Real Analysis Exchange Vol. 7 (1981-82)

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Modification of Functions

It has been known for more than a century that there are continuous functions whose Fourier series do not converge everywhere [3]. This defect may be removed in several ways. The most important manner of obtaining a convergence result is the one of Fejer [6]. This consists in showing that the $(C,1)$ sums of the Fourier series of f converge uniformly to f if f is continuous and periodic of period 2π. The main point here is that the Fejer kernels, which are the averages of the Dirichlet kernels, turn out to be positive. The Dirichlet kernels themselves are oscillatory. It might further be mentioned that there are functions in L_1 whose Fourier series diverge everywhere, whereas for each $f \in L_1$, the $(C,1)$ sums of the Fourier series of f converge almost everywhere.

We are concerned with two other ways in which the above defect has been removed. The first is the so called Bohr-Pal theorem which says that, for each f, continuous and of period 2π, there is a homeomorphism g of $[-\pi,\pi]$ with itself such that the Fourier series of $f \circ g$ converges uniformly. The standard proof of this result is elegant and uses several facts from complex variables. An outline of the proof is given in [4]. Until very recently no real variable proof was known. The second method is due to Menchov [1] who showed that if f is continuous on $[-\pi,\pi]$, for each $\varepsilon > 0$ there is a g such that $f(x) = g(x)$, except on a set of measure less than ε, such that the Fourier series of g converges uniformly on $[-\pi,\pi]$. By Lusin's theorem, the same fact holds even if f is merely measurable. We make some remarks related to each of these modes of adjusting a function

In regard to the Bohr-Pal theorem, a study has not been made of other orthonormal systems of functions. As long as the only viable proof of the theorem involves complex methods there is no chance of generalizing the proof to cover other systems. A proof using only real variable methods is welcome but does not necessarily lend itself to generalization. A different sort of extension of the Bohr-Pal theorem also is suggested. Accordingly, it is plausible that if f is regulated there is a homeomorphism g such that $f \circ g$ has an everywhere convergent Fourier series. An inplausible, but interesting, conjecture is that if f is just measurable then there is a homeomorphism g such that $f \circ g$ has everywhere convergent Fourier series. The falseness of this conjecture follows from a fact which has little to do with Fourier series. Observe that if a Fourier series converges everywhere the limit is a Baire one function. Accordingly, in order for the conjecture to have a chance, for every measurable function f, there must be a homeomorphism g, such that $f \circ g$ is equivalent to a Baire 1 function. This is true for functions which assume only finitely many values but is false for some functions which assume infinitely many values [5]. On examination, these functions are not fully satisfactory. Although measurable, it turns out that, for each function of this sort, there is a homeomorphism g such that $f \circ g$ is not measurable. We say that f

is absolutely measurable if $f \circ g$ is measurable for every homeomorphism g. Then, for each absolutely measurable f there is a homeomorphism g such that $f \circ g$ is equivalent to a function of type Baire 1, [2].

It follows from this discussion that to show the conjecture false for absolutely measurable functions it would require some difficult analysis.

Related conjectures occur which may be easier to handle. Since the $(C,1)$ sums of the Fourier series converge almost everywhere for every $f \in L_1$, it may be more plausible to conjecture that, for every absolutely measurable f, there is a homeomorphism g such that the $(C,1)$ sums of $f \circ g$ converge everywhere. This suggests a more general problem. Let k_h, $h > 0$, be mollifiers. For every absolutely measurable f is there a homeomorphism g such that $\lim_{h \to 0} [(f \circ g) * k_h](x) = (f \circ g)(x)$ for every x? Perhaps the simplest example is that for which $k_h(x) = \frac{1}{2h}$, $-h \leq x \leq h$, and $k_h(x) = 0$ elsewhere? Restricted to characteristic functions of measurable sets this gives the following question: For every measurable set S, is there a homeomorphism g such that the symmetric density of $g(S)$ exists at every point? A related question concerns the right, or left, density.

We now say a word about the Menchov modification. The result holds for Walsh functions but little more is known. We conjecture that there is a complete orthonormal system for which the Menchov theorem fails, but that if the system is the sequence of characters of a group then the theorem holds.

References

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