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ON A SPACE OF FUNCTIONS REPRESENTABLE BY DERIVATIVES

The present note was inspired by the original version of [1] where the first three authors proved that

(A) the characteristic function of a closed set of real numbers can be expressed as the product of two derivatives.

This assertion, however, played only an auxiliary role and is not contained in the final version of [1]. Various generalizations of (A) will appear elsewhere; some of the corresponding results have been mentioned in [2], part IV.

This note deals with the space $\mathcal{Q}_0$ of bounded functions on the interval $[0,1]$ which have only a finite number of discontinuities and with its uniform closure $\mathcal{Q}$. It is easy to see that $\mathcal{Q}$ is an algebra containing all monotone functions. The main result says, in particular, that for each $f \in \mathcal{Q}$ there are derivatives $f_j \in \mathcal{Q}$ such that $f = f_1 + f_2 f_3$.

Notation. 1° The word function means a mapping to $R = (-\infty,\infty)$. For each nondegenerate interval $J \subset R$ let

$$\mathcal{D}(J) = \{f';\ f \text{ is a function differentiable on } J\} .$$

2° Let f be a bounded function on a nondegenerate interval J. For each $x \in J$ and each $h \in (0,\infty)$ set

$$\omega^*(f,x,h) = \sup f(M),\quad \omega_*(f,x,h) = \inf f(M)\quad (M = J \cap [x-h,x+h]),$$

$$\omega^*(f,x) = \lim \omega^*(f,x,h),\quad \omega_*(f,x) = \lim \omega_*(f,x,h)\quad (h \searrow 0),$$

$$\omega_f(x) = \omega^*(f,x) - \omega_*(f,x),\quad \sigma_f(x) = \tfrac{1}{2}(\omega^*(f,x) + \omega_*(f,x)) .$$

Lemma 1. Let f,g be bounded functions on R. Then $\omega_{f+g} \leqq \omega_f + \omega_g$. If f is continuous at a point x, then $\omega_{fg}(x) = |f(x)| \cdot \omega_g(x)$.

(The proof is left to the reader.)

Lemma 2. Let $J = (0,\infty)$ and let $\delta \in J$. Let f be a nonnegative function continuous on J. Then there are functions $\gamma,\ \varphi \in \mathcal{D}([0,\infty))$ which are continuous on J such that $\gamma(0) = \varphi(0) = 0$, $\varphi = 0$ on (δ,∞) and $\gamma \geqq 0$, $f = \gamma + \varphi^2$ on J.

Proof. Choose numbers y_n such that $\delta = y_0 > y_1 > \cdots$, $y_n \to 0$, $y_{n-1}/y_n \to 1$. Set $J_n = [y_n, y_{n-1}]$, $M_n = \max\{f(x); x \in J_n\}$. For $n = 1,2,\cdots$ find a $\delta_n \in J$ such that $\delta_n M_n^{1/2} < y_n/n$ and that $|f(y) - f(x)| < 1/n$, whenever $x,y \in J_n$ and $|y-x| < \delta_n$; further choose an integer r_n such that $y_{n-1} - y_n < r_n\delta_n$. Let

$$S = \{y_n + k(y_{n-1} - y_n)/r_n;\ n = 1,2,\ldots,\ k = 1,\ldots,r_n\}.$$

We have $S = \{x_0, x_1, \cdots\}$, where $\delta = x_0 > x_1 > \cdots$. For $k = 1,2,\ldots$ set $L_k = [x_k, x_{k-1}]$, $d_k = x_{k-1} - x_k$, $w_k = \max\{|f(y) - f(x)|; x,y \in L_k\}$, $m_k = \min\{f(x); x \in L_k\}$, $s_k = \frac{1}{2}(x_k + x_{k-1})$. It is easy to construct a function φ continuous on L_k such that $\varphi \geqq 0$ on $[x_k, s_k]$, $\varphi \leqq 0$ on $[s_k, x_{k-1}]$, $\varphi = 0$ on $\{x_k, x_{k-1}\}$, $\varphi^2 \leqq m_k$ on L_k, $\int_{L_k} \varphi = 0$, $\int_{L_k} \varphi^2 > m_k d_k - 2^{-k}x_k$. In this way we define a function φ on $(0,\delta]$. Further we set $\varphi = 0$ on $\{0\} \cup (\delta,\infty)$, $\gamma = f - \varphi^2$ on J, $\gamma(0) = 0$. Then φ, γ are continuous on J. Obviously $\gamma \geqq 0$, $\int_{L_k} \gamma < w_k d_k + 2^{-k}x_k$. Let $x \in (0,\delta]$. There are k, n such that $y_n \leqq x_k < x \leqq x_{k-1} \leqq y_{n-1}$. It follows from the choice of $r_n, r_{n+1}, \cdots$ that $w_j < 1/n$ $(j = k, k+1, \ldots)$. Therefore $\int_0^x \gamma \leqq \sum_{j=k}^{\infty} (w_j d_j + 2^{-j}x_j) \leqq x_{k-1}/n + 2^{-k+1}x_k < x(y_{n-1}/(ny_n) + 2^{-k+1})$, $0 \leqq \int_0^x \varphi =$

$= \int_{x_k}^{x} \varphi \leqq \frac{1}{2} d_k m_k^{1/2} \leqq \delta_n M_n^{1/2} < y_n/n < x/n$. For $x \to 0$ we have $k \to \infty$, $n \to \infty$ so that $x^{-1} \int_0^x \gamma \to 0$, $x^{-1} \int_0^x \varphi \to 0$. This completes the proof.

Lemma 3. Let $\delta \in (0,\infty)$. Let f be a function bounded on R and continuous on $R\setminus\{0\}$. Then there are functions $\gamma, \varphi, \psi \in \mathcal{B}(R)$ which are continuous on $R\setminus\{0\}$ such that $\varphi = \psi = 0$ on $R\setminus(-\delta,\delta)$,

(1) $f = \gamma + \varphi\psi$, $f \wedge f(0) \leqq \gamma \leqq f \vee f(0)$, $\varphi^2 = \psi^2 \leqq |f - f(0)|$,

(2) $\omega_\gamma \leqq \omega_f$, $\omega_\varphi \vee \omega_\psi \leqq 2\,\omega_f^{1/2}$.

Proof. We may suppose that $f(0) = 0$. Let $f_1 = f \vee 0$, $f_2 = (-f) \vee 0$. By Lemma 2 there are functions $\gamma_j, \varphi_j \in \mathcal{B}(R)$ which are continuous on $R\setminus\{0\}$ such that $\gamma_j(0) = 0$, $\varphi_j = 0$ on $R\setminus(-\delta,\delta)$, $\gamma_j \geqq 0$ and that $f_j = \gamma_j + \varphi_j^2$ $(j = 1,2)$. Set $\gamma = \gamma_1 - \gamma_2$, $\varphi = \varphi_1 + \varphi_2$, $\psi = \varphi_1 - \varphi_2$, $A = \{x \in R;\ f(x) \geqq 0\}$, $B = R\setminus A$. On A we have $\gamma_2 = \varphi_2 = 0$, $0 \leqq \gamma = \gamma_1 \leqq f_1$, $\varphi = \varphi_1 = \psi$, $\varphi_1^2 \leqq f_1 = |f|$; on B we have $\gamma_1 = \varphi_1 = 0$, $-f_2 \leqq -\gamma_2 = \gamma \leqq 0$, $\varphi = \varphi_2 = -\psi$, $\varphi_2^2 \leqq f_2 = |f|$. Obviously $f = \gamma + \varphi\psi$. This proves (1).

Now let $\theta^*(t) = \omega^*(f,0,|t|)$, $\theta_*(t) = \omega_*(f,0,|t|)$ $(t \neq 0)$, $\theta^*(0) = \omega^*(f,0)$, $\theta_*(0) = \omega_*(f,0)$. It follows from (1) that $\theta_* \leqq \gamma \leqq \theta^*$, $\varphi^2 = \psi^2 \leqq \theta^* - \theta_*$. This easily implies (2).

Lemma 4. Let $a,b \in R$ and let A be a finite set of real numbers. Let f be a function on R which is continuous on $R\setminus A$ such that $a < f < b$. Then there are functions $\gamma,\varphi,\psi \in \mathcal{B}(R)$ such that

$$f = \gamma + \varphi\psi, \quad a < \gamma < b, \quad \varphi^2 = \psi^2 < b - a,$$

$$\omega_\gamma \leqq \omega_f, \quad \omega_\varphi \vee \omega_\psi \leqq 2\, \omega_f^{1/2} .$$

(This follows easily from Lemma 3.)

Lemma 5. Let $\varphi,\psi \in \mathcal{B}(R)$. Suppose that φ,ψ are bounded and that at each point of R at least one of the functions φ,ψ is continuous. Then $\varphi\psi \in \mathcal{B}(R)$.

(The proof is left to the reader.)

Lemma 6. Let f be a bounded function on the interval $J = [0,1]$; let $\eta \in (0,\infty)$ and let $\omega_f < 2\eta$ on J. Then there is a function h continuous on J such that $h(0) = \sigma_f(0)$, $h(1) = \sigma_f(1)$ and $|h - f| < \eta$.

Proof. Let Y be the set of all points $y \in (0,1]$ with the following property:

(P) There is a function g continuous on $[0,y]$ such that $g(0) = \sigma_f(0)$ and $|g - f| < \eta$ on $[0,y]$.

There is a $y_1 \in (0,1]$ such that $|f - \sigma_f(0)| < \eta$ on $[0,y_1]$. Setting $g = \sigma_f(0)$ on $[0,y_1]$ we see that $y_1 \in Y$. Therefore $s = \sup Y > 0$. There are s_1, s_2 such that $0 < s_1 < s < s_2$ and that $|f - \sigma_f(s)| < \eta$ on

$J \cap [s_1, s_2]$. Choose a $y \in Y \cap (s_1, s]$ and a function g as in (P). Let λ be a linear function such that $\lambda(s_1) = 0$, $\lambda(y) = 1$. Define a function h on $[0,s_2]$ setting $h = g$ on $[0,s_1]$, $h = g + \lambda\cdot(\sigma_f(s) - g)$ on $[s_1, y]$, $h = \sigma_f(s)$ on $[y,s_2]$. Then h is continuous on $[0,s_2]$, $h(0) = \sigma_f(0)$ and $|h - f| < \eta$ on $[0,s_2] \cap J$. This shows that $s = 1$ and that our assertion holds.

Lemma 7. Let f be a bounded function on the interval $J = (0,\infty)$. Then $\omega_f(0+) = 0$ if and only if there is a function h continuous on J such that $(f - h)(0+) = 0$.

Proof. If there is such an h, then, for each $x \in J$, $\omega_f(x) = \omega_{f-h}(x)$ and, obviously, $\omega_{f-h}(0+) = 0$. Now suppose that $\omega_f(0+) = 0$. Let $x_n \in J$, $x_0 > x_1 > \cdots$, $x_n \to 0$. Let $\eta_n = 1/n + \sup\{\omega_f(x);\ x_n \leqq x \leqq x_{n-1}\}$. It follows from Lemma 6 that there is a function h continuous on J such that $|f - h| < \eta_n$ on $[x_n, x_{n-1}]$ $(n = 1,2,\cdots)$. It is easy to see that $(f - h)(0+) = 0$.

Lemma 8. Let f be a bounded function on R, let $\eta \in (0,\infty)$ and let A be a finite set of real numbers. Suppose that $\omega_f(t) \to 0$ $(t \to x)$ and $\omega_f(x) < 2\eta$ for each $x \in A$. Then there is a function h on R which is continuous on $R\backslash A$ such that $|h| < \eta$ and that $f - h$ is continuous at each point of A.

Proof. There are disjoint open intervals I_x such that $x \in I_x$ and $|f - \sigma_f(x)| < \eta$ on I_x $(x \in A)$. Let $x \in A$. By Lemma 7 there is a function h_x on R which is continuous on $R \setminus \{x\}$ such that $h_x(x) = f(x)$ and that $f - h_x$ is continuous at x. There is a closed interval J_x containing x in its interior such that $J_x \subset I_x$ and that $|h_x - \sigma_f(x)| < \eta$ on J_x. Now it is easy to construct a function h on R which is continuous on $R \setminus A$ such that $|h| < \eta$ and that $h = h_x - \sigma_f(x)$ on J_x for each $x \in A$. It is obvious that $f - h$ is continuous at each point of A.

Notation. In the rest of this note, J will stand for $[0,1]$. Let $\mathcal{A}_0$ be the system of all bounded functions f on J such that f is continuous on $J \setminus A$ for some finite set A. Let $\mathcal{A}$ be the system of all bounded functions f on J such that

$$(3) \qquad \omega_f(t) \to 0 \quad (t \in J,\ t \to x) \quad \text{for each} \quad x \in J \ .$$

Proposition 1. Let f be a bounded function on J. Then $f \in \mathcal{A}$ if and only if the set $\{x \in J;\ \omega_f(x) > \eta\}$ is finite for each $\eta \in (0,\infty)$.

(The proof is left to the reader.)

Proposition 2. Let f be a bounded function on J. Then $f \in \mathcal{A}$ if and only if for each $x \in J$ there is a function h on J which is continuous on $J\setminus\{x\}$ such that $f - h$ is continuous at x.

(This follows easily from Lemma 7.)

Proposition 3. Let $f_n \in \mathcal{A}$ $(n = 1,2,\ldots)$ and let $f_n \to f$ uniformly. Then $f \in \mathcal{A}$.

Proof. It is easy to see that $\omega_{f_n} \to \omega_f$ uniformly. This implies (3).

Remark. Obviously $\mathcal{A}_0 \subset \mathcal{A}$. If f is a bounded function on J such that $f(x+)$ exists for each $x \in [0,1)$ and that $f(x-)$ exists for each $x \in (0,1]$, then $f \in \mathcal{A}$. In particular, each function of bounded variation on J is in $\mathcal{A}$.

Lemma 9. Let $f \in \mathcal{A}$. Let $\alpha_n \in (0,\infty)$, $2\,\alpha_0 > \max\{\omega_f(x);\ x \in J\}$, $\alpha_0 > \alpha_1 > \cdots$, $\sum_{n=0}^{\infty} \alpha_n < \infty$. Let

$$A_n = \{x \in J;\ \alpha_{n+1} \leqq \omega_f(x)/2 < \alpha_n\} .$$

Then there are functions $g, h_0, h_1, \cdots$ on J such that g is continuous on J, h_n is continuous on $J\setminus A_n$, $f - h_n$ is continuous at each point of A_n,

$$(4) \qquad |h_n| < \alpha_n \ (n = 0,1,\ldots) \quad \text{and} \quad f = g + \sum_{n=0}^{\infty} h_n .$$

Proof. For $n = 0,1,\dots$ let h_n be a function constructed according to Lemma 8 where we take $A = A_n$, $\eta = \alpha_n$. It is easy to see that the function $f - \sum_{n=0}^{\infty} h_n$ is continuous.

Proposition 4. Let f be a function on J. Then $f \in \mathcal{A}$ if and only if there are $f_n \in \mathcal{A}_0$ such that $f_n \to f$ uniformly.

(This follows from Lemma 9 and Proposition 3.)

Proposition 5. Let $f,g \in \mathcal{A}$. Then $f+g$, $fg \in \mathcal{A}$.

Proof. If $f,g \in \mathcal{A}_0$, then, obviously, $f+g$, $fg \in \mathcal{A}_0$. Now we apply Proposition 4.

Theorem. Let $f \in \mathcal{A}$, $\varepsilon \in (0,\infty)$, $\varepsilon > \max\{\omega_f(x);\ x \in J\}$. Then there is a function g continuous on J and functions $\lambda,\varphi,\psi \in \mathcal{A} \cap \mathcal{B}(J)$ such that

$$(5) \qquad f = g + \lambda + \varphi\psi, \quad |f-g| < \varepsilon/2, \quad |\lambda| < \varepsilon/2, \quad \varphi^2 \vee \psi^2 < \varepsilon,$$

$$(6) \qquad \omega_\lambda \leqq 5(\varepsilon\omega_f)^{1/2}, \quad \omega_\varphi \vee \omega_\psi \leqq 2\omega_f^{1/2} .$$

Proof. Set $\eta = \varepsilon/2$, $\mathcal{B} = \mathcal{B}(J)$. Choose a $\beta_0 \in (0,\infty)$ such that $\max\{\omega_f(x);\ x \in J\} < 2\beta_0^2 < \varepsilon$ (hence $\beta_0^2 < \eta$) and then a $b \in (0, \frac{1}{2})$ such that

$$(7) \qquad \beta_0^2 + b(2+4\beta_0) < \eta .$$

Further choose $\beta_n \in (0,\infty)$ such that $\beta_0 > \beta_1 > \cdots$ and $\sum_{n=1}^{\infty} \beta_n < b$. Notice that

(8) $$(\beta_0 + b)^2 < \eta .$$

Set $\alpha_n = \beta_n^2$ and find sets A_n and functions g, h_n according to Lemma 9. It follows from (4) and (7) that $|f - g| < \sum_{n=0}^{\infty} \alpha_n < \alpha_0 + b < \eta$.

By Lemma 4 there are $\gamma_n, \varphi_n, \psi_n \in \mathcal{B}$ such that

(9) $$h_n = \gamma_n + \varphi_n\psi_n ,$$

(10) $$|\gamma_n| < \alpha_n, \quad |\varphi_n| = |\psi_n| < \beta_n \cdot 2^{1/2} ,$$

(11) $$\omega_{\gamma_n} \leqq \omega_{h_n}, \quad \omega_{\varphi_n} \vee \omega_{\psi_n} \leqq 2\, \omega_{h_n}^{1/2} .$$

Because of (10) we may define $\gamma = \sum_{n=0}^{\infty} \gamma_n$, $\varphi = \sum_{n=0}^{\infty} \varphi_n$, $\psi = \sum_{n=0}^{\infty} \psi_n$, $\rho = \varphi\psi - \sum_{n=0}^{\infty} \varphi_n\psi_n$; further set $\lambda = \gamma - \rho$. Obviously $\gamma, \varphi, \psi \in \mathcal{B}$. Since $|\varphi_j\psi_k| \leqq 2\, \beta_j\beta_k$ and, according to Lemma 5, $\varphi_j\psi_k \in \mathcal{B}$ whenever $j \neq k$, we have $\rho \in \mathcal{B}$. Therefore $\lambda \in \mathcal{B}$. By (4) and (9),

$$f = g + \gamma + \sum_{n=0}^{\infty} \varphi_n\psi_n = g + \lambda + \varphi\psi .$$

For each nonnegative integer p let S_p be the set of all pairs (j,k) of nonnegative integers such that $j \neq p \neq k \neq j$ and let $\sigma_p = \sum \varphi_j\psi_k \ ((j,k) \in S_p)$. Obviously

(12) $$|\sigma_0| < 2b^2 < b \ .$$

Set $\gamma_p^* = \gamma - \gamma_p$, $\varphi_p^* = \varphi - \varphi_p$, $\psi_p^* = \psi - \psi_p$. We have (see (10))

(13) $$|\gamma| \vee |\gamma_p^*| < \alpha_0 + b \ ,$$

(14) $$|\varphi_0^*| \vee |\psi_0^*| < b \cdot 2^{1/2}$$

and, by (8),

(15) $$|\varphi| \vee |\varphi_p^*| \vee |\psi| \vee |\psi_p^*| < (\beta_0 + b) \cdot 2^{1/2} < \varepsilon^{1/2} \ .$$

It is easy to show that

(16) $$\rho = \varphi_p \psi_p^* + \psi_p \varphi_p^* + \sigma_p \ .$$

Taking $p = 0$ we get, according to (10), (14) and (12), $|\rho| < 2 \cdot 2\ \beta_0 b + b$ so that, by (13) and (7), $|\lambda| < \alpha_0 + b + 4\beta_0 b + b < \eta$. This proves (5).

Now let $x \in J$. If f is continuous at x, then all the functions h_n, γ_n, $\cdots$, λ, φ, ψ are continuous at x. Otherwise we choose a p such that $x \in A_p$. Then the functions γ_p^*, φ_p^*, ψ_p^*, σ_p are continuous at x. Since $f - h_p$ is continuous at x, we have $\omega_{h_p}(x) = \omega_f(x)$. Now it follows from (11) that $\omega_\gamma(x) = \omega_{\gamma_p}(x) \leqq \omega_f(x) \leqq (\varepsilon\omega_f(x))^{1/2}$, $\omega_\varphi(x) = \omega_{\varphi_p}(x) \leqq 2(\omega_f(x))^{1/2}$, $\omega_\psi(x) = \omega_{\psi_p}(x) \leqq 2(\omega_f(x))^{1/2}$. By (16), Lemma 1 and (15) we get

$\omega_\rho(x) \leqq \omega_{\varphi_p}(x)\cdot|\psi_p^*(x)| + \omega_{\psi_p}(x)\cdot|\varphi_p^*(x)| \leqq 4(\omega_f(x))^{1/2}\cdot\varepsilon^{1/2}$.
Therefore $\omega_\lambda(x) \leqq 5(\varepsilon\omega_f(x))^{1/2}$. This proves (6) which implies that $\lambda, \varphi, \psi \in \mathcal{A}$.

<u>Remark</u>. It is natural to ask whether each function $f \in \mathcal{A}$ can be expressed as $\gamma + \varphi\psi$, where $\gamma, \varphi, \psi \in \mathcal{B} = \mathcal{B}(J)$ and where at least one of these three functions is continuous. The answer to this question is contained in Proposition 6. We need four lemmas.

<u>Lemma 10</u>. Let $a,b,c \in R$, $0 \leq a < b \leq 1$, $c > 0$. Let $\varphi, \psi \in \mathcal{B}$ and let $\varphi\psi \geqq c$ on (a,b). Then $\varphi(a)\psi(a) \geqq c$.

<u>Proof</u>. For each $x \in (a,b)$ we have $(x-a)^2\cdot c \leqq (\int_L (\varphi\psi)^{1/2})^2 \leqq \int_L |\varphi| \cdot \int_L |\psi| = \int_L \varphi \cdot \int_L \psi \quad (L = (a,x))$.

<u>Lemma 11</u>. Let $a \in (0,1)$. Let γ be a function on J which is continuous at a and let $\varphi, \psi \in \mathcal{B}$. Define $f = \gamma + \varphi\psi$,

(17) $\Lambda_+ = \limsup f(x)$, $\lambda_+ = \liminf f(x) \quad (x \searrow a)$,

(18) $\Lambda_- = \limsup f(x)$, $\lambda_- = \liminf f(x) \quad (x \nearrow a)$,

(19) $M = \max(\lambda_+, \lambda_-)$, $m = \min(\Lambda_+, \Lambda_-)$.

1° If $\gamma(a) < M$, then $f(a) \geqq M$.

2° If $\gamma(a) > m$, then $f(a) \leqq m$.

Proof. Suppose that $\gamma(a) < M$. Let, e.g., $M = \lambda_+$. Choose numbers p, q such that $\gamma(a) < p < q < M$. There is a $b \in (a,1)$ such that $\gamma < p$, $q < f$ on (a,b). Then $\varphi\psi = f - \gamma > q - p$ on (a,b). By Lemma 10 we have $(\varphi\psi)(a) \geqq q - p$ whence $(\varphi\psi)(a) \geqq M - \gamma(a)$, $f(a) \geqq M$. This proves 1° which easily implies 2°.

Lemma 12. Let $a \in (0,1)$ and let f be a function on J. Define m, M by (17)-(19). If $m < f(a) < M$, then f cannot be expressed as $\gamma + \varphi\psi$, where γ is a function continuous at a and $\varphi, \psi \in \mathcal{B}$.

(This follows at once from Lemma 11.)

Lemma 13. Let $\psi \in \mathcal{B}$ and let φ be a function continuous on J. Then there is an interval $L \subset J$ such that the restriction of the function $\varphi\psi$ to L is in $\mathcal{B}(L)$.

(This follows from [1], Theorem 4.)

Proposition 6. Let f be an increasing function on J. Suppose that the set $\{x \in (0,1); f(x-) < f(x+)\}$ is dense in J and that $f(x-) < f(x) < f(x+)$ for some $x \in (0,1)$. Let $f = \gamma + \varphi\psi$, where $\gamma, \varphi, \psi \in \mathcal{B}$. Then none of the functions γ, φ, ψ is continuous on J.

(This follows from Lemmas 12 and 13.)

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