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# A NEW NOTION OF DERIVATIVE[1)]

Introduction. As we know since Weierstrass, there are continuous functions which are not derivable at any point. The same is true for the various known generalizations of derivative, e.g. the unilateral, approximate and symmetric derivatives.

Let us denote by $C$ the Banach space of continuous real-valued functions on the interval $[0,1]$ with the uniform norm. When any function-theoretic property $P$ holds for a residual set of functions in $C$, this situation is frequently described by saying that a typical continuous function has property $P$. It has been proved by Evans that a typical continuous function is neither approximately derivable nor symmetrically derivable at any point.

We discuss here a new notion of derivative in terms of which every continuous function $f$ is derivable at uncountably many points in each interval, and the properties of $f$ can in turn be investigated in terms of the values of its new derivative wherever it exists. The results on the new derivative are found to unify numerous known results in the theory of differentiation, and they yield some interesting new results on Zygmund functions.

In the same connection two new theorems on unilateral derivates are presented which together unify most of the known results on such derivates. Many of the results are indeed strengthened in this process.

## 1. Definition.

We present here three equivalent definitions of the new derivative. Let us begin with a geometrical definition.

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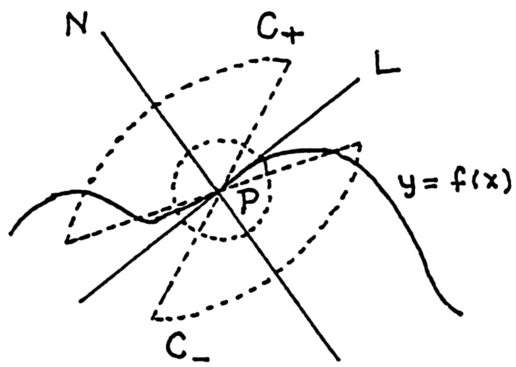
[1)] This is a report of the two lectures presented at the Real Analysis Symposium held at Syracuse last summer. It deals with some of the main features of a forthcoming work under a similar title. A few of these results were announced earlier in Bull. Amer. Math. Soc. 82 (1976), 768-770.

For this, we redefine the ordinary derivative first in an appropriate manner. Given $f:R \to R$ and $x, \alpha \in R$, let $P = (x,f(x))$, $L$ be the straight line through $P$ with slope $\alpha$, and $N$ be the normal to $L$ through $P$. Then $\alpha$ is the derivative of $f$ at $x$ iff for each double cone $C$ with vertex $P$ and axis $N$ there is a neighborhood of $P$ in which no point of the graph of $f$ is in the interior of $C$. On replacing the double cone $C$ by its lower and upper parts $C_-$ and $C_+$ we obtain the notions of lower and upper gradients respectively as follows:




Definition 1. $\alpha$ is a lower gradient of $f$ at $x$ if for each lower cone $C_-$ with vertex $P$ and axis $N$ there is a neighborhood of $P$ in which no point of the graph of $f$ is in the interior of $C_-$.

What is unusual about lower gradient is that it is not always unique like the derivative. For example, for the norm function $f(x) = |x|$ each element of the interval $[-1,1]$ is a lower gradient of $f$ at 0.

We call $f$ lower derivable at a point $x$ if it has at least one lower gradient at $x$, and then the set of all lower gradients of $f$ at $x$ will be called the lower derivative of $f$ at $x$, and denoted by $Lf'(x)$. It is clearly a closed convex set in $R$. When it is singleton, $f$ will be said to have a unique lower derivative at $x$, and then $Lf'(x)$ is also employed to denote its unique element.

The above definitions extend without difficulty to real-valued functions on any real TVS $X$. In case $X$ is a normed space, we have further the following analytical definition:

Definition 2. A linear functional $x' \in X'$ is a lower gradient of $f$ at a point $x_0 \in X$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $f(x) > f(x_0) + (x', x-x_0) - \varepsilon\|x-x_0\|$ whenever $0 < \|x-x_0\| < \delta$.

It is now easy to see that lower derivability is considerably more general than the notion of subdifferentiability used in Convex Analysis. In the case of convex functions, however, these two notions of derivative become identical.

Applying the above definitions to the function $-f$, we obtain the definitions of upper gradient and upper derivative. It can be verified without difficulty that $f$ is derivable at a point iff it is lower and upper derivable there.

We assume now on that $f: X \to R$, where $X \subset R$. In this case the lower and upper derivatives of $f$ can also be defined in terms of its unilateral derivates, and this definition extends in a natural manner to extended real numbers as well.

Definition 3. An $\alpha \in \overline{R} \equiv [-\infty, +\infty]$ is a lower gradient of $f$ at $x \in X$ if $D^-f(x) \leq \alpha \leq D_+f(x)$. The function $f$ is thus lower derivable at $x$ iff $D^-f(x) \leq D_+f(x)$, and then $Lf'(x) = [D^-f(x), D_+f(x)]$. We call $f$, further, lower differentiable at $x$ if it has a finite unique lower derivative at $x$. Similar statements hold for the upper derivative.

As the lower and upper derivatives are set-valued, and they are not defined in terms of a limit, the nature of results on these derivatives is quite different from the usual results. They do include, however, most of the results on the ordinary derivative.

## 2. Two unified theorems on unilateral derivates.

Let $E \subset X \subset R$, $f: X \to R$ and $c \in \overline{R}$.

2.1. THEOREM. Each of the sets $\{x \in E: D_+f(x) \leq c\}$ and $\{x \in E: D^-f(x) \geq c\}$ is a set of the form $G_\delta \cup (G_\delta \sim S)$ relative to $E$, where $S$ is the set of points in $E$ where $f$ is LSC (lower semicontinuous) relative to any given set $A \supset E$.

Given a countable ordinal number $\alpha$, let us use $B_\alpha$ to denote the Baire class $\alpha$. We use further $\underline{B}_\alpha$ and $\overline{B}_\alpha$ to denote the <u>lower</u> and <u>upper Baire classes</u> $\alpha$ which are defined as follows: A function $g: E \to \overline{R}$ is in $\underline{\mathcal{B}}_\alpha$ or $\overline{\mathcal{B}}_\alpha$ if for each $c \in R$ the set $\{x \in E: g(x) > c\}$ or $\{x \in E: g(x) < c\}$ respectively is of additive class $\alpha$ in $E$. Clearly $\mathcal{B}_\alpha = \underline{\mathcal{B}}_\alpha \cap \overline{\mathcal{B}}_\alpha$, and it is further easy to see that $\underline{\mathcal{B}}_\alpha \cup \overline{\mathcal{B}}_\alpha \subset \mathcal{B}_{\alpha+1}$.

Solving an old problem of Sierpiński, the above theorem yields the following best possible result on the Baire class (and measurability) of unilateral derivates:

2.2. <u>Corollary</u>. (a) If $f \in \underline{\mathcal{B}}_0$ on $E$, then $D_+f \in \underline{\mathcal{B}}_1$ and $D^-f \in \overline{\mathcal{B}}_1$ on $E$.

(b) If $f \in \underline{\mathcal{B}}_1$ on $E$, then $D_+f$, $D^-f \in \mathcal{B}_2$ on $E$.

(c) If $\alpha > 1$ and $f \in \underline{\mathcal{B}}_\alpha$ on $E$, then $D_+f \in \overline{\mathcal{B}}_\alpha$ and $D^-f \in \underline{\mathcal{B}}_\alpha$ on $E$.

(d) If $f$ is (Lebesgue) measurable on $E$, then so are its derivates on $E$.

We have consequently the following result which strengthens the existing results of Sierpiński, Banach and Misik:

2.3. <u>Corollary</u>. If $f \in \mathcal{B}_\alpha$ on $E$, where $0 < \alpha < \Omega$, then each unilateral derivate of $f$ is in $\mathcal{B}_{\alpha+1}$ on $E$.

The above theorem unifies further some other theorems of W.H. Young, Denjoy, Zahorski and Bruckner and Goffman. The next theorem unifies similarly many of the other important theorems on unilateral derivates. It is similar to a theorem obtained earlier in [4, p. 301] on bilateral derivates.

Generalizing the notion of LSC, let $f$ be called <u>lower quasicontinuous</u> (or LQC) if for each point $x \in X$, and for each pair of real numbers $\varepsilon, \delta > 0$, there is a portion $X_0$ of $X \cap (x-\delta, x+\delta)$ such that $f(t) > f(x) - \varepsilon$ for $t \in X_0$. Let $f$ be called, further, <u>right increasing</u> on a set $E \subset X$ if $f(t) > f(x)$ whenever $x, y \in E$, $t \in X$ and $x \le t \le y$. We call $f$ <u>strongly right increasing</u> on $E$ if there is a $c > 0$ such that the function $f(x) - cx$, $x \in X$, is right increasing on $E$.

2.4. THEOREM. Suppose $f$ is LQC relative to a set $E \subset X$. If $D_+f > 0$ at a nonmeager set of points in $E$, then $f$ is strongly right increasing on some portion of $E$.

We state three consequences of this theorem which are directly related to lower and upper derivatives. Some nomenclature is necessary here. If $x \in X$ is a bilateral limit point of $X$, we call $f$ nonangular at $x$ if $\vee D_{\pm}f(x) \leq \wedge D^{\pm}f(x)$. When $f$ is nonangular at each point of $X$ that is a bilateral limit point of $X$, $f$ will be called simply nonangular.

2.5. Corollary (G.C. Young). Every function $f:X \to R$ is nonangular at all but a countable set of points.

Consequently, the set of points where $f$ has a non-unique lower or upper derivative is always countable.

Further, as proved by Járnik, a typical continuous function is nonangular. Consequently, the lower derivative of most of the continuous functions is unique at all of the points where it exists.

Let $x \in X$ be called (i) a lower knot point of $f$ if $D_+f(x) = -\infty$ and $D^-f(x) = +\infty$, (ii) an upper knot point of $f$ if $D_-f(x) = -\infty$ and $D^+f(x) = +\infty$, and (iii) a bilateral knot point of $f$ if $f$ has simultaneously a lower and an upper knot at $x$. For example, the continuous function $f(x) = \sqrt{|x|} \sin 1/x$, $f(0) = 0$, has a bilateral knot at $0$.

We call $\alpha \in \overline{R}$ a knotted lower derivative of $f$ at $x$ if $f$ has a unique lower derivative $\alpha$ and an upper knot at $x$. For example, the function $f(x) = \sqrt{|x|} \sin^2 1/x$, $f(0) = 0$, has a knotted lower derivative $0$ at $0$. Let $\alpha \in \overline{R}$ be called further a normal lower derivative of $f$ at $x$ if it is either an ordinary derivative or a knotted lower derivative of $f$ at $x$. The terms knotted and normal upper derivatives are defined analogously.

Finally, we call $f$ semiderivable at $x$ if it is lower or upper derivable at $x$, and then its lower or upper derivative at x is called the semiderivative of f at x, and denoted by $Sf'(x)$. The terms unique, knotted and normal semiderivatives are defined in a similar manner.

2.6. Corollary (Denjoy-Young-Saks). Every function $f:X \to R$ is lower differentiable at almost all of the points where it does not have a lower knot.

Equivalently, f has a finite normal semiderivative at almost all of the points where it does not have a bilateral knot.

Let any property P be said to hold for almost every level set of f if it holds for $f^{-1}(y)$ for almost every y in R. Following is an analogue of the above result, obtained in [3], which seems to have considerably more applications:

2.7. Corollary (Denjoy analogue)[2]. Almost every level set of f is differentially normal in the following sense: it consists of only points where either f has a nonzero normal semiderivative or it has a bilateral knot.

There are several other applications of Theorem 2.2. For example, it yields strengthened versions of some theorems of

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[2] Several applications of this theorem were given in some earlier papers. As a new application we obtain from it the following extension of a theorem of Saks [6, p. 280] and Bruckner [1]. Given a function f on a closed interval $I = [a,b]$, let f be called [4] lower internal if $f(x) \leq \overline{f}(x+0)$ for $a \leq x < b$ and $f(x) \geq \underline{f}(x-0)$ for $a < x \leq b$. Clearly, every Darboux function is lower and upper internal, where f is called upper internal if $-f$ is lower internal.

"If f is a lower internal function in $\mathcal{B}_1$ with property $(T_2)$, then $f(a) - f(b) \leq m^*(f(E))$, where E is the set of points in I where f has a negative derivative."

Using further an extension of Banach's first theorem in [6, p. 284] to measurable functions, we obtain from the above result the following extension of his third theorem [6, p. 286]:

"If f is a lower or upper internal function in $\mathcal{B}_1$ with property (N), then f is derivable at a metrically dense set of points in I."

The above result leads further to a new version of Bruckner's reduction theorem [1] which extends many of the monotonicity theorems to lower internal functions in $\mathcal{B}_1$.

Denjoy (see [2, pp. 187, 191] and [6, pp. 234-239]), three theorems of Saks [6, pp. 271-272] and generalizations of some other theorems on derivates due to Morse, Bruckner and Goffman, Young and Neugebauer.

## 3. The existence of points of lower derivability.

When f has a finite lower gradient at some point $x \in X$, it is easy to see that f is LSC at x. The following result in the opposite direction is more significant.

3.1. THEOREM. Every LSC function f on an interval I has a finite lower gradient at a dense set of points in I, and this set is c-dense in I (viz. its cardinality is c in each subinterval of I) when f is further nonangular.

A typical continuous function has thus a 'finite' unique lower derivative at a c-dense set of points. This is not true for every continuous function as it is illustrated by the Weierstrass function. However, on dropping the word 'finite', the result becomes valid for every continuous function:

3.2. THEOREM. If f is a continuous function on an interval I, then f has a normal lower derivative at a dense set of points in almost all of its level sets, and at a c-dense set of points in I.

The proof of Theorem 3.1 is rather simple, but Theorem 3.2 is more difficult and it is deduced from the Denjoy analogue 2.7. Using a theorem of Saks [6, p. 271], which can also be deduced from Theorem 2.4, we obtain from Theorem 3.2 the following result on the existence of finite lower derivative which is similar to Banach's third theorem (see footnote 2):

3.3. Corollary. If a continuous function f, on I, maps the points where it has an infinite lower derivative into a set of measure zero, then f has a finite normal lower derivative at a metrically dense set of points in I.

The hypothesis of this result holds for every continuous function which has nowhere an infinite lower derivative, e.g. the Besicovitch function. Although there are continuous functions that are lower derivable only at a set of points of measure zero, e.g. the Weierstrass function, there are also continuous functions that are lower differentiable a.e. (almost everywhere) but derivable only at a set of points of measure zero. We define later in §10 a wide class of functions that are lower differentiable a.e.

Let us consider, next, two applications of the above two theorems. A function $f$, on $R$, is called a <u>smooth</u> [<u>Zygmund</u>] <u>function</u> if, for each $x \in R$,

$$f(x+h) + f(x-h) - 2f(x) = o(h)[O(h)] \quad \text{as} \quad h \to 0.$$

Since a smooth function has symmetrical derivates everywhere, we obtain immediately from Theorem 3.1,

3.4. <u>Corollary</u>. Every LSC smooth function $f$ has a finite derivative at a c-dense set of points.

This result is well-known for continuous smooth functions, and it can be extended to measurable or bounded smooth functions. It is easy to see, on the other hand, that a Zygmund function does not have a knotted semiderivative at any point. Hence from Theorem 3.2 we obtain the following new result which is somewhat surprising:

3.5. <u>Corollary</u>. Every continuous Zygmund function $f$ is derivable at a dense set of points in almost all of its level sets, and at a c-dense set of points in $R$.

There are known examples of continuous Zygmund functions which do not have a finite derivative at any point. But if a continuous Zygmund (or smooth) function $f$ maps the points where it has an infinite derivative into a set of measure zero, it follows from Corollary 3.3 that $f$ has a finite derivative at a metrically dense set of points.

4. <u>The calculus of new derivatives</u>.

Unifying the notion of semiderivative with unilateral derivatives, let us call a function $f$, on $X$, <u>weakly derivable</u> at a point $x \in X$ if $\wedge D^{\pm} f(x) \leq \vee D_{\pm} f(x)$, and then the interval $[\wedge D^{\pm} f(x), \vee D_{\pm} f(x)]$ will be called the <u>weak derivative</u> of $f$ at $x$ which is denoted by $Wf'(x)$. The elements of $Wf'(x)$ are called as before the <u>weak gradients</u> of $f$ at $x$.

It is easy to see that $f$ is weakly derivable at $x$ iff it is either semiderivable or unilaterally derivable at $x$. When $f$ is semiderivable at $x$, we have $Sf'(x) = Wf'(x)$, and when $f$ is unilaterally derivable at $x$, the (or each) unilateral derivative of $f$ at $x$ is a weak gradient of $f$ at $x$.

Let us assume that $f$ and $g$ are two real-valued functions defined on a common set $X \subset R$, and that $x$ is a point of $X$ that is a bilateral limit point of the set. When $f$ and $g$ are both semiderivable at $x$, let their semiderivatives at $x$ be called (i) <u>similar</u> if they are simultaneously lower or upper derivatives, (ii) <u>dissimilar</u> if one of them is a lower derivative and the other one is an upper derivative. When $f$ and $g$ are weakly derivable at $x$, the weak derivative of $f$ will be said to be <u>totally similar</u> [<u>dissimilar</u>] to that of $g$ if the following conditions hold: (i) if $g$ is left or right derivable at $x$, then so is $f$, (ii) if $g$ is lower or upper derivable at $x$, then $f$ is lower [upper] or upper [lower] derivable respectively at $x$, and (iii) if $g$ has a unique or normal semiderivative at $x$, then so has $f$.

4.1. THEOREM. If $f$ has a finite derivative at $x$, then $f + g$ is weakly derivable at $x$ iff $g$ is so, and when it is so, the two weak derivatives are totally similar (to each other) and $W(f+g)'(x) = f'(x) + Wg'(x)$.

4.2. THEOREM. Suppose $f$ and $g$ are both lower derivable at $x$.

(a) If $Lf'(x) + Lg'(x) \neq \phi$, then $f + g$ is lower derivable at $x$ and

$$Lf'(x) + Lg'(x) \subset L(f+g)'(x). \tag{1}$$

(b) If from each side one of the functions $f$ and $g$ has a finite unilateral derivative at $x$, then $f + g$ is lower derivable at $x$, identity holds in place of the inclusion (1) and, further, if the lower derivatives of $f$ and $g$ are simultaneously unique or normal at $x$, so is the lower derivative of $f + g$.

The reverse inclusion of (1) does not hold in general. However, when the lower derivative of $f + g$ is unique at $x$, the present inclusion does become an identity. Consequently, (1) can fail to be an identity only at a countable set of points. From the part (b) we obtain

4.3. Corollary. Suppose $f$ is convex and $X = R$. If $g$ is lower derivable at $x$, then so is $f + g$ and in place of (1) the identity holds.

When $f$ and $g$ have dissimilar semiderivatives at $x$, the function $f + g$ is not always semiderivable at $x$. However, if $f + g$ is also known to be semiderivable at $x$, the semiderivatives of the three functions at $x$ are related as follows. This relation also has some important applications.

4.4. THEOREM. Suppose $f$, $g$ and $f + g$ are all weakly derivable at $x$, the weak derivatives of $f$ and $g$ are unique and that their sum is defined. Then

$$Wf'(x) + Wg'(x) \in W(f+g)'(x). \tag{2}$$

In case $f$ and $g$ have either (i) dissimilar semiderivatives or (ii) unilateral derivatives from opposite sides at $x$, then (2) becomes an identity.

4.5. THEOREM. Let $\alpha \in R$. If $f$ is weakly derivable at $x$, then so is $\alpha f$, the weak derivative of $\alpha f$ is totally similar or dissimilar to that of $f$ according as $\alpha \geq 0$ or $\leq 0$ and $W(\alpha f)'(x) = \alpha Wf'(x)$.

Combining this result with the previous three theorems, the results on linear combinations of $f$ and $g$ are obtained. Similar results hold for the product $fg$. Let us state the one similar to Theorem 4.2.

4.6. THEOREM. Suppose $f$ is continuous at $x$, $f(x) \neq 0$ and that $f$ and $g$ have similar or dissimilar semiderivatives at $x$ according as $f(x)g(x) \geq 0$ or $\leq 0$.

(a) If $f(x)Sg'(x) + g(x)Sf'(x) \neq \phi$, then $fg$ has a semiderivative at $x$ similar or dissimilar to that of $g$ according as $f(x) \geq 0$ or $\leq 0$ and

$$f(x)Sg'(x) + g(x)Sf'(x) \subset S(fg)'(x). \tag{3}$$

(b) If from each side one of the functions $f$ and $g$ has a finite unilateral derivative at $x$, then $fg$ has a semiderivative at $x$ as in (a), identity holds in place of (3) and, further, if the semiderivatives of $f$ and $g$ are simultaneously unique or normal at $x$, so is the semiderivative of $fg$.

Moreover, the above holds also when $f(x) = 0$ provided $f$ has a finite derivative at $x$ and $g$ is continuous at $x$.

Next, we come to the <u>chain rules</u>. We now assume $f$ to be defined on some set containing $g(X)$, and set $h = f \circ g$ and $y = g(x)$.

4.7. THEOREM. Suppose $f$ has a finite nonzero derivative at $y$ and that $g$ is continuous at $x$. Then $h$ is weakly derivable at $x$ iff $g$ is so, and when it is so, the two weak derivatives are totally similar or dissimilar according as $f'(y) > 0$ or $< 0$ and $Wh'(x) = f'(y)Wg'(x)$.

4.8. THEOREM. Suppose $f$ is weakly derivable at $y$ and that $g$ has a finite derivative at $x$. Then $h$ is weakly derivable at $x$ and

$$g'(x)Wf'(y) \subset Wh'(x). \tag{4}$$

Further, when $f$ is semiderivable at $y$, $h$ has a semiderivative at $x$ similar to that of $f$.

In case $g$ is a continuous function and $X$ is connected, then (4) becomes an identity and, further, if $f$ is semiderivable at $y$, the semiderivative of $h$ is totally similar to that of $f$.

Moreover, if $g'(x) = 0$ and $f$ has finite derivates at $y$, then $h$ has a zero derivative at $x$.

4.9. THEOREM. Suppose $f$ is semiderivable at $y$, $g$ is continuous and semiderivable at $x$, $0 \notin Sf'(y) \cup Sg'(x)$ and that the semiderivatives of $f$ and $g$ are similar or dissimilar according as $Sf'(y) > 0$ or $< 0$. Then $h$ has a semiderivative at $x$ similar to that of $f$ at $y$ and

(5) $$Sf'(y)Sg'(x) \subset Sh'(x).$$

In case $g$ is continuous, $X$ is connected and either (i) $f$ or $g$ has finite unilateral derivatives from both sides at $y$ or $x$ respectively, or (ii) $f$ and $g$ have finite unilateral derivatives at $y$ and $x$ from the same or opposite sides according as $Sg'(x) < 0$ or $> 0$, then (5) becomes an identity and, further, if the semiderivatives of $f$ and $g$ are simultaneously unique or normal, so is the semiderivative of $h$.

Moreover, the above holds also when $0 \in Sg'(x)$ provided $Sf'(y) \subset R$.

4.10. THEOREM. Suppose $X$ is connected, $g$ is continuous, $f$ and $g$ have unique nonzero weak derivatives at $y$ and $x$ respectively and that $h$ is weakly derivable at $x$. Then

(6) $$Wf'(y)Wg'(x) \in Wh'(x).$$

In case $f$ and $g$ have either (i) similar or dissimilar semiderivatives at $y$ and $x$ according as $Wf'(y) < 0$ or $> 0$, or (ii) unilateral derivatives at $y$ and $x$ from the same or opposite sides according as $Wg'(x) < 0$ or $> 0$, then (6) becomes an identity.

5. Mean-value theorems.

Let $f$ be a function on $R$, $I = [a,b]$ and set

$$\rho = \{f(b) - f(a)\}/(b-a).$$

5.1. THEOREM. If $f$ is continuous relative to $I$, then there is a point $x \in I^o$ such that $f$ is semiderivable at $x$ and $\rho \in Sf'(x)$.

Moreover, if $f$ is further nonangular on $I^o$, then there is a point $x \in I^o$ such that $f$ is lower differentiable at $x$ and $\rho = Lf'(x)$.

As we do not assume here the semiderivability of $f$ at any point of $I$, the above two mean-value theorems are of a much more general nature than the known mean-value theorems on other generalized derivatives. This has some far reaching consequences as it will be seen subsequently.

5.2. THEOREM. If $f$ is lower differentiable on $I$, then there is a point $x \in I^o$ where $\rho = LF'(x)$.

5.3. THEOREM (Taylor's formula). Suppose $f$ is continuous and $n$ ($> 0$) times weakly differentiable on $I$ and that its first $n$ weak derivatives are continuous on $I$. Then $f$ has an $(n+1)^{th}$ semigradient $\alpha$ at some point in $I^o$, where

$$f(b) = f(a) + (b-a)Wf'(a) + \dots + \frac{(b-a)^n}{n!} Wf^{(n)}(a) + \frac{(b-a)^{n+1}}{(n+1)!} \alpha.$$

Moreover, when $f^{(n)}$ is further nonangular on $I^o$, $\alpha$ is the $(n+1)^{th}$ lower derivative of $f$ at some point in $I^o$.

6. Term-by-term differentiation.

6.1. THEOREM. Let $\{f_n\}$ be a sequence of functions on $X$ that converges pointwise to $f$. Let $x \in X$ be a bilateral limit point of $X$ and suppose each function $f_n$ is weakly derivable at $x$. If for each $\varepsilon > 0$ there is a positive integer $k$ such that to each integer $n > k$ there corresponds a $\delta > 0$ so that

$$\left| \frac{f_n(t)-f_n(x)}{t-x} - \frac{f(t)-f(x)}{t-x} \right| < \varepsilon \quad \text{for} \quad t \in X,\ 0 < |t-x| < \delta,$$

then f is weakly derivable at x and the sequence $\{Wf_n'(x)\}$ converges to $Wf'(x)$ relative to the Vietoris topology of $2^{\overline{R}}$.

Moreover, if all the functions $f_n$ are simultaneously lower or upper derivable, nonangular or lower or upper knotted at x, then so is f.

As an application of this theorem we obtain the following result with the help of Theorems 4.4 and 5.2:

6.2. THEOREM. Let $\{f_n\}$ be a sequence of continuous, lower differentiable functions on an open interval I which converges pointwise to f. If $x \in I$ is a point of uniform convergence of the sequence $\{Lf_n'\}$, then f is lower differentiable at x and $Lf'(x) = \lim Lf_n'(x)$.

Moreover, if all the functions $f_n$ are derivable or strongly derivable at x, then so is f.

7. Measurability of new derivability sets.

Given a function f on X, let $\underline{\Delta}(f)$ denote the set of points in X where f is lower derivable, $\underline{\Delta}_u(f)$ and $\underline{\Delta}_n(f)$ be the sets of points in $\underline{\Delta}(f)$ where the lower derivative of f is unique or normal respectively, and $\underline{\Delta}^*(f)$, $\underline{\Delta}_u^*(f)$ and $\underline{\Delta}_n^*(f)$ be the sets of points in the earlier three sets where each lower gradient or lower derivative of f is finite. All these six sets will be called the lower derivability sets of f. The six semiderivability sets of f, viz. $\Delta_s(f)$, $\Delta_{su}(f)$ etc., are defined analogously.

Unlike the ordinary derivability sets $\Delta(f)$ and $\Delta^*(f)$ which are always $F_{\sigma\delta}$-sets, the lower derivability sets are not even measurable in general. Using Theorem 2.1, we obtain

7.1. THEOREM. (a) Suppose $2 \leq \alpha < \Omega$. If $f \in \underline{B}_\alpha$, then all the lower derivability sets of f are of multiplicative class $\alpha$, and when $f \in B_\alpha$, the semiderivability sets of f are also of multiplicative class $\alpha$.

(b) If f is measurable, then all the lower and semiderivability sets of f are measurable.

Let a set be called a Zahorski set if it is a set of the form $G_\delta \cup N$ where $m(N) = 0$. Zahorski proved that any subset of R is the complement of any ordinary derivability set of a function iff it is a Zahorski $G_{\delta\sigma}$-set. Let $S_*(f)$ and $S^*(f)$ denote further the sets of points in X where f is LSC or USC respectively.

7.2. THEOREM. If the complement of the set $S^*(f)$ is a Zahorski set [$G_{\delta\sigma}$-set] in X, then so is the complement of each lower derivability set of f, and in case the complement of $S^*(f)$ is also a Zahorski set [$G_{\delta\sigma}$-set] in X, the same holds for the complement of each semiderivability set of f.

8. Properties of functions in terms of lower derivative.

The following theorem is also proved with the help of Denjoy analogue 2.7:

8.1. THEOREM. Suppose f is a function on $I = [a,b]$ such that $\underline{f}(x-0) \leq f(x) \leq \underline{f}(x+0)$ for $x \in I$. Then

$$f(a) - f(b) \leq m^*(f(E)), \text{ where } E = \{x \in \underline{\Delta}_n(f) : Lf'(x) < 0\}.$$

The following monotonicity theorem follows directly from this result.

8.2. Corollary. A function f, on an interval I, is nondecreasing iff the following conditions hold:

(i) $\underline{f}(x-0) \leq f(x) \leq \underline{f}(x+0)$ for each $x \in I$,

(ii) the set $\{x \in \underline{\Delta}_n(f) : Lf'(x) < 0\}$ is of measure zero and

(iii) the set $\{x \in \underline{\Delta}_n(f) : Lf'(x) = -\infty\}$ is countable.

When f is a Zygmund function, the above two results remain valid on replacing the set $\underline{\Delta}_n(f)$ by the ordinary derivability set $\Delta(f)$. These results are new even for smooth functions.

Results in the above form cannot hold for other generalizations of derivative, for there are continuous nowhere monotone functions that are nowhere derivable in those senses.

It is easy to deduce from Corollary 8.2 a general test for the points of local minima which does not involve any differentiability hypothesis. Also, it is enough to consider the points where the function has a zero lower gradient as critical points.

The following theorem provides an extension of the Goldowski-Tonelli theorem to lower, semi- and weak derivatives at the same time:

8.3 THEOREM. Let $f$ be a function on an interval $I$ such that $\underline{f}(x-0) \leq f(x) \leq \overline{f}(x+0)$ for $x \in I$ and $f \in B_1$. If $f$ is weakly derivable n.e. (nearly everywhere, or at all but a countable set of points), and the set of points where $f$ has a negative (ordinary) derivative is of measure zero, then $f$ is nondecreasing.

## 9. Properties of new derivatives.

In terms of the Vietoris topology on the space $2^{\overline{R}}$ of all closed subsets of $\overline{R}$, we have the following results:

9.1. THEOREM. The lower derivative of a function $f$, on $X$, is always in $B_2$ relative to $\underline{\Delta}(f)$, and it is in $B_1$ relative to $\underline{\Delta}^*_u(f)$.

9.2. THEOREM. If $f$ is measurable on $X$, or it is in $B_\alpha$ with $\alpha \geq 2$, then the weak and semiderivatives of $f$ have the same property and they have further selections with that property.

9.3. THEOREM. The weak and semiderivatives of every continuous function $f$, on $I$, possess the Darboux property in the following sense: for each connected set $C$ in $R$ the set $\cup\{Wf'(x): x \in C\cap\Delta_w(t)\}$ or $\cup\{Sf'(x): x \in C\cap\Delta_s(f)\}$ is connected in $\overline{R}$.

Moreover, if $f$ is nonangular, then its lower derivative also possesses the Darboux property.

9.4. THEOREM. If $f$ is lower differentiable on $I$, then its lower derivative possesses the Darboux property.

Given a set $X \subset R$, let a multifunction $\phi: X \to 2^{\overline{R}}$ be said to have the Denjoy property if for each open set $U$ in $\overline{R}$ the set $\{x \in X: \phi(x) \subset U\}$ is either empty or it has a positive measure.

9.5. THEOREM. Let $f$ be a LSC function on $I$ which has nowhere an ordinary discontinuity from any side. If $f$ is nonangular and lower derivable n.e., then the ordinary and lower derivatives of $f$ both possess the Denjoy property.

Consequently, every finite unique lower derivative possesses the Denjoy property.

9.6. THEOREM. If a continuous function $f$, on $I$, is nonangular and semiderivable n.e., then its ordinary and semi-derivatives both possess the Denjoy property.

## 10. $BV_-$ and $BVG_-$ functions.

We now define a wide class of functions that are lower differentiable a.e.

Given a function $f$ on $X$, let the lower oscillation of $f$ on an interval $I = [a,b]$ with endpoints in $X$ be defined as

$$O_-(f;I) \equiv f(a) \vee f(b) - \wedge_{x \in I \cap X} f(x).$$

Given a set $E \subset X$, let us define the lower strong variation $V_-(f;E)$ of $f$ on $E$ to be the supremum of all the numbers $\Sigma_n O_-(f;I_n)$, where $\{I_n\}$ is any sequence of nonoverlapping intervals whose endpoints are in $E$.

We say $f$ is of bounded lower strong variation on $E$, or simply $BV_-$ on $E$, if $V_-(f;E) < +\infty$. In case $E$ is the union of a sequence of sets $\{E_n\}$ on each of which $f$ is $BV_-$, $f$ will be said to be $BVG_-$ on $E$.

Applying the above definitions to the function $-f$ we obtain the notions of $BV^-$ and $BVG^-$ functions on $E$. It is easy to see that a function $f$ is $BV_*$ on $E$ (see [6, p. 228]) iff it is $BV_-$ and $BV^-$ on $E$, and $f$ is similarly $BVG_*$ on $E$ iff it is $BVG_-$ and $BVG^-$ on $E$.

10.1. THEOREM. If a function f, on X, is $BVG_-$ on a set $E \subset X$, then f has a finite normal lower derivative a.e. in E, and $m(f(N)) = 0$ where $N = E \sim \underline{\Delta}_n(f)$.

10.2. Corollary (Denjoy-Lusin-Saks). If a function f, on X, is $BVG_*$ on a set $E \subset X$, then f has a finite derivative a.e. in E, and $m(f(N)) = 0$ where N is the set of points in E where f is not derivable.

The class of $BVG_-$ functions is indeed much wider than that of $BVG_*$ functions, but it is contained in the class of BVG functions.

## 11. The fundamental theorem of calculus.

Using the derivate $D^-$ in place of $\overline{D}$ and $\underline{D}$ in the definition of Perron integral, we obtain a new integral which we denote by $P_-$. This integral has been announced earlier by Ionescu Tulcea [5] without proofs. It is indeed more general than the Denjoy-Khintchine integral $\mathcal{D}^k$, but not as general as the general Denjoy integral $\mathcal{D}$.

Employing the lower oscillation $O_-$ in place of the usual oscillation in the definitions of $AC_*$ and $ACG_*$ functions [6, p. 231], we obtain the definitions of $AC_-$ and $ACG_-$ functions. In terms of $ACG_-$ functions we obtain the following descriptive definition of the integral $P_-$:

11.1. THEOREM. A function f, on $I = [a,b]$, is $P_-$-integrable iff there is an $ACG_-$ function F on I such that $f = LF'$ a.e., and then

$$P_- \int_a^x f = F(x) - F(a), \quad x \in I.$$

## 12. Applications to other generalizations of derivative.

It is easy to construct examples to show that none of the known generalizations of derivative (except for subdifferential) is comparable with lower derivative. Curiously enough, many of the results on the other generalized derivatives can still be deduced from the results on lower derivative.

Let any notion G of generalized derivative be called compatible with lower derivative if whenever any function f is simultaneously G and lower derivable at some point x, the lower derivative of f at x is unique and equal to Gf'(x).

The following generalized notions of derivative are all compatible with lower derivative: (i) approximate and preponderant derivatives of Denjoy, (ii) qualitative derivative of Marcus, (iii) Császár derivative with respect to a family of negligible sets, (iv) mean derivative of Borel, (v) Cesàro derivative, (vi) derivative relative to a set, (vii) congruent derivative of Sindalovskii, (viii) selective derivative of O'Malley and (ix) the general and generalized derivatives of Khintchine (local versions). Hence for each of these generalized derivatives, the mean-value theorem, the monotonicity theorem, the extension of Goldowski-Tonelli theorem and their Darboux and Denjoy properties follow from the results on lower derivative. Many of these results are known but some are new.

The unilateral and symmetric derivatives are compatible with lower derivative only in the case of nonangular functions. Hence the above mentioned results are obtained for these derivatives only when the function under consideration is nonangular.

The results on lower derivative lead further to some new classes of functions that are derivable in the above senses at c-dense sets of points.

Further, the notions of lower and upper derivatives can in turn be generalized using the various limits employed in the definitions of above generalized derivatives. For example, the approximate lower derivative generalizes simultaneously the approximate and lower derivatives.

Finally, many of the results presented here are valid under more general continuity hypotheses. To keep the new nomenclature to the minimum we have avoided the necessary generalizations of continuity.

Besides the extension of the present work to higher dimensions, there are some other basic problems that arise naturally from it. These will be discussed in the forthcoming work.

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