $f(x)$, it suffices to show that $F(x)$ is ACG*. For this, it will suffice to show that on each set E_n, $F(x)$ is AC*. Let a natural number n and $\epsilon > 0$ be given. Choose $M > n$ so that $2^{-M} < \varepsilon/3$. Since $\sum_1^M F_m(x)$ is absolutely continuous, it is AC* on E_n and there exists $\delta > 0$ so that if I_j is any sequence of non-overlapping intervals with endpoints in E_n and $\Sigma|I_j| < \delta$, $\sum_j \mathcal{O}\left(\sum_1^M F_m(x); I_j\right) < \varepsilon/3$. Let $G(x) = F(x) - \sum_1^M F_m(x) = \sum_{M+1}^\infty F_m(x)$.

Then if I_j is any sequence of intervals with endpoints in E_n, Since $V(F_m; E_n) = 0$ when $m > n$, it follows that

$$\sum_j \mathcal{O}(G_M; I_j) \le \sum_j \sum_{M+1}^\infty \mathcal{O}(F_m; I_j)$$

$$\le \sum_j \sum_{M+1}^\infty \left(V(F_m; E_n) + 2\sum_k \mathcal{O}(F_m; I_{mk} \cap I_j)\right)$$

$$\le 2\sum_j \sum_{M+1}^\infty \sum_k \mathcal{O}(F_m; I_{mk} \cap I_j)$$

$$\leq 2 \sum_{M+1}^{\infty} 2^{-m} \leq 2 \cdot 2^{-M} \leq \frac{2\varepsilon}{3}$$

It follows that for this δ and any sequence of intervals which are pairwise nonoverlapping with endpoints in E_n that $\Sigma\, \mathcal{O}(F, I_j) < \varepsilon$. Thus F is AC* on each E_n, F is ACG*, and f is $\mathcal{D}^*$ integrable with $F(x) = \mathcal{D}^* \int_0^x f(t)\, dt$.

Note. Both $g_{nk}^{-1}(1)$ and $g_{nk}^{-1}(-1)$ are F_σ sets. This is because they are a finite union of sets of the form

$$(E_{n+1} \setminus X_n) \cap [x_{i-1}, x_i)$$

where both E_{n+1} and X_n are closed. Thus $g(x) = \sum_n \sum_k g_{nk}(x)$, which takes on only the values 1 and -1, satisfies $g^{-1}(1)$ and $g^{-1}(-1)$ are F_σ sets. Consequently g belongs to Baire class 1.

References

1. S. Saks, Theory of the Integral, Dover Publications, New York.

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