Real Analysis Exchange Vol. 7 (1981-82)

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Some Problems in Need of Solution

How presumptuous of the author to include the word "need" in the title of this paper, especially when the contents concern problems he is interested in! But, perhaps, the need is merely his own and perhaps the problems to be described below are not meant to be the most important, or the deepest, or even the most intriguing, but rather only a set of three problems with which the author has found some fascination. And, I should add, problems he would enjoy knowing the solutions to. Kindness demands that in reading this paper we adopt this latter attitude and henceforth we shall axiomatize said adoption.

The problems discussed here have three quite distinctive personalities which reflect, in some sense, three aspects of our discipline. The first problem is **something old.** That is, many of the techniques used in working on it are classical, but, moreover, the entire flavor of the study has deep traditions. The second problem is **something borrowed.** Real analysts have always been useful to mathematicians in other areas, and indeed, much of the motivation for studying a particular real variables concept often stems from its interest elsewhere. The early Fundamentas are a rich source for such material, and the second problem described here is of that mold. In specific, this second problem has direct motivational ties with ordinary differential equations. The final problem, or problem area, is something new, it concerns the study of σ-porous sets.

Something Old

In [G], Z. Grande considered the relationship between the classes of pointwise limits of approximately continuous functions, $B_1(A)$, pointwise limits of almost everywhere continuous functions, $B_1(P)$, and pointwise limits of functions which are both approximately continuous and almost everywhere continuous functions, $B_1(A \cap P)$. Now, the characteristic function of the rational numbers is an example which shows that $B_1(A \cap P)$ is not merely the intersection, $B_1(A) \cap B_1(P)$, and the reason is that the rationals are sewn to their complement in a topological-measuretheoretic way. To delimit the functions in $B_1(A) \cap B_1(P)$ Grande defined a new class of functions AP_1 (see [N]) as the set of all those functions f such that whenever $a < b$ and U and V are nonempty sets with $U \subset \{x:f(x) < a\}$ and $V \subset \{x:f(x) > b\}$, then $U \cup V \not\subset d^*(\mathrm{cl}U \cap \mathrm{cl}V)$, $d^*(X)$ denotes the set of points of upper density of the set X, cl denotes Euclidean closure. Although this is not the original definition of AP_1, it is equivalent, [CH_1], and is a bit more geometric. Grande shows that

$$* \qquad B_1(A \cap P) \subset B_1(A) \cap B_1(P) \cap AP_1$$

and asks whether the inequality in * is actually an equality. Because a fairly general theory of Baire functions is known (see [K]), a great deal of machinery can immediately be brought to baire(?) on this question. In particular, if F is the class of those functions which are continuous relative to a topology on R^n, then there is a characterization of $B_\alpha(F)$ in terms of the Baire sets associated with that topology in the same way that the classical Baire functions are characterized by the usual Baire sets. That is, $f \in B_\alpha(F)$ if and only if the preimage of every open set is in the α^{th} additive Baire class. Of the classes considered here A and $A \cap P$ are the continuous functions for specific topologies and, as a consequence, are characterized by the associated Baire sets as described above. The approximately continuous functions are precisely those functions which are continuous with respect to the density topology [GW] which consists of those measurable sets for which each point of the set is a point of full density of the set. The topology for $A \cap P$ (in R^1) was introduced by R.J. O'Malley in the course of his investigation of the approximate derivative, [O_1] and [O_2], and consists of those density open sets which can be decomposed as $G \cup Z$ where G is Euclidean open and Z is a null set. Neither of these topologies is normal and as a consequence, the Baire sets and the Borel sets are different; this, perhaps, adds to their charisma. In addition to the general study of Baire functions, there are the following two specific and quite lovely characterizations of two of the spaces under consideration. The first is a theorem of D. Preiss, [P], and the second is a theorem of D. Mauldin, [M]. In these theorems, C denotes the continuous functions.

THEOREM P. $B_1(A) = B_2(C)$.

THEOREM M. A function $f \in B_1(P)$ iff there is a function $g \in B_1(C)$ such that $\{x: f(x) \neq g(x)\}$ is contained in an F_σ null set.

In [N], T. Nishiura studied the topology of $A \cap P$ (in R^n) in some detail and gleaned from this more general study a number of results which were pertinent to Grande's question. Nishiura's work led him to define a new class of functions, AP_2, whose definition is quite technically linked to the topology of $A \cap P$ and subsequently he proved

$$B_1(A \cap P) = B_1(A) \cap B_1(P) \cap AP_1 \cap \pm AP_2$$

However, it is not clear whether $\pm AP_2$ plays an essential role. Then in [CH_1] and [CH_2] G. Cox and P. Humke studied the spaces in question with the specific goal of settling Grande's question. Their basic approach was classical and the machinery they set up in conjunction with Nishiura's topological work may prove to be the context of an eventual solution, but maybe not? In particular, they show that every function in one of the many spaces under consideration is the uniform limit of simple functions from the same space, and that the properties involved are retained when taking uniform limits. These results, and the linear structure of the spaces allow them to conclude both:

THEOREM CH. If $B_1(A \cap P) \neq B_1(A) \cap B_1(P) \cap AP_1$ then there is a characteristic function which shows this.

THEOREM HC. $AP_1 \subset B_1(P)$.

Hence, Grandes question becomes:

$$\text{Is } B_1(A \cap P) = B_1(A) \cap AP_1 \ ?$$

And this problem is still in need of a solution.

Something Borrowed

This borrowed problem is borrowed from ordinary differential equations and was shown to me by Keith Schrader [S] from Missouri. But there is nothing ordinary concerning this problem, for its solution would not only provide a nice theorem (or example) in the theory of boundary value problem's, but would be of interest just as a real variables problem. Although I'll spare the reader the details of the problem's background, I should mention that the problem comes from an area of boundary value problems labeled "uniqueness implies existence" and a nice overview of the subject with informative examples can be found in L. K. Jackson's article, [J]. In particular, the problem discussed here appears in conjunction with Kamke's Convergence Theorem in which one is able to assert that for a given sequence of solutions to a certain differential equation that

$$* \qquad |y_n'| + |y_n''| + \ldots + |y_n^{(k)}| \to \infty$$

uniformly on a set of positive measure. Is it then true that the graph of y_n intersects some set of polynomials (or lines) with uniformly bounded coefficients a large number of times on short intervals? Let me be more specific and hopefully more precise. For definiteness I'll consider just one function $f:[0,1]\to\mathbb{R}$ which could be any term of the original sequence, $\{y_n\}$. If k=1 then we can normalize f to be bounded by 1 and then we can interpret * to say that $|f'|$ is very large on, say half of [0,1]. It then follows from a familiar Banach indicatrix argument that some horizontal line intersects the graph of f very may times on [0,1]. Let's consider the second case. Again suppose that f is bounded by 1, but now assume that $|f'| + |f''|$ is very large on a set of measure 1/2. Is it true that some line (not necessarily horizontal) intersects the graph of f many times? To my knowledge, the answer to this is not known. Such is also the state of the higher degree problems. To conclude this section I'd like to give two variations of this problem.

PROBLEM 1. Let F_n denote the set of all twice differentiable functions, f, such that

a. $|f(x)| \leq 1/n$ for $0 \leq x \leq 1$

b. $|f''(x)| > 1$ for $x \in S \subset [0,1]$ where $m(S) \geq 1/2$.

Let i(f,L) denote the number of intersections of the graph of f with the line L, and let

$i(n) = \inf\{i(f,L) : f \in F_n \text{ and } L \text{ is a line}\}$.

Compute a reasonable estimate of i(n), or at least lim sup i(n).

PROBLEM 2. Let S denote a set of n disjoint segments in $[0,1] \times [0,\varepsilon]$ each of length, say 1/2n. Suppose also that the segments are increasing in the sense that if one segment lies to the right of another it also lies above. Is there a line L which intersects "many" (like $n^{1/2}$) of these segments?

I find these problems both interesting and important, a nice combination.

Something New

The notion of σ-porosity is no longer infant in its applications to real function theory, but it is new enough that many potential applications and many ideas that appear fundamental to the notion remain unresolved or at least unfocused. In [EFH] Foran and Humke studied a few of the connections between the notion of σ-porous sets and some of the more standard set theoretic properties. In particular, they show that every σ-porous set is contained in a $G_{\delta\sigma}$ σ-porous set, but that there are σ-porous sets which are contained in no σ-porous F_σ set. This deprives the user of one of the tools normally used to show a set is of the first Baire category. The notion of σ-porosity is quite distinct from that of either measure zero or first category, so one might expect that the measure theoretic companion to the category result is also false. This is, as of this writing, unknown, and would be quite useful to know.

Is every σ-porous set contained in a G_δ σ-porous set?

There are other areas of the set theoretic study needing work, and the first I'll mention is this.

Does there exist a closed σ-porous set, S, such that S+S contains an interval?

OR

Is it true that if E is closed and not σ-porous then E+E contains an interval

Although these two questions appear analogous, they are not, for the Cantor Ternary Set provides an affirmative answer to the first question, but to my knowledge the second question is unresolved. An affirmative answer to the second question would show that every non σ-porous set would be a basis. That is, if a Fourier Series converges absolutely on such a set, then it converges absolutely a.e.. A partial converse to this is found below.

THEOREM. Suppose $\sum \rho_n \sin\alpha_n x$ $(\rho_n > 0)$ has the properties that:

1. $\sum \rho_n |\sin\alpha_n x| < +\infty$ on E
2. $\sum \rho_n |\sin\alpha_n x| = +\infty$ a.e.
3. $\rho_n > \rho > 0$ for every n. Then E is σ-porous.

Proof. If there is a subsequence $\{\alpha_{n_k}\}$, of $\{\alpha_n\}$ which converges to α, then $E \subset \{n\pi/\alpha : n=0,+1,\ldots\}$ if $\alpha \neq 0$ and $\{0\}$ if $\alpha=0$ so in either case, E is porous. Hence, the only interesting case is when $\{\alpha_n\} \to \infty$ Let

$$E_m = \{x: \sum_m^\infty \rho_n |\sin\alpha_n x| < \rho/2\},$$

and let $x_0 \varepsilon E_m$. As $\rho_n |\sin\alpha_n x_0| < \rho/2$ for $m<n$, it follows that $|\sin\alpha_n x_0| < 1/2$ and consequently,

$$x_0 \varepsilon \bigcup_{k=-\infty}^{\infty} [k\pi/\alpha_n - \pi/6\alpha_n \ , \ k\pi/\alpha_n + \pi/6\alpha_n] = I.$$

If $x\varepsilon I$, then $|\sin\alpha_n x| > 1/2$ and so $\rho_n |\sin\alpha_n x| > \rho |\sin\alpha_n x| > \rho/2$, or $x\varepsilon E_m$. It follows, then, that the porosity of E_m in $[x_0, x_0+\pi/\alpha_n]$ is at least 2/3 for each $n>m$ and as $\{\alpha_n\} \to \infty$, E_m has porosity at least 2/3 at x_0. This completes the proof.

It would be nice to know whether the condition $\rho_n > \rho_0 > 0$ for infinitely many n could be eliminated from the hypothesis of the result stated above.

Two other questions concerning σ-porous sets were asked by L. Zajicek, [Z], and, for a variety of reasons, I'd like to restate them here. First, I find them intersting in their own right, and second, I have re-bumped into them in a obtuse fashion while studying discontinuities of the derivative. A bit of notation is necessary. Let f be a continuous, increasing real valued function. If $M \subset \mathbf{R}$ and $x\varepsilon\mathbf{R}$, we let $\gamma(x,\varepsilon,M)$ denote the supremum of the lengths of the intervals in $(x-\varepsilon, x+\varepsilon) \cap (\mathbf{R}-M)$ and then define the f-porosity of M at x to be

$$\limsup_{\varepsilon \to 0^+} f(\gamma(x,\varepsilon,M))/\varepsilon$$

The point x is a point of f-porosity of M if the f-porosity of M at x is positive, and M is an f-porous set if it is f-porous at each of its points. Two surprising results (both due to Zajicek) are:

THEOREM Z_1. Let $0<q<p<1$ and let $M\subset R$. Then M is a set of (x^q)-σ-porosity iff M is a set of (x^p)-σ-porosity.

THEOREM Z_2. If $0<q<1$ then there is a perfect set of (x^q)-σ-porosity which has positive Lebesgue measure.

In light of the first result we see that the notion of "polynomially σ-porous" makes good sense, and from the second that there are sets of positive measure which are polynomially σ-porous. In a real sense, however, most sets which have positive measure are not polynomially σ-porous. Two questions remain.

Does there exist a (perfect) set which is both of measure zero and of the first Baire category which is not polynomially σ-porous?

Does there exist a continuous increasing function, f, such that any (perfect) set of measure zero and of the first Baire category is a set of f-σ-porosity?

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