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# SUMS OF CONTINUOUS AND DARBOUX FUNCTIONS

## 1 Introduction

The class $\mathcal{C} + \mathcal{D}$ of the real functions that are the sum of a continuous and a Darboux function has received some atention, [2, 5, 6, 7], due to the fact that its exact characterization is not known. The results obtained so far yield comparisons with some classes of functions having generalized Darboux properties. For instance, one knows that $\mathcal{C}+\mathcal{D} \subset \mathcal{U}$, the class of uniform limits of Darboux functions and the inclusion is strict.

Given an interval $I$ and a set $A \subset \mathbf{R}$, denote by $\mathcal{D}^*(I, A)$ the set of all $f : I \longrightarrow \mathbf{R}$ such that range$(f) = A$ and $cl(f^{-1}(y)) = I$ for any $y \in A$ (we will frequently omit $I$ from this notation in the case $I = \mathbf{R}$). In their paper [5] Natkaniec and Kircheim have provided an $f \in \mathcal{D}^*(\mathbf{R}\backslash\mathbf{Q})$ such that $f \notin \mathcal{C}+\mathcal{D}$. The following question arises naturally: *characterize those sets $A \subset \mathbf{R}$ such that $\mathcal{D}^*(I, A) \subset \mathcal{C}+\mathcal{D}$.* Refining the result from [5], we will settle this question.

## 2 Our result

Clearly any interval (including $\mathbf{R}$ or singleton sets) is a solution of the previous problem. The interesting fact is that there are no other solutions:

**Theorem 1** *The only sets $A \subset \mathbf{R}$ for which $\mathcal{D}^*(I, A) \subset \mathcal{C} + \mathcal{D}$ are the intervals.*

PROOF. Suppose there were a set $A$, other than an interval, having the desired property. Then $\mathcal{D}^*(I, A) \subset \mathcal{U}[I]$. Since the functions from $\mathcal{U}[I]$

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are characterized (cf. [1]) by the following property: for any subinterval $J \subset I$ and every set C of cardinality less than $\mathbf{c}$ the set $f(J\backslash C)$ is dense in $[inf_{x\in J}f(x), sup_{x\in J}f(x)]$, it follows that $A$ is c-dense in $co(A)$, the convex hull of A.

Indeed, suppose there were an interval $(a,b) \subset co(A)$ such that $card((a,b)\cap A) < \mathbf{c}$. By restricting ourselves to a subinterval of $(a,b)$ we may suppose that $A$ meets both $(-\infty,a)$ and $(b,\infty)$. We will construct a function $f \in \mathcal{D}^*(I,A)\backslash\mathcal{U}$. There are two cases:

- $card(A) = \mathbf{c}$.

  Let $\sim$ be the mod $\mathbf{Q}$ equivalence on I. Since $card(I/\sim) = \mathbf{c}$, there is a bijection $\overline{f} : I/\sim \longrightarrow A$. Take $f(x) = \overline{f}(\overline{x})$, where $\overline{x}$ is the equivalence class of $x$. Then $f \in \mathcal{D}^*(I,A)\backslash\mathcal{U}$ (to see the second part take $C = f^{-1}((a,b)\cap A)$).

- $card(A) < \mathbf{c}$.

  We proceed analogously. Take $K \subset I/\sim$ such that $card(K) = card(A)$, take $\overline{f} : K \longrightarrow A$ and $z_0 \in A\backslash(a,b)$. Now define

$$f(x) = \begin{cases} \overline{f}(\overline{x}) & \text{if } \overline{x} \in K, \\ z_0 & \text{otherwise.} \end{cases} \tag{1}$$

Since either case leads to a contradiction, it follows that $A$ is c-dense in $co(A)$; in particular $card(A) = \mathbf{c}$. We will construct (in the hypothesis that $A$ is not an interval) a function $f \in \mathcal{D}^*(I,A)\backslash\mathcal{C}+\mathcal{D}$. The construction is a variation of the one from [5].

Let $\mathcal{M}$ be the family of continuous nowhere constant functions $g$ defined on intervals, together with their domains. Since any continuous $g : J \longrightarrow \mathbf{R}$ is uniquely determined by its values on $J \cap \mathbf{Q}$, it follows that $card(\mathcal{M}) = \mathbf{c}$, hence (using the Axiom of Choice) we may well-order $\mathcal{M}$:

$$\mathcal{M} = \{(g_\alpha, I_\alpha) | \alpha < \mathbf{c}\}. \tag{2}$$

Let $\{U_i : i < \omega\}$ be a countable basis for the Euclidian topology and $\{x_\alpha : \alpha < \mathbf{c}\}$ be a well-ordering of $\mathbf{R}$. We will define the sequences $t_{\alpha,i} \in U_i, y_\alpha \in \mathbf{R}, p_\alpha \in g_\alpha(I_\alpha), z_\alpha \in A$, for any $\alpha < \mathbf{c}$,and $i < \omega$, such that:

$$t_{\alpha,i} = t_{\beta,j} \Longrightarrow (\alpha,i) = (\beta,j) \tag{3}$$

$$t_{\beta,i} \in I_\alpha \Longrightarrow x_\beta + g_\alpha(t_{\beta,i}) \neq y_\alpha \tag{4}$$

$$x_\beta \in I_\alpha \Longrightarrow z_\beta + g_\alpha(x_\beta) \neq y_\alpha \tag{5}$$

$$(6) \qquad y_\alpha - p_\alpha \in int(co(A))$$

Note that $p_\alpha$ can be constructed simply by the Axiom of Choice. Let $\alpha < \mathfrak{c}$ and suppose that we have already defined $t_{\beta,i}, y_\beta, z_\beta$ for any $\beta < \alpha$ and any $i < \omega$. As the set $\{\beta | \beta < \alpha\}$ is at most countable, $\{y_\beta - g_\beta(x_\alpha) | \beta < \alpha, x_\alpha \in I_\beta\}$ is at most countable too, hence we may choose $z_\alpha \in A \backslash \{y_\beta - g_\beta(x_\alpha) : \beta < \alpha\}$ (for $A$ is uncountable).

Let us now consider the set $E^\beta = \{x \in I_\beta | g_\beta(x) = y_\beta - x_\alpha\}$ for a fixed $\beta < \alpha$. It is a closed set, and, since $g_\beta$ is nowhere constant, it is a nowhere dense set. Hence $E_\alpha = \bigcup_{\beta < \alpha} E^\beta$ is a set of the first Baire category.

Fix now $i < \omega$ and define the set $F_{\alpha,i} = \{t_{\beta,j} | \beta < \alpha \text{ or } \beta = \alpha \text{ and } j < i\}$. $F_{\alpha,i}$ is at most countable, hence $E_\alpha \cup F_{\alpha,i}$ is of the first Baire category. Therefore we may choose $t_{\alpha,i} \in U_i \backslash (E_\alpha \cup F_{\alpha,i})$. Let us finally consider the set $K_\alpha = \bigcup_{\beta \leq \alpha}(\{x_\beta + g_\alpha(t_{\beta,i}) : t_{\beta,i} \in I_\alpha\} \cup \{z_\beta + g_\alpha(x_\beta) : x_\beta \in I_\alpha\})$. $K_\alpha$ is at most countable, hence we may take $y_\alpha \notin K_\alpha$ verifying (6).

Consider $\alpha, \beta < \mathfrak{c}$ and $i < \omega$. We must verify (3)-(5):

(3) This is true because of the way we have chosen $t_{\alpha,i}$.

(4) Suppose that $t_{\beta,i} \in I_\alpha$.
**Case 1** $\beta \leq \alpha \Longrightarrow y_\alpha \neq x_\beta + g_\alpha(t_{\beta,i})$ (from the way we have chosen $y_\alpha$).
**Case 2** $\beta > \alpha \Longrightarrow g_\alpha(t_{\beta,i}) \neq y_\alpha - x_\beta$ (from the way we have chosen $t_{\beta,i}$).

(5) **Case 1** $\beta \leq \alpha \Longrightarrow y_\alpha \neq z_\beta + g_\alpha(x_\beta)$ (from the way we have chosen $y_\alpha$).
**Case 2** $\beta > \alpha \Longrightarrow g_\alpha(x_\beta) \neq y_\alpha - z_\beta$ (from the way we have chosen $z_\beta$).

Hence (3), (4) and (5) are verified. Now define $\overline{f} : \mathbf{R} \longrightarrow \mathbf{R}$:

$$(7) \qquad \overline{f}(x) = \begin{cases} x_\alpha & \text{if } x = t_{\alpha,i} \text{ and } x_\alpha \in A \text{ for some } \alpha < c, i < \omega, \\ z_\alpha & \text{if } x = x_\alpha \notin \{t_{\beta,i} : x_\beta \in A, i < \omega, \beta < \mathfrak{c}\} \end{cases}$$

It is clear that range$(\overline{f}) = A$. Take $f = \overline{f}|_I$. Since for any fixed $\alpha < \mathfrak{c}$ the set $\{t_{\alpha,i}\}_{i<\omega}$ is dense in $\mathbf{R}$, it follows that $f \in \mathcal{D}^*(I, A)$. Let us suppose that $f = g + h$ with $g$ continuous and $h$ Darboux:

**Case 1** $g$ is constant on a subinterval $[a, b] \subset I$. It follows that $f$ is Darboux on $[a, b]$. But this is false, since $f([a, b]) = A$, which is not an interval.

**Case 2** $g$ is nowhere constant. Then there exists $\alpha < \mathfrak{c}$ such that $I = I_\alpha$ and $g = -g_\alpha$. We will show that:

$$(8) \qquad y_\alpha \notin h(I)$$

$$(9) \qquad y_\alpha \in (inf(h(I)), sup(h(I))),$$

hence $h$ is not a Darboux function, a contradiction. Indeed, suppose that $y_\alpha = h(x)$ for some $x \in I$.

**Case 1** $x = t_{\beta,i} \in I = I_\alpha$ and $x_\beta \in A$.

Then $f(x) = x_\beta$, hence $g(x) = f(x)-h(x) = x_\beta - y_\alpha \Longrightarrow x_\beta + g_\alpha(t_{\beta,i}) = y_\alpha$, contradicting (4).

**Case 2** $x = x_\gamma \notin \{t_{\beta,i} : x_\beta \in A, i < \omega, \beta < \mathfrak{c}\}$.

Then $f(x) = z_\gamma$, so $g(x) = f(x) - h(x) = z_\gamma - y_\alpha \Longrightarrow y_\alpha = z_\gamma + g_\alpha(x_\gamma)$, contradicting (5).

Hence (8) holds. Since $A$ is dense in $int(co(A))$, we may choose $v, w \in A, v < y_\alpha - p_\alpha < w$. Since $p_\alpha \in g_\alpha(I)$, we may find $s \in I$ such that $p_\alpha = g_\alpha(s) = -g(s)$. As $f \in \mathcal{D}^*(I, A)$, we may find $a_i, b_i \in I$ with $\lim_i a_i = \lim_i b_i = s$ such that $f(a_i) = v, f(b_i) = w$. It follows that

$$\lim_i h(a_i) = -g(s) + \lim_i f(a_i) = v - g(s) = v + p_\alpha < y_\alpha \tag{10}$$

$$\lim_i h(b_i) = w + p_\alpha > y_\alpha, \tag{11}$$

hence (9) also holds. Since we reached a contradiction, $f$ cannot be in $\mathcal{C} + \mathcal{D}$, so $f \in \mathcal{D}^*(I, A) \setminus (\mathcal{C} + \mathcal{D})$. □

**Remark 1** *Our proof yields slightly more: for any disconnected uncountable set $A \subset \mathbf{R}$ dense in its convex hull there exists $f \in \mathcal{D}^*(I, A)$ such that $f|_I$ is not in $\mathcal{C} + \mathcal{D}$ for any subinterval $I$.*

**Remark 2** *Our result does not follow from the theorem in [5]. That result implies only that any non-interval set $A$ having the desired property must be of the first Baire category. But there are sets which are* $\mathfrak{c}$*-dense in* $\mathbf{R}$ *and of the first Baire category. P. Erdös has given [3] an example of an additive subgroup of* $\mathbf{R}$ *which is of the first Baire category but is not a Lebesgue null set. Such a subgroup is the required counterexample.*

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