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SOME THEOREMS ON DINI DERIVATES

The relationships between the density of sets of points where the various Dini derivates of a function are nonnegative are studied.

Theorem 1: If f(x) is a real valued function of a real variable, λ any real number and $\{ x : D_-f(x) \geq \lambda \}$ is dense, then $\{ x : D^+f(x) \geq \lambda \}$ is dense.

Proof: Without loss of generality, assume $\lambda=0$. Let (a,b) be any interval. There exists x_1 in (a,b) such that $D_-f(x_1) > -1$. Therefore, there is some $\delta_1 > 0$ such that for every t in $(x_1-\delta_1,x_1)$,

$$f(t) < f(x_1) + (x_1-t).$$

Choose $\delta_1 < 1$ and such that $x_1-\delta_1 > a$. There is x_2 in $(x_1-\delta_1,x_1)$ such that $D_-f(x_2) > -1/2$. Therefore, there is some $\delta_2 > 0$ such that for every t in $(x_2-\delta_2,x_2)$,

$$f(t) < f(x_2) + (1/2)(x_2-t).$$

Choose $\delta_2 < 1/2$ and such that $x_2-\delta_2 > x_1-\delta_1$.

Continuing in this manner we obtain a decreasing sequence of intervals $\{(x_n-\delta_n,x_n)\}$ such that $\delta_n < 1/n$.

Let $w = \bigcap_{n=1}^{\infty} (x_n-\delta_n, x_n)$.

Then $f(w) < f(x_n) + (1/n)(x_n-w)$ for every n.

i.e. $\dfrac{f(w) - f(x_n)}{w - x_n} > -1/n$ for every n.

Since $x_n \to w^+$, we conclude that $D^+f(w) \geq 0$.

The converse of the above theorem is not true as can be seen by considering

$$f(x) = \begin{cases} 1-x & x \text{ rational} \\ -x & x \text{ irrational.} \end{cases}$$

Example: In the context of the above theorem we mention the following example.

Let $\{a_m\}$, $m \geq 0$ denote the following sequence:

$0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{16}, \frac{3}{16}, \ldots$

The set $\{ x : x=a_m \}$ is dense in $[0,1]$.

For each $m \geq 1$, there is a positive integer n_m such that

$$2^{n_m-1} \leq m < 2^{n_m}.$$

Thus

$$a_m = \frac{2m + 1 - 2^{n_m}}{2^{n_m}} \qquad \text{for } m \geq 1.$$

Set

$$t_m = \frac{2m - 2^{n_m}}{2^{n_m}} \qquad \text{for } m \geq 1.$$

We define a sequence of functions $\{ f_m(x) \}$ for x in $[0,1]$ by induction, as follows:

Set $f_0(x) = (x-1)^2$ and for $m \geq 1$, set

$$f_m(x) = \begin{cases} f_{m-1}(x) & x \notin [t_m,a_m] \\ f_{m-1}(a_m) + \dfrac{f_{m-1}(t_m)-f_{m-1}(a_m)}{(t_m-a_m)^2}(x-a_m)^2 & x \in [t_m,a_m] \end{cases}$$

$\{ f_m(x) \}$ is a monotonically decreasing sequence of bounded, continuous functions on $[0,1]$.

Let $\lim_{m\to\infty} f_m(x) = f(x)$.

We note that for each $m \geq 0$, the right hand derivative of $f_m(x)$ is negative for x in $[0,1)$ and the left hand derivative is zero whenever $x = a_k$, $k \leq m$ and negative otherwise, for x in $(0,1]$.

Since the left hand derivative of $f(x)$ is zero whenever $x = a_m$, $m \geq 1$, it is interesting to note that the above theorem implies that $\{ x : D^+f(x) = 0 \}$ is dense in $[0,1]$.

Lemma: If $f(x)$ is a real valued function of a real variable which is continuous on the left and if $\{ x : D^+f(x) \geq \lambda \}$ is dense for some real number λ, then $\{ x : D^-f(x) \geq \lambda \}$ is dense.

Proof: Without loss of generality, assume $\lambda = 0$. Let (a,b) be any interval. There exists x_1 in (a,b) such that $D^+f(x_1) > -1$. Therefore, there exists $\delta_1 > 0$ such that for some t in $(x_1, x_1+\delta_1)$,

$$f(t) > f(x_1) - (t-x_1).$$

Since $f(t) = \lim_{u \to t^-} f(u)$, it can be seen that there exists $r > 0$ such that for every u in $(t-r,t)$

$$f(u) > f(x_1) - (u-x_1).$$

Choose r such that $(t-r,t) \subset (x_1, x_1+\delta_1)$. There exists x_2 in $(t-r,t)$ such that $D^+f(x_2) > -1/2$. Therefore, there exists $\delta_2 > 0$ such that for some v in $(x_2, x_2+\delta_2)$,

$$f(v) > f(x_2) - (1/2)(v-x_2).$$

The number δ_2 can be so chosen that $0 < \delta_2 < 1/2$ and $x_2+\delta_2 < x_1+\delta_1$.

We can find x_3 in $(x_2, x_2+\delta_2)$ such that $D^+f(x_3) > -1/3$ and $f(x_3) > f(x_2) - (1/2)(x_3-x_2)$.

Continuing in this manner we obtain a decreasing sequence of intervals $\{ (x_n, x_n+\delta_n) \}$ where $\delta_n < 1/n$ and

$f(x_{n+1}) > f(x_n) - (1/n)(x_{n+1}-x_n)$ for each positive integer n.

Let $$w = \bigcap_{n=1}^{\infty} (x_n, x_n+\delta_n).$$

If m and p are positive integers such that $p > m$, then it can be seen that

$$f(x_p) > f(x_m) - (1/m)(x_p-x_m).$$

Keeping m fixed and taking the limit as p goes to infinity and noting that $x_n \to w^-$ so that $f(w)=\lim_{n\to\infty} f(x_n)$

we get $f(w) \geq f(x_m) - (1/m)(w-x_m)$.

i.e. $$\frac{f(w) - f(x_m)}{w - x_m} \geq -1/m$$

Since this is true for each m, we get $D^{-}f(w) \geq 0$.

Theorem 2: *If f(x) is a continuous, real valued function of a real variable and λ any real number, then the following statements are equivalent.*

(a) $\{ x : D^{-}f(x) \geq \lambda \}$ *is dense.*

(b) $\{ x : D^{+}f(x) \geq \lambda \}$ *is dense.*

Proof: Follows immediately from the above lemma.

Remark: Let f(x) be a continuous, real valued function of a real variable. The following six statements are equivalent.

(1) $\{ x : D_{-}f(x) > -\alpha \}$ is dense for each $\alpha > 0$.

(2) $\{ x : D^{-}f(x) > -\alpha \}$ is dense for each $\alpha > 0$.

(3) $\{ x : D_{+}f(x) > -\alpha \}$ is dense for each $\alpha > 0$.

(4) $\{ x : D^{+}f(x) > -\alpha \}$ is dense for each $\alpha > 0$.

(5) $\{ x : D^{-}f(x) \geq 0 \}$ is dense.

(6) $\{ x : D^{+}f(x) \geq 0 \}$ is dense.

The first four statements are equivalent because of Dini's theorem [1]. In the proof of theorem 1 we have actually proved that statement 1 implies statement 6. Clearly statement 6 implies statement 4. Statement 6 and statement 5 are equivalent by theorem 2.

Whether the following 2 statements for continuous functions:

(1) $\{ x : D_-f(x) \geq 0 \}$ is dense,

(2) $\{ x : D^+f(x) \geq 0 \}$ is dense,

are (a) equivalent to each other and (b) to the above mentioned six statements remain open questions.

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REFERENCES

1. S. Saks, Theory of the Integral, Dover Publications.

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