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SECTIONWISE PROPERTIES AND AVERAGING PROCESSES

The material presented here is based on two papers of ours jointly written with M. Laczkovich /[11], [12]/.

1. Let $\mathfrak{F}$ and $\mathcal{G}$ be function classes on $[0,1]$. We denote by $\mathfrak{F}\times\mathcal{G}$ the class of functions f defined on $Q = [0,1]\times[0,1]$ with the property

$$f_x \in \mathcal{G} \quad \text{and} \quad f^y \in \mathfrak{F}$$

for every $x,y \in [0,1]$ that is all the horizontal sections $f^y(x) = f(x,y)$ belong to $\mathfrak{F}$ and all the vertical sections $f_x(y) = f(x,y)$ belong to $\mathcal{G}$.

We deal with the measurabiliy properties of $\mathfrak{F}\times\mathcal{G}$ where $\mathfrak{F}$ and $\mathcal{G}$ run through the following classes defined on $[0,1]$:

$C = C[0,1] = \{f\ ;\ f \text{ is continuous}\}$,

$\mathcal{A} = \{f\ ;\ f \text{ is approximately continuous}\}$,

$b_1\Delta = \{f\ ;\ |f| \le 1 \text{ and } f \text{ is a derivative}\}$,

$\Delta = \{f\ ;\ f \text{ is a derivative}\}$,

$\mathcal{D}\mathcal{B}_1 = \{f\ ;\ f \text{ is Darboux Baire 1}\}$,

$\mathcal{B}_\alpha$: the α'th class of Baire, $\alpha = 1,2,\ldots$.

We also use the notation $\mathcal{B}_\alpha$ for the Baire classes of functions defined on Q . $\mathcal{M}$ denotes the class of Lebesgue measurable functions on Q .

The following chart makes easy to look through the measurability results.

	C	$\mathcal{A}$	$b_1\Delta$	Δ	$\mathcal{DB}_1$	$\mathcal{B}_1$	$\mathcal{B}_2$
C	$\mathcal{B}_1$	$\mathcal{B}_1$	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_2$	$\mathcal{B}_2$	$\mathcal{B}_3$
$\mathcal{A}$		$\mathcal{B}_2$	$\mathcal{B}_2$	$\mathcal{B}_2$	$\mathcal{B}_2$	$\mathcal{B}_2$	$*$
$b_1\Delta$			$\mathcal{B}_2$	$\mathcal{B}_2$	$\mathcal{B}_2$	$\mathcal{B}_2$	$*$
Δ				$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$*$
$\mathcal{DB}_1$					$*$	$*$	$*$
$\mathcal{B}_1$						$*$	$*$
$\mathcal{B}_2$							$*$

$*$ indicates that the corresponding classes $\mathcal{F}\times\mathcal{G}$ contain non-measurable functions.

The reader should realize that our classes are listed in increasing order apart from the independent classes $\mathcal{A}$ and $b_1\Delta$. It is obvious that a measurability property of $\mathcal{F}\times\mathcal{G}$ also holds for $\mathcal{F}'\times\mathcal{G}'$ if only $\mathcal{F}'\subset\mathcal{F}$, $\mathcal{G}'\subset\mathcal{G}$.

<u>Comments on the chart</u>

(i) $C\times C\subset\mathcal{B}_1$ and, in general, $C\times\mathcal{B}_\alpha\subset\mathcal{B}_{\alpha+1}$ is a piece of classic due to Lebesgue, [7], §27, V. Théorème 2, p. 285.

(ii) $C \times b_1\Delta \subset \mathcal{B}_1$ might possibly be known. We could not find a reference and hence a simple proof is provided as follows. Actually I prove here $C \times b\Delta \subset \mathcal{B}_1$, slighty stronger result then in [11] /$b\Delta$ denotes the class of bounded derivatives/. Let $f \in C \times b\Delta$ and put

$$F_n(x,y) = \int_0^y f_n(x,t)\,dt \qquad ((x,y) \in Q),$$

where $f_n = \min[n, \max(f,-n)]$ $(n=1,2,\ldots)$.
It is immediate by Lebesgue's convergence theorem that the sections $(F_n)^y$ are continuous. On the other hand, the sections $(F_n)_x$ are uniformly Lipschitz 1 functions and these imply the continuity of F_n on Q. For any fixed x $f_n \equiv f$, if n is large enough and hence

$$f(x,y) = \lim_{n\to\infty} n\left(F_n(x,y+\tfrac{1}{n}) - F_n(x,y)\right)$$

is Baire 1 on Q.

(iii) If $f \in C \times \mathcal{A}$, then $\frac{2}{\pi}\,\mathrm{arctg}\, f \in C \times b_1\Delta \subset \mathcal{B}_1$ and hence $f \in \mathcal{B}_1$. /We used here the fact that bounded approximately continuous functions are derivatives [1] p. 21./

(iv) $C \times \Delta \subset C \times \mathcal{D}\mathcal{B}_1 \subset C \times \mathcal{B}_1 \subset \mathcal{B}_2$ and $C \times \mathcal{B}_2 \subset \mathcal{B}_3$ follows by (i) and hence the assertions in the first row are verified.

(v) $\mathcal{A} \times \mathcal{A} \subset \mathcal{B}_2$ was proved by R.O. Davies [2]. $\mathcal{A} \times \mathcal{B}_1 \subset \mathcal{B}_2$ is proved in [9] and this implies everything in the second row apart from *.

(vi) If $2^{\aleph_0} = \aleph_1$ then there exists a non-measurable function in $\mathcal{A} \times \mathcal{B}_2$. A construction can be found in [3]

Theorem 11 and [4], Théorème 3. The authors of these papers claim only the Lebesgue measurability of the sections f_x, but their constructions actually give Baire 2 functions. We do not know whether the continuum hypothesis is necessary for $\mathcal{A}\times\mathcal{B}_2 \not\subset \mathcal{M}$.

(vii) $b_1\Delta \times b_1\Delta \subset \mathcal{B}_2$ due to Z. Grande /[6], Théorème 3/. The stronger assertion $b_1\Delta \times \mathcal{B}_1 \subset \mathcal{B}_2$ is contained in [9].

(viii) All the stars in the last column follow from (vi). In fact, if $f \in (\mathcal{A}\times\mathcal{B}_2) - \mathcal{M}$ then $\frac{2}{\pi}$ arctg $f \in$ $\in (b_1\Delta \times \mathcal{B}_2) - \mathcal{M} \subset \ldots \subset (\mathcal{B}_2 \times \mathcal{B}_2) - \mathcal{M}$. All these relations relies upon $2^{\aleph_0} = \aleph_1$.

(ix) $\Delta \times \mathcal{B}_1 \subset \mathcal{M}$ was proved by M. Laczkovich in [8]. We do not know whether or not stronger measurability properties /e.g. $\Delta\times\Delta \subset \mathcal{B}_4$/ hold in the fourth row.

(x) $\mathcal{D}\mathcal{B}_1 \times \mathcal{D}\mathcal{B}_1 \not\subset \mathcal{M}$ is a theorem of J.S. Lipiński [10]. This implies all the remaining stars in the chart. We remark that the first result in this topic is a theorem of Sierpiński stating $\mathcal{B}_1 \times \mathcal{B}_1 \not\subset \mathcal{M}$ /[14], p. 147./. It is remarkable that Lipiński's counterexample is sectionwise approximately continuous with at most one exceptional point for each section. The sharp contrast between this fact and $\mathcal{A}\times\mathcal{A} \subset \mathcal{B}_2$ shows that the two dimensional measurability very delicately depends on those of the sections.

(xi) As we mentioned above, it is not known whether in the fourth row we have sharp results. On the other hand all the positive results in the first three rows are sharp. $\mathcal{A} \times \mathcal{A} \not\subset \mathcal{B}_1$ was proved by Davies /[2], Theorem 2/. He constructed a bounded function, thus he also proved $\mathcal{A} \times b_1\Delta \subset \mathcal{B}_1$ and $b_1\Delta \times b_1\Delta \not\subset \mathcal{B}_1$. Thus the second and the third row can not be improved.

(xii) $C \times \mathcal{D}\mathcal{B}_1 \not\subset \mathcal{B}_1$ was shown by Z. Grande /[5], Théorème 3/. The only gap remained to fill up is to show $C \times \Delta \not\subset \mathcal{B}_1$. The construction to prove this, is the bulk of [11].

Our result is stated in the following

Theorem. There exists a function $f(x,y)$ defined on the unit square Q such that the section f^y is continuous for every $y \in [0,1]$, the section f_x is a derivative for every $x \in [0,1]$ and f does not belong to the first class of Baire.

The details of the proof are of course omitted here.

We represent our function f as a sum $f = g+h$ where g and h satisfy the following properties.

$(1)_g$ g^y is continuous for every $y \in [0,1]$;

$(2)_g$ the function

$$g_x^*(y) = \begin{cases} g_x(y) & \text{if } \quad y \neq x \\ 0 & \text{if } \quad y = x \end{cases}$$

is a derivative for every $x \in [0,1]$;

$(3)_g$

$$g(x,x) = \begin{cases} 1 & \text{if } x = s_k \quad (k=1,2,\ldots) , \\ 0 & \text{otherwise} ; \end{cases}$$

where $\{s_k\}_{k=1}^{\infty}$ is a suitable sequence everywhere dense in $[0,1]$;

$(1)_h$ h^y is continuous for every $y \in [0,1]$;

$(2)_h$ If $x \in [0,1] - \{s_k\}_{k=1}^{\infty}$ then the section h_x is a derivative. Furthermore, the function

$$\begin{cases} h_{s_k}(y) & \text{if } y \neq s_k \\ 1 & \text{if } y = s_k \end{cases}$$

is a derivative for every $k=1,2,\ldots$;

$(3)_h$ $h(x,x) = 0$ for every $x \in [0,1]$. Having these properties above, the function $f \overset{\text{def}}{=} g+h$ obviously has the required continuous and derivative sections, respectively. Since

$$f(x,x) = \begin{cases} 1 & \text{if } x=s_k \quad (k=1,2,\ldots) , \\ 0 & \text{otherwise} \end{cases}$$

and $\{s_k\}_{k=1}^{\infty}$ is everywhere dense in $[0,1]$, f can not be a Baire 1 function.

2. Averaging processes. Let Φ be a class of real valued functions defined on an arbitrary set $X \neq \emptyset$. The simplest way to form averages of elements of Φ is taking weighted arithmetical means

$$\sum_{i=1}^{n} \alpha_i f_i \quad \left(\sum_{i=1}^{n} \alpha_i = 1 \ , \quad \alpha_i \geqq 0 \ , \quad f_i \in \Phi \right).$$

The family of all such finite sums is the convex hull of Φ

We can extend the averaging process by considering all sums of form

$$\sum_{i=1}^{\infty} \alpha_i f_i \quad \left(\sum_{i=1}^{\infty} \alpha_i = 1 \ , \quad \alpha_i \geqq 0 \ , \quad f_i \in \Phi \ , \quad \|f_i\| \leqq K \right).$$

/We denote by $\| f \|$ the sup norm of f./ If Φ is a vector lattice /with the usual pointwise operations/ then it is easily seen that the family of all such infinite sums is the uniform closure of Φ.

Instead of summation we may also consider integrals and this leads to a more general averaging process as follows.

Definition 1. Let $\mathbb{P} = (S, \mathfrak{A}, \mu)$ be a probability space and let $F : X \times S \to R$. This function F is said to be admissible /for an averaging process concerning the class Φ / if

(i) F is bounded,

(ii) for any fixed $s \in S$, $F(.,s) \in \Phi$,

(iii) for any fixed $x \in X$, $F(x,.)$ is measurable on S.

In this case

$$f(x) = \int_S F(x,s)\,d\,\mu(s) \qquad (x \in X)$$

is meaningful and the average function f is represented by $\mathbb{P}$ and F.

Definition 2. For a given Φ the set of all possible averages is denoted by Φ^a.

/We could further generalize the averaging method, if integration with respect to finitely additive measures were allowed. This kind of averaging, however, would yield to an over-abundant collection of averages as shown by Remark 1./

Our aim is to study the class Φ^a. Dealing with the averaging process described by Definition 1, we always consider classes Φ consisting of bounded functions only.

The relations $\Phi \subset \Phi^a$ and

$$\Phi \subset \Psi \implies \Phi^a \subset \Psi^a$$

are obvious. Natural questions arise like these:

1. For a given Φ how can we characterize Φ^a ?
2. How large can Φ^a be compared with Φ ?
3. What conditions imply $\Phi = \Phi^a$?
4. What conditions imply $\Phi^a = \Phi^{aa}$?

These of course are too general to be answered completely. Some of the details are answered in this paper and many open problems are raised in the last section.

Here are some instructive examples.

A/ Let X be a metrisable space and let Φ denote the family of bounded continuous real valued functions on X . Then $\Phi^a = \Phi$. This is an easy consequence of Lebesgue's convergence theorem.

B/ Let X be a compact metric space and let Φ denote the set of bounded Baire 1 functions on X , $\Phi = b\mathfrak{B}^1$. Then again $\Phi^a = \Phi$. This is rather difficult to prove /see [9]/.

C/ Let $X = [0,1]$ and let Φ be the class of bounded Baire 2 functions on $[0,1]$. Then Φ^a is the set of <u>all</u> bounded real functions on $[0,1]$. Assuming the continuum hypothesis there exists a set $S \subset [0,1] \times [0,1]$ such that all horizontal sections are countable and all vertical sections have countable complement [15]. Given an arbitrary function $0 \leqq f(x) \leqq 1$ we delete from S the segments $\{(x,y) : f(x) \leqq y \leqq 1\}$ and obtain a subset S' such that the horizontal sections are /at most/ countable a fortiori. Therefore its characteristic function $F(x,y) = \chi_{S'}(x,y)$ is Baire 2 for every fixed y, and

$$\int_0^1 F(x,y)dy = \int_0^{f(x)} F(x,y)dy = f(x)$$

since $F(x,y) = 1$ almost everywhere on the segment $\{(x,y);\ 0 \leqq y \leqq f(x)\}$. We prove this assertion on Φ^a without CH, but we need some machinery developed later.

<u>Averaging and uniform approximation.</u> Our main result in this section is Theorem 7 which states that an average $f \in \Phi^a$ can always be represented by a suitable admissible function F such that $\|F\| < \|f\| + \varepsilon$.

<u>Lemma 3.</u> Let Φ be a linear space and let $f \in \Phi^a$, that is

$$f(x) = \int_S F(x,s)\, d\mu(s) \qquad (x \in X).$$

Then there exists a probability space (T, ν) and for every $\varepsilon > 0$, $\delta > 0$ there exists an admissible function $F_{\varepsilon,\delta} : X \times T \to R$ such that

(i) $\| F_{\varepsilon,\delta} \| \leqq \| F \|$,

(ii) $f(x) = \int_T F_{\varepsilon,\delta}(x,t)\, d\nu(t) \qquad (x \in X)$,

(iii) $\nu\left(\left\{t \in T \;;\; F_{\varepsilon,\delta}(x,t) > \|f\|(1+\delta)\right\}\right) < \varepsilon$

for every $x \in X$.

Theorem 4. Let Φ be a vector lattice and let $f \in \Phi^a$ and $\eta > 0$ be given. Then there exists a $g \in \Phi^a$ such that

(i) $\|f-g\| < \eta$,

(ii) $g(x) = \int_T G(x,t)\, d\nu(t) \qquad (x \in X)$,

where G is admissible and $\|G\| \leqq \|f\|$.

Lemma 5. Let Φ be a class of functions containing the constant multiples cf for each $f \in \Phi$ and real c. Let the sequence $f_n \in \Phi^a$ be given such that

$$f_n(x) = \int_{S_n} F_n(x,s)\, d\mu_n(s) \qquad (x \in X,\ n=1,2,\dots),$$

where $\| F_n \| \leqq \alpha_n$ and $\alpha = \sum_{n=1}^{\infty} \alpha_n < \infty$. Then $f = \sum_{n=1}^{\infty} f_n \in \Phi^a$, moreover

$$f(x) = \int_S F(x,s)d\mu(s) \qquad (x \in X),$$

where $\| F \| \leqq \alpha$.

<u>Corollary 6</u>. If Φ contains the constant multiples of its elements, then Φ^a is a linear space.

<u>Theorem 7</u>. Let Φ be a vector lattice on X, $f \in \Phi^a$ and $\varepsilon > 0$. Then there exist a probability space (T, ν) and an admissible function $F : X \times T \to R$ such that

$$f(x) = \int_T F(x,t)d\nu(t) \qquad (x \in X)$$

and

$$\| F \| < \| f \| + \varepsilon .$$

<u>Theorem 8</u>. If Φ is a vector lattice, then Φ^a is a uniformly closed linear space.

It was observed by A. Bárdossy that our method actually proves that Φ^a is a uniformly closed vector lattice.

The universal representation.

Theorem 9. Let Φ be a given vector lattice on X. Then there exists a universal measurable space $(\Omega, \mathcal{A})$ and a universal function $F_o : X \times \Omega \rightarrow R$ such that $\|F_o\| \leqq 2$ and for every $f \in \Phi^a$, $\|f\| \leqq 1$ there exists a probability measure μ on $\mathcal{A}$ giving the representation

$$f(x) = \int_{\Omega} F_o(x,\omega)\, d\mu(\omega) \qquad (x \in X).$$

Averages and other operations. In this section we introduce some other extension operations defined on an arbitrary family of functions.

Definition 10. Let Φ be an arbitrary class of functions defined on X. We denote by Φ^u the class of limits of sequences from Φ uniformly convergent on X. Furthermore Φ^o consists of all functions f defined on X such that for every countable subset $H \subset X$ there exists $g \in \Phi$ with $f|_H = g|_H$.

Lemma 11.

(i) For any Φ, $\Phi \subset \Phi^u \cap \Phi^o$.

(ii) For any $\Phi \subset \Psi$, $\Phi^u \subset \Psi^u$ and $\Phi^o \subset \Psi^o$.

(iii) For any Φ, $\Phi^{uu} = \Phi^u$ and $\Phi^{oo} = \Phi^o$.

(iv) If Φ consists of bounded functions, then the same holds for Φ^u and Φ^o.

(v) If Φ is a linear space, or a lattice, or a vector lattice, then the same property holds for both Φ^u and Φ^o.

The easy proofs are left to the reader.

Lemma 12. If Φ is a vector lattice, then $\Phi^{uou} = \Phi^{uo}$.

Corollary 13. If Φ is a vector lattice, then

(i) $\Phi^{ou} \subset \Phi^{uo}$,

(ii) $\Phi^{uouo} = \Phi^{uo}$, that is operation "uo" is a closure operation for vector lattices.

Lemma 14. If $Y \neq \emptyset$ is an arbitrary set and $\mathcal{H}$ is an arbitrary system of subsets of Y such that $Y - \bigcup_{n=1}^{\infty} H_n \neq \emptyset$ for every sequence $H_1, H_2, \ldots$ from $\mathcal{H}$, then there exists a σ-field $\mathcal{A} \supset \mathcal{H}$ and a 0-1 measure μ on $\mathcal{A}$ with $\mu(Y) = 1$ and $\mu(H) = 0$ $(H \in \mathcal{H})$.

Theorem 15. For an arbitrary class Φ of bounded functions we always have $\Phi^a \supset \Phi^o$.

Theorem 16. If Φ is a vector lattice of bounded functions, then $\Phi^a \supset \Phi^{ou}$.

Remarks and problems.

1. As it was mentioned in the introduction, averaging with finitely additive measures results in too many averages. Namely, we prove that if Φ is a vector lattice separating the points in X then every bounded function is an average with respect to a finitely additive measure.

Theorem 17. Let Φ be a vector lattice of functions defined on X and suppose that Φ separates the points in X . Let φ be an arbitrary bounded function on X . Then there exists a set $S \neq \emptyset$, a finitely additive 0-1 measure μ defined on all subset of S and a bounded function $F : X \times S \to R$ such that for any $s \in S$ $F(\cdot,s) \in \Phi$ and for any $x \in X$,

$$\mu\left(\{s \; ; \; \varphi(x) = F(x,s)\}\right) = 1 .$$

2. Theorem 18. There exists a vector lattice Φ on a countable set X such that Φ^u is a strict subclass of Φ^a .

It should be noticed that since X is countable, operation "o" does not affect any class defined on X, hence our theorem also shows that $\Phi^{uo} = \Phi^{a}$ can not hold in general. In particular, the inclusion $\Phi^{ou} \subset \Phi^{a}$ in Theorem 16 can not be improved to equality.

3. Let $\Phi = b\mathfrak{A}$ denote the family of bounded approximately continuous functions on $[0,1]$. As an application we prove $\Phi^{a} = b\mathcal{B}^{1}$. Indeed, it follows immediately from Theorem 2.2 of [13] that any bounded Baire 1 function restricted to a countable set can be extended to get a function in Φ. That is $\Phi^{o} \supset b\mathcal{B}^{1}$, and by Theorem 15, $\Phi^{a} \supset b\mathcal{B}^{1}$ also holds. On the other hand, $\Phi \subset b\mathcal{B}^{1}$ implies $\Phi^{a} \subset (b\mathcal{B}^{1})^{a} = b\mathcal{B}^{1}$ /see [9], Theorem 2/.

We remark, that $\Phi^{a} = b\mathcal{B}^{1}$ holds also if Φ is the class of bounded derivatives, $\Phi = b\Delta$, or the class of bounded Darboux Baire 1 functions $\Phi = b\mathcal{D}\mathcal{B}_{1}$. This is obvious from $b\mathfrak{A} \subset b\Delta \subset b\mathcal{D}\mathcal{B}_{1} \subset b\mathcal{B}^{1}$.

4. <u>Problems</u>.

(i) What conditions on Φ imply $\Phi^{uo} = \Phi^{ou}$? We know no examples of vector lattices where this equality fails to hold.

(ii) Determine Φ^a if

$\Phi = \{$functions of bounded variation on $[0,1]\}$ or

$\Phi = \{$bounded functions which are continuous on $[0,1]$ with at most countably many exceptional points$\}$.

(iii) Is $\Phi^{aa} = \Phi^a$ true for any vector lattice Φ ? If not, then can the sequence $\Phi^a \subset \Phi^{aa} \subset \dots$ be strictly increasing?

(iv) Is the weaker equality $\Phi^{ao} = \Phi^a$, or $\Phi^{oa} = \Phi^a$, or $\Phi^{ua} = \Phi^a$ true for all vector lattices? /$\Phi^{au} = \Phi^a$ holds true, according to Theorem 8./

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