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On Dini derivates of continuous and monotone functions.

1. Introduction. In [10] H.H. Pu, J.D. Chen and H.W. Pu proved the following:

Lemma A. Let f be a continuous function on R. Then the set of all x at which $\overline{f}^{+}_{ap}(x) < \overline{f}^{-}(x)$ is a first category set.

They used this lemma to obtain the approximate analogue of the well known Neugebauer theorem [9] on the symmetry of derivates of continuous functions. M.J. Evans and P.D. Humke [4] pointed out that Lemma A immediately yields the following:

Theorem B. Let f be a continuous function. Then at all points x except a first category set $\overline{f}^{+}(x) = \overline{f}^{-}(x) = \overline{f}^{+}_{ap}(x) = \overline{f}^{-}_{ap}(x)$ and $\underline{f}^{+}(x) = \underline{f}^{-}(x) = \underline{f}^{+}_{ap}(x) = \underline{f}^{-}_{ap}(x)$.

In [5] M.J. Evans and P.D. Humke proved that the following assertion holds.

Theorem C. Let f be a monotone function on R. Then at all points x except a σ-porous set $\overline{f}^{+}(x) = \overline{f}^{-}(x) = \overline{f}^{+}_{ap}(x) = \overline{f}^{-}_{ap}(x)$ and $\underline{f}^{+}(x) = \underline{f}^{-}(x) = \underline{f}^{+}_{ap}(x) = \underline{f}^{-}_{ap}(x)$.

In the present article we show that Theorems B, C can be strengthened. In fact, in these theorems we can assert instead of $\overline{f}^{+}(x) = \overline{f}^{+}_{ap}(x)$ the stronger fact that $\overline{f}^{+}(x)$ is a right essential derived number. We say (see [7], p.4 and [2], p. 51) that an extended real number e is a right essential derived number (resp. an essential derived number) of f at x if there exists a measurable set E having right upper density 1 at x (resp. symmetric upper density 1 at x) such that $\lim_{y\to x, y\in E} (f(y)-f(x))(y-x)^{-1}=e$. As consequences some results on preponderant derivatives are obtained. Note that Corollary 4 gives a partial answer to a question to Problem 4 of [1].

For the definitions of σ-porous sets and related notions see e.g. [5] or [6]. The symbol λ stands for the Lebesgue measure on the real line R.

2. Theorems. The following simple fact is essentially well known (c.f.[7] and [10]).

Lemma. *Let* f *be a continuous function on* R. *Let* $a \geqq 0$, $h > 0$, $r \in R$, $x_n \to x$ *and* $\lambda\{y\in(x_n, x_n+h): (f(y) - f(x_n))(y-x_n)^{-1} \leq r\} \geq a$. *Then* $\lambda\{y\in(x,x+h): (f(y)-f(x))(y-x)^{-1} \leq r\} \geq a$.

Proof. It is easy to see that

$\operatorname{Lim\,sup}_{n\to\infty} \{t\in(0,h): (f(x_n+t) - f(x_n))/t \leq r\} \subset \{t\in(0,h): (f(x+t) - f(x))/ t \leq r\}$. From this fact the assertion of Lemma immediately follows.

Theorem 1. *Let* f *be a continuous function on* R *and* w *an arbitrary positive function such that* $\lim_{h\to 0+} w(h) = 0$. *Then for all points* x *except a first category set there exists a measurable set* M *such that*

$$\liminf_{h\to 0+} \lambda((x,x+h)-M) / w(h) = 0 \quad \textit{and}$$

$$\lim_{y\to x, y\in M} (f(y)-f(x))\cdot(y-x)^{-1} = \overline{f}^{+}(x)$$

Proof. It is easy to verify that it is sufficient to prove that the set A of all x for which there exists a rational number r such that $\overline{f}^{+}(x) > r$ and

$$\liminf_{h\to 0+} \lambda((x,x+h) - \{y:(f(y)-f(x))(y-x)^{-1} > r\}) / w(h) > 0$$

is a first category set. For a rational r and a positive integer n let $B_{r,n}$ be the set of all x for which

$\lambda(\{y\in(x,x+h):\ (f(y) - f(x))(y-x)^{-1} \le r\}) \ge w(h) / n$ whenever $0 < h < 1/n$. Put $A_{r,n} = B_{r,n} \cap \{x:\ \overline{f}^{+}(x) > r\}$. Since $A = \bigcup A_{r,n}$, it is sufficient to prove that all $A_{r,n}$ are nowhere dense sets. Suppose that an $A_{r,n}$ is dense in (a,b), $a < b$. From the Lemma it immediately follows that $B_{r,n}$ is a closed set. Consequently $(a,b) \subset B_{r,n}$ and thus $\underline{f}^{+}(y) \le r$ for any $y\in(a,b)$. By Dini's Theorem ([11], p.204) we obtain that $\overline{f}^{+}(x) \le r$ on (a,b), which is a contradiction.

Corollary 1. Let f be a continuous function on R. Then at all points x except a first category set $\overline{f}^{+}(x) = \overline{f}^{-}(x)$, $\underline{f}^{+}(x) = \underline{f}^{-}(x)$ are essential derived numbers of f.

Corollary 2. Let f be a continuous function. Then the set of all points at which the preponderant derivative of f (resp. the right preponderant derivative) exists but the derivative (resp. the right derivative) does not exist is a first category set. (We can clearly choose any definition of the (right) preponderant derivative contained in [3], [8], [1].)

Theorem 2. Let f be a monotone function. Then at all points x except a σ-porous set $\overline{f}^{+}(x) = \overline{f}^{-}(x)$ and $\underline{f}^{+}(x) = \underline{f}^{-}(x)$ are

<u>essential</u> <u>derived</u> <u>numbers</u> <u>of</u> f.

<u>Proof</u>. From Theorem C and from the fact that the functions $-f(x)$, $f(-x)$, $-f(-x)$ are monotone it follows that it is sufficient to prove that the set of all x at which $\overline{f}^{-}(x) = \overline{f}^{+}(x)$ and $\overline{f}^{-}(x)$ is not a left essential derived number is σ-porous. For any positive integer n and rational numbers $s < r$ let $A_{n,s,r}$ be the set of all points x at which f is continuous, $\overline{f}^{+}(x) > r$ and $\lambda(\{y \epsilon (x-h,x): (f(y) - f(x))(y-x)^{-1} < s\})/h > 1/n$ whenever $0<h<1/n$. It is clearly sufficient to prove that all $A_{n,s,r}$ are porous. Let $x \epsilon A_{n,s,r}$ be given. Choose $p > 0$ such that

(1) $$8p(1 + |r|/(r-s)) < 1/n .$$

We shall prove that the porosity of $A_{n,s,r}$ at x is at least p. Let $0<d<1/2n$. Choose $y \epsilon (x,x+d)$ such that

(2) $$(f(y) - f(x))(y-x)^{-1} > r .$$

Define the auxiliary linear function $g(z) = f(y) + (z-y)r$ and put $a = \inf\{z \epsilon [x,y] : f(z) \geq g(z)\}$. Since f is continuous at x, (2) yields $a > x$. Put $J = (a,a + 4p(a-x))$ if f is nondecreasing and $J = (a - 4p(a-x), a)$ if f is nonincreasing. Since $J \subset (x,x + 2(a-x))$ it is clearly sufficient to prove that $J \cap A_{n,s,r} = \emptyset$. Suppose on the contrary that there exists $z \epsilon J \cap A_{n,s,r}$. Then $f(z) \geq g(a)$ and therefore

(3) $$g(z) - f(z) \leq 4p|r|(a-x) .$$

Define the further auxiliary linear function $h(t) = f(z) + (t-z)s$. From (3) easily follows that $h(t) > g(t)$ for $t < z - 4|r|p(a-x)/(r-s)$. Therefore $C \equiv \{t \epsilon (x,z):$

$(f(t) - f(z))/(t-z) < s\} = \{t\epsilon(x,z): f(t) > h(t)\} \subset$
$[z - 4|r|(a-x)p/(r-s), z) \cup \{t\epsilon(x,z - 4|r|(a-x)p/(r-s)):$
$f(t) > g(t)\}$. By the definition of a the last set is a subset of J and therefore $\lambda C \leq 4p(a-x)+4|r|(a-x)p/(r-s)<(a-x)/2n<(z-x)/n$ and this is a contradiction since $(z-x) < 1/n$ and $z\epsilon A_{n,s,r}$.

Corollary 3. *Let f be a Lipschitz function. Then the conclusion of Theorem 2 holds.*

Proof. Consider the function $g(x) = f(x) + Kx$, where K is a Lipschitz constant of f.

Corollary 4. *Let f be a monotone or Lipschitz function. Then the set of all points at which the preponderant derivative of f(resp. the right preponderant derivative) exists but the derivative does not exist is a σ-porous set.*

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