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Peano Derivatives: A Survey

M.J. Evans and C.E. Weil

§1. Introduction and Early Historical Notes

The notion of a k^{th} derivative of a real valued function of a real variable is well understood by most undergraduate calculus students. The definition is iterative in nature and thus easily comprehended if one initially understands what a first derivative is. This nice feature can present a problem, however, because in order to find the k^{th} derivative of a function f at a point x, one must know all the previous derivatives, not only at x, but at every point in a neighborhood of x. One type of generalized k^{th} order differentiation, having Taylor's theorem as its motivation, attempts to skirt this drawback. This kind of differentiation is called Peano differentiation and is the topic of this survey article.

Definition 1.1. A function f is said to have a k^{th} Peano derivative at x if there exist numbers $f_1(x)$, $f_2(x)$, ..., $f_k(x)$ such that

$$f(x+h) = f(x) + hf_1(x) + (h^2/2)f_2(x) + \ldots + (h^k/k!)f_k(x) + o(h^k)$$

as $h \to 0$. The number $f_k(x)$ is called the k^{th} Peano derivative of f at x.

This concept was presented in 1891 by G. Peano in [34]. (A German translation of this Italian article may be found as

Appendix III in [16].) There Peano introduced this type of derivative, obtained a product rule, a quotient rule, and pointed out that if a function f has an ordinary k^{th} derivative at x, $f^{(k)}(x)$, then it must have a k^{th} Peano derivative at x and $f^{(k)}(x) = f_k(x)$. The converse is not true for $k \geq 2$ as is exhibited by the function

$$f(x) = \begin{cases} x^3 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 . \end{cases}$$

Here $f_2(0) = 0$, but $f^{(2)}(0)$ does not exist. Thus k^{th} order Peano differentiation is a true generalization of ordinary k^{th} order differentiation although obviously there is no difference for $k = 1$.

Reintroducing the Peano derivative in 1935 [12], A. Denjoy refined the example mentioned above to show that given any nowhere dense closed set E and an integer $k \geq 2$, there is a function which has a k^{th} Peano derivative at each point of the real line R, but for which no point of E is a point of ordinary k^{th} order differentiability; in fact, the set of points where $f^{(k)}(x)$ fails to exist turns out to be precisely E. Denjoy then showed that this function is in a sense typical of functions having k^{th} Peano derivatives by showing that if f has a k^{th} Peano derivative at each point of an interval I, then there is an open, dense set in I at each point of which f is k times differentiable in the ordinary sense. He further showed that a function is determined (up to a polynomial of degree $k - 1$) by its k^{th} Peano derivative by proving that if $f_k(x) = 0$ for each x in an interval, then f is a polynomial of degree at most $k - 1$ on that interval.

Before proceeding to a more in-depth examination of the properties of a k^{th} Peano derivative, we should mention an early modification of it due to Ch. de la Vallee-Poussin [35] which proved useful in the study of trigonometric series in the early part of this century.

Definition 1.2. *A function f is said to have a k^{th} symmetric derivative at x if there is a polynomial $P(h)$ of degree k or smaller such that*

$$\{f(x+h) + (-1)^k f(x-h)\}/2 = P(h) + o(h^k) \quad \text{as} \quad h \to 0 .$$

If $\alpha_k/k!$ denotes the coefficient of h^k in $P(h)$, then α_k is called the k^{th} symmetric derivative of f at x.

It is clear that if k is even, then P has only even powers of h, and if k is odd, only odd powers. It is not difficult to see that if f has a k^{th} Peano derivative at x, then it has a k^{th} symmetric derivative at x and the two are equal. The converse is false, even for $k = 1$, as the absolute value function demonstrates at the origin. However, J. Marcinkiewicz and A. Zygmund [25] showed that if f has a k^{th} symmetric derivative at each point of a set E, then f has a k^{th} Peano derivative almost everywhere in E.

The k^{th} symmetric derivative occurs in the study of trigonometric series as one considers the Cesàro summability of repeatedly differentiated series. Marcinkiewicz and Zygmund [25] showed that if the k^{th} symmetric derivative of f at x

exists, then the Fourier series for f differentiated term by term k times is summable (C,α) at x to the k^{th} symmetric derivative of f at x, provided $\alpha > k$. They further showed that if one supposes that the series

$$a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is summable (C,α), $\alpha > -1$, at x_0 to a finite sum s and if the series obtained by integrating termwise k times converges in a neighborhood of x_0 to sum $F(x)$, then the k^{th} symmetric derivative of F at x_0 exists and equals s.

Marcinkiewicz [24] obtained the following decomposition as a tool for the study of trigonometric series, which, like the previously mentioned theorem of Denjoy, sheds considerable light on the structure of functions possessing k^{th} Peano derivative everywhere. He showed that if $f_k(x)$ exists at each point of a measurable set E, then there is a closed set F in E of measure arbitrarily close to that of E and a decomposition

$$f(x) = g(x) + h(x)$$

such that g has a continuous k^{th} ordinary derivative throughout the interval of definition of f and $h(x) = 0$ for each x in F. One application of this decomposition to Fourier series is to prove that if f has a k^{th} Peano derivative at each point of a measurable set E, then the series obtained by differentiating the Fourier series of f termwise k times is summable (C,k) almost everywhere in E. Other applications of this decomposition to the study of Fourier series as well as other results dealing with the almost everywhere existence of k^{th} Peano derivatives may be found in [44]. In the next section of this survey we wish to concentrate on the properties of k^{th} Peano derivatives which exist on an interval.

§2. Properties of the k^{th} Peano Derivative

In 1954 H.W. Oliver [31] published the first extensive work devoted exclusively to exhibiting properties of k^{th} Peano derivatives. He showed that such a derivative has several of the properties known to be possessed by an ordinary derivative. To be more specific, suppose that at each point of an interval I a function f has a k^{th} Peano derivative. Oliver established that f_k is of Baire class one by noting that the k^{th} difference

$$h^{-k} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x+ih)$$

converges pointwise to $f_k(x)$ as $h \to 0$. Denjoy [12] had obtained this result earlier in the more general setting where f_k is defined relative to a perfect set H for a continuous f. Oliver also showed that f_k must have the Darboux property on I, another property well known and easily verified for ordinary derivatives. Moreover, he showed that if f_k is bounded above or below on I, then f is the ordinary k^{th} derivative of f on I. In particular, this yields the monotonicity theorem which states that if $f_k \geq 0$ on I, then f_{k-1} is nondecreasing and continuous on I. Combining this with the fact that f_k is of Baire class one, it follows that f_k is an ordinary k^{th} derivative on an open, dense subset of I, a fact previously established by Denjoy via different means as we noted in §1.

Before continuing with the work of Oliver, we wish to interject that f_k possesses the Darboux property in a rather interesting fashion with respect to the open, dense set on which it is an ordinary derivative. The ordinary k^{th} derivative has the Darboux property on this open dense set; in fact, a much stronger result was established by R.J. O'Malley and C.E. Weil [32]. They showed that if f_k attains both values $-M$ and M on I, then there is an open interval J on which $f_k = f^{(k)}$ and $f^{(k)}$ attains both $-M$ and M on J. (In this paper we shall say that any generalized derivative possessing this property has the $-M,M$ property.)

In [13] Denjoy showed that if g is an ordinary derivative on I, then for any open interval, (a,b), $g^{-1}(a,b)$ either is empty or has positive Lebesque measure. A function having this property is said to have the Denjoy property. Oliver [31] showed that f_k possesses the Denjoy property on I.

In a monumental study of properties of the ordinary derivative Z. Zahorski [43] proved that the following property is possessed by every ordinary derivative.

Definition 2.1. A function g is said to have the Zahorski property if for each open interval (a,b), for each x in $g^{-1}(a,b)$, and for each sequence of intervals I_n converging to x, (The endpoints of the $\{I_n\}$ converge to x, but x belongs to no I_n.) $\lim_{n\to\infty} |I_n|/\mathrm{dist}(x,I_n) = 0$, where $|\cdot|$ denotes Lebesgue measure and $\mathrm{dist}(x,I_n)$ denotes the distance between x and I_n.

C.E. Weil [42] showed that a k^{th} Peano derivative also has the Zahorski property. In a subsequent paper [40] he then introduced a property somewhat stronger that the Zahorski property, which he called property Z.

<u>Definition 2.2</u>. <u>A function</u> g <u>defined on an interval</u> I <u>is said to have property</u> Z <u>if for every</u> $\epsilon > 0$, <u>each</u> x <u>in</u> I, <u>and each sequence of intervals</u> $\{I_n\}$ <u>converging to</u> x <u>such that</u> $g(y) \geq g(x)$ <u>on</u> I_n <u>or</u> $g(y) \leq g(x)$ <u>on</u> I_n <u>for each</u> n, <u>we have</u>

$$\lim_{n\to\infty} \frac{|\{y \in I_n: |g(y) - g(x)| \geq \epsilon\}|}{|I_n| + \text{dist}(x, I_n)} = 0 .$$

Weil showed that this property is strictly stronger than the Zahorski property, yet still is possessed by every k^{th} Peano derivative.

Before closing this section on properties of k^{th} Peano derivatives we wish to mention an interesting generalization recently introduced by M. Laczkovich [18], namely, absolute Peano derivatives.

<u>Definition 2.3</u>. <u>Let</u> f <u>be defined in a neighborhood of</u> x. <u>We say that the absolute Peano derivative of</u> f <u>at</u> x <u>exists and is</u> A (<u>in symbols</u> $f^*(x) = A$) <u>if there is a function</u> g, <u>a nonnegative integer</u> k, <u>and a</u> $\delta > 0$ <u>such that</u>

(i) $g_k = f$ <u>on</u> $(x - \delta, x + \delta)$, <u>and</u>

(ii) $g_{k+1}(x) = A$.

Laczkovich showed that this concept is unambiguously defined, that if f^* exists on an interval it is a function of Baire class one, that it has the Darboux and Denjoy properties, and if

f^* is bounded above or below on an interval, then $f^* = f'$ on that interval. It is not known whether f^* has the $-M,M$ property, the Zahorski property, or property Z.

§3. k^{th} L_p Derivatives

Notice that we can reinterpret the definition of the k^{th} Peano derivative, Definition 1.1, as follows:

$$\|f(x+t) - f(x) - tf_1(x) - \ldots - (t^k/k!)f_k(x)\|_{\infty,[0,h]} = o(h^k) .$$

Now if we replace L_∞ convergence by convergence in L_p, $0 < p < \infty$, we arrive at the definition of the k^{th} L_p derivative.

Definition 3.1. A function f has a k^{th} L_p derivative, $0 < p < \infty$, at x if there exist numbers $f_{p,1}(x), \ldots, f_{p,k}(x)$ such that

$$\{\frac{1}{h}\int_0^h |f(x+t) - f(x) - tf_{p,1}(x) - \ldots - (t^k/k!)f_{p,k}(x)|^p dt\}^{1/p} = o(h^k) \quad \text{as} \quad h \to 0 .$$

The number $f_{p,k}(x)$ is called the k^{th} L_p derivative of f at x.

This type of natural generalization of Peano differentiability was introduced and studied in the several variable setting by A.P. Calderon and A. Zygmund [8]. It has the advantage of being preserved under singular integral transformations.

The role of the $p = 0$ case can now nicely be assumed by the approximate k^{th} Peano derivative which was originally defined by Denjoy [13].

Definition 3.2. A function f has a k^{th} approximate Peano (L_0) derivative at x if there exist numbers $f_{(1)}(x), \ldots, f_{(k)}(x)$ and a set E having density 1 at 0 such *that*

$$f(x+h) - f(x) - hf_{(1)}(x) - \dots - (h^k/k!)f_{(k)}(x)$$
$$= o(h^k) \quad \textit{as} \quad h \to 0 \quad \textit{through} \quad E .$$

The number $f_{(k)}(x)$ *is called the* k^{th} *approximate Peano derivative of* f *at* x.

The notation $f_{p,k}(x)$ will be used to unify these definitions. It will denote the k^{th} L_p derivative of f at x, $0 \le p \le \infty$. Thus $f_k(x) = f_{\infty,k}(x)$ and $f_{(k)}(x) = f_{0,k}(x)$. It is a routine matter to show that if $0 \le q < p \le \infty$ and f has a k^{th} L_p derivative at x, then it has a k^{th} L_q derivative at x and $f_{p,k}(x) = f_{q,k}(x)$ [8,14]. It is also easily verified that if f has a k^{th} L_p derivative at x, where $p \ge 1$, then $f_{p,k}$ is the $(k+1)^{th}$ Peano derivative of the integral of f [14]. Thus for $p \ge 1$, k^{th} L_p derivatives are seen to have all the properties mentioned for Peano derivatives in §2.

Let us take up the case $p = 0$. The situation where $k = 1$ is that of what is classically called the approximate derivative of f at x and is customarily denoted by $f'_{ap}(x)$. (Thus $f'_{ap}(x) = f_{0,1}(x)$.) One of the first investigations of the properties of the approximate derivative is due to G. Tolstoff [38]. He showed that if f'_{ap} exists on an interval I, then it is of Baire class one, it has the Darboux property, and if $f'_{ap} \ge 0$ on I, then f is nondecreasing on I. Tolstoff's proofs are difficult and lengthy. The results were subsequently established in a more concise fashion by C. Goffman and

C. Neugebauer [17] by utilizing the axiom of choice in an adroit manner. The Denjoy property for f'_{ap} was established independently by S. Marcus [26] and Weil [42]. Weil further showed that f'_{ap} has the Zahorski property [42], and subsequently that it has property Z [40]. O'Malley and Weil [32] verified that f'_{ap} has the -M,M property.

Taking up the case of the general k, M.J. Evans [14] showed that $f_{0,k}$ is of Baire class one. B.S. Babcock [3] and S.N. Mukhopadhyay [30] independently established the Darboux, Denjoy, and Zahorski properties as well as the fact that if $f_{0,k}$ is positive on an interval I, then $f_{0,k-1}$ is nondecreasing on I, and more generally that if $f_{0,k}$ is bounded above or below on I, then $f_{0,k} = f^{(k)}$ on I. Utilizing certain techniques of S. Verblunsky [39] dealing with Peano derivates, which are defined in a natural manner, Mukhopadhyay's approach yielded a more general monotonicity result in derivate form. Babcock further showed that $f_{0,k}$ has property Z. C.-M. Lee [19] has established the -M,M property for $f_{0,k}$ as well.

As mentioned in the introduction, the ordinary k^{th} derivative is defined in an iterative fashion, whereas the $f_{p,k}$ derivatives are not. It is natural to inquire whether these derivatives can be obtained in an iterative manner. The first major result of this nature was obtained by Marcinkiewicz and Zygmund [25], who showed that if $f_{\infty,k+1}$ exists on a measurable set E, then $f_{\infty,k+1}(x) = (f_{\infty,k})_{0,1}(x)$ almost everywhere in E. Evans and Weil [15] have recently extended this to show that if $f_{p,k+j}, p \geq 1$, exists on E, then $f_{p,k+j}(x) = (f_{p,k})_{0,j}$ almost everywhere in E. Further, it is shown that the 0 in this result cannot be replaced by any $q > 0$. H.H. Pu and H.W. Pu [37] have obtained a pointwise result on iterated Peano differentiation. They have shown that if $f_{\infty,k+j-1}$ is monotone or Lipschitz on I and $f_{\infty,k+j}(x)$ exists for some x in I, then $(f_{\infty,k})_{\infty,j}(x)$ exists and equals $f_{\infty,k+j}(x)$. O'Malley and Weil [33] have examined the reverse problem by assuming that $f_{p,k}$, $0 \leq p \leq \infty$, exists on I and if $(f_{p,k})_{q,j}(x)$, $1 \leq q \leq \infty$, exists for some x in I, then $f_{\infty,k+j}(x)$ exists and equals $(f_{p,k})_{q,j}(x)$. (Evans and Weil [15] noted that under these conditions $(f_{p,k})_{q,j}(x)$ can exist only at a point x at which f is k times differentiable in the ordinary sense.) In addition O'Malley and Weil provided two examples to show that the condition $1 \leq q \leq \infty$ is essential. Let $0 < q < 1$. First, there is a function f defined on a neighborhood of zero such that $(f_{\infty,1})_{q,1}(0)$ exists but $f_{0,2}(0)$ does not. Second, there is a function g such that $(g_{\infty,1})_{q,1}(0)$ and $g_{\infty,2}(0)$ both exist but are not equal. Lee and O'Malley [22] had earlier provided two analogous examples for the situation where $q = 0$.

In reference to the notion of an absolute Peano derivative mentioned in §2., Laczkovich [18] observed that because of functions like those presented by Lee and O'Malley, the notion of an approximate absolute Peano derivative cannot be unambiguously defined. Using the above function g of O'Malley and Weil, it follows by the same reasoning that an absolute L_p derivative, $0 < p < 1$, cannot be well defined. However, for $p \geq 1$ an absolute notion is feasible and indeed possesses all the properties of the absolute Peano derivative. In truth it is an absolute Peano derivative, as follows from the fact that any k^{th} L_p derivative, $p \geq 1$, must also be a $(k+1)^{th}$ Peano derivative [14].

§4. Related Topics

The centrality of the notion of the Peano derivative was perhaps best demonstrated by J.M. Ash [1]. He defined a generalized k^{th} Riemann derivative in the following fashion. Let $A = \{a_0, a_1, \ldots, a_{k+\ell}; A_0, \ldots, A_{k+\ell}\}$ be a set of real numbers with $a_i \neq a_j$ and

$$\sum_{i=0}^{k+\ell} A_i a_i^j = 0, \; j = 0, 1, \ldots, k-1 = k!, \; j = k \; .$$

The function f is said to have a k^{th} generalized derivative with respect to A at x if there exists a number $f_{[k]}(x)$ such that

$$\sum_{i=0}^{k+\ell} A_i f(x + a_i h) = f_{[k]}(x) h^k + o(h^k) \text{ as } h \to 0 \; .$$

He then showed that if a function f had such a k^{th} generalized derivative for a given A at each point of a measurable set E, then f must have a k^{th} Peano derivative at almost every point of E. In a subsequent study [2] Ash characterized the Peano derivative in terms of these generalized derivatives.

We should point out that throughout this paper we have assumed all derivatives mentioned to be finite. By making the obvious changes in the definitions one can permit $+\infty$ and $-\infty$ as values for the derivatives discussed. Properties do not automatically carry over, however. Zahorski [43] showed that such an ordinary derivative will still be of Baire class one. It was not until 1970 that the corresponding result was established for the approximate derivative in a very clever fashion by D. Preiss [36]. P.S. Bullen and S.N. Mukhopadhyay [5] have presented a proof for the statement that if f is continuous on I and f_k exists finitely or infinitely on I, then f_k is of Baire class one. (Indeed, they concluded that f_k has the Denjoy property). Unfortunately, their proof rests on the claim that

$$f_k(x) = \lim_{h\to 0} h^{-k} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x+ih),$$

which, as noted in §2., is valid if $f_k(x)$ is finite. However, this is not necessarily true if $f_k(x)$ is infinite, as the following example, constructed with the assistance of C.-M. Lee and R.J. O'Malley, shows. Decompose the interval $(0,1]$ into $[1/4^{n+1}, 1/4^n]$ for $n = 1,2,3,\ldots$, and further decompose each of these intervals into $[1/4^{n+1}, 1/(2\cdot 4^n)]$ and $[1/(2\cdot 4^n), 1/4^n]$. On each second interval let $f(x) = x/n$. On each first interval let f be linear. Finally let $f(0) = 0$. Note that for $x \in [0,1/4^n]$, $f(x) \leq x/n$. Thus $f_1(0) = 0$. Next note that for $x \in [1/4^{n+1}, 1/4^n]$, $f(x) \geq 1/[(n+1)4^{n+1}]$ and hence $f(x)/x^2 \geq 4^{2n}/[(n+1)4^{n+1}] = 4^{n-1}/(n+1)$. It follows that $f_2(0) = +\infty$. For $h = 1/(2\cdot 4^n)$, $\sum_{i=0}^{2} (-1)^{2-i}\binom{2}{i} f(0+ih) = -2f(1/(2\cdot 4^n)) + f(2/(2\cdot 4^n))$ $= -2(1/(2\cdot 4^n\cdot n)) + 1/(4^n\cdot n) = 0$. Consequently $\lim_{h\to 0} \sum_{i=0}^{2} (-1)^{2-i}\binom{2}{i} f(0+ih) \neq +\infty$. So the question remains, if $f_k(x)$ exists for all x in an interval where $\pm\infty$ are allowed, is f_k of Baire class one?

In this survey we have not dealt with the application of the Peano notion of differentiation to the theory of non-absolutely continuous integrals. The interested reader is referred to [4].

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Department of Mathematics
Western Illinois University
Macomb, Illinois 61455

Department of Mathematics
Michigan State University
East Lansing, Michigan 48824