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SOME EQUIVALENTS OF THE *AP* CONTROLLED CONVERGENCE THEOREM, THEIR GENERALIZATIONS AND A RIESZ-TYPE DEFINITION OF THE *AP*-INTEGRAL

In this paper, the author will propose the definitions of *ap* variational convergence and an *ap* equi-integrable sequence. Their corresponding convergence theorems will be proved to be equivalent to the *AP* Controlled Convergence Theorem. By their equivalency, we prove the condition (3) of the *AP* Controlled Convergence Theorem is actually implied in other conditions. Then we will give some generalizations.

Finally, a Riesz-type definition of the *AP*-integral will be given.

These definitions and theorems are extensions of the oscillation convergence, equi-integrable sequence, Riesz-type definition, and their corresponding convergence theorems with respect to Henstock Integration (see [4], [8]).

1 Prerequisites and Explanation

Our problems are concerned with one dimensional *AP*-integration. The sets and functions involved are assumed to be Lebesgue measurable. The notation $\mathbb{N}$ means all natural numbers, $\mathbb{R}$ denotes all real numbers, $[a,b]$ stands for a bounded real closed interval, and (a,b) is bounded real open interval.

The details of the following definitions and theorems are mainly from [1], [4] Section 22, [5] and [7] Chapter 7.8.

$S = \{S_x : x \in E\}$: We call a measurable set $D_x \subset [a,b]$ an approximate neighbourhood (*ap* neighbourhood) if it has density 1 at x (or has x as a point of density, see [7]) and includes x. Given a measurable set $E \subset [a,b]$, if for every $x \in E$, and *ap* neighbourhood of x, $S_x \subset [a,b]$ is chosen, then we say

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the set $S = \{S_x : x \in E\}$ is a choice of *ap* neighbourhoods on E, or a choice for short. (Gordon first defined it and called it a 'distribution', see [3]).

$S_\delta = \{S_{\delta,x} : x \in E\}$: the choice given by any S and $\delta : E \to (0, +\infty)$, by $S_{\delta,x} = S_x \cap (x - \delta(x), x + \delta(x))$.

AFC on E: Given a choice S on E, if $u, v \in S_x$, $x \in [u, v]$, we call x associated point of $[u, v]$. The set of all intervals having an associated point $x \in E$ is called an approximate full cover (AFC) on D given by S, denoted by Δ.

$\{[u_i, v_i];\ x_i : i = 1, 2, \ldots, k\}$: We call a finite set of nonoverlapping $[u_i, v_i] \in \Delta$; $i = 1, 2, \ldots, k$ together with associated points $x_i \in E$ a partial division of S on E and denote it by $\{[u_i, v_i];\ x_i : i = 1, 2, \ldots, k\}$ or $\{[u_i, v_i]; x_i\}$ or $\{[u, v]; x\}$ for short. If $\bigcup_{n=1}^{k} [u_i, v_i] \supset E$ we call $\{[u, v]; x\}$ a partition of S on E. IF a partial division (a partition) is of S_δ on E, we call it δ-fine.

Let $F_1, F_2, \ldots, F_n, \ldots, F$; $f_1, f_2, \ldots, f_n, \ldots, f$ be functions from $[a, b]$ to $\mathbb{R}$, $E \subset [a, b]$. We have the following definitions:

$$\lim_{x \to x_0} ap], f(x) = A \text{ means there is } S_{x_0} \text{ such that } \lim_{\substack{x \to x_0 \\ x \in S_{x_0}}} = A.$$

If $\lim_{x \to x_0} ap\, f(x) = f(x_0)$ we say f is *ap* continuous at x_0. If f is *ap* continuous at every $x \in E$, we say f is *ap* continuous on E, denoted by $f \in C_{ap}(E)$.

$$f'_{ap}(x_0) = A \text{ means } \lim_{x \to x_0} ap \frac{f(x) - f(x_0)}{x - x_0} = A.$$

$F \in (\delta)AC^*_{ap}(E)$ if there exists a choice S on E such that for every $\varepsilon > 0$, there exists an $\eta > 0$ and $\delta : E \to (0, \infty)$ such that for any partial division of S_δ on E : $\{[u, v]; x\}$, whenever $\sum (v - u) < \eta$ we have $\sum |F(u, v)| < \varepsilon$, where $F(u, v) = F(v) - F(u)$.

$$F \in (\delta)ACG^*_{ap}(E) \text{ if } E = \bigcup_{i=1}^{\infty} E_i \text{ and } F \in (\delta)AC^*_{ap}(E_i),\ i = 1, 2, \ldots .$$

$\{F_n\} \in U(\delta)ACG^*_{ap}(E)$ (U means uniformly) if $F_n \in ACG^*_{ap}(E)$, $n = 1, 2, \ldots$ with the same η, S_δ and E_i in their definitions.

$f \in (\delta)D^*_{ap}([a, b])$, if there exists F such that $F \in (\delta)ACG^*_{ap}([a, b])$ and $F'_{ap}(x) = f(x)$ almost everywhere. We say F is a primitive of f and denote it by $F = (\delta)D^*_{ap} - \int f$.

$f \in (\delta)R^*_{ap}([a,b])$ if there exist a constant I and a choice S on $[a,b]$ such that for any $\varepsilon > 0$, there exists $\delta : [a,b] \to (0,\infty)$ such that for any δ-fine partition of $S : \{[u,v];x\}$, we have

$$|I - \sum f(x)(v-u)| < \varepsilon.$$

We denote this by $(\delta)R^*_{ap} - \int_a^b f = I$.

Remark 1.1

(1) *We note [3] and [5] that $(\delta)D^*_{ap}([a,b]) = (\delta)R^*_{ap}([a,b]) = AP([a,b])$, the last is defined by Burkill ([1], [2]).*

(2) *The definitions of $(\delta)AC^*_{ap}(E), \ldots$ are equivalent to the Definitions of $AC^*_{ap}(E), \ldots$ respectively. $F \in AC^*_{ap}(E)$ means for every $\varepsilon > 0$, there exists an $\eta > 0$ and a choice S such that for any partial division of S on $E : \{[u,v];x\}$ whenever $\sum(v-u) < \eta$ we have $\sum|F(u,v)| < \varepsilon$, and likewise $F \in ACG^*_{ap}(E)$, and so on. For details see [5]. Hence for convenience sake we also denote $F \in (\delta)AC^*_{ap}(E)$ by $F \in AC^*_{ap}(E)$ and sometimes we say F is $AC^*_{ap}(E)$, taking AC^*_{ap} as a property as well as a set of functions and likewise for the other definitions.*

(3) *The definitions of $AC^*_{ap}, ACG^*_{ap}, D^*_{ap}$ are the revised versions of those defined in [1], and appearing in [6].*

ASL: We say F satisfies the Approximately Strong Lusin Conditions, ASL on a set $H \subset \mathbb{R}$, if and only if there exists a choice S on H such that for every set E of measure zero and every $\varepsilon > 0$, there exists S_δ on H such that for any partial divisions $\{[u,v];x\}$ of S_δ on $E \cap H$, we have

$$\sum |F(u,v)| < \varepsilon.$$

We also denote this by $F \in ASL$ on H or "F is ASL on H".

$UASL$: We say $\{F_n\} \in UASL$ if and only if $F_n \in ASL$, $n = 1,2,\ldots$, with the same S and δ in their definitions.

In the following, for convenience sake, we take the primitive of any AP-integrable function $f : [a,b] \to (-\infty,+\infty)$ toe F, given by $F(x) = AP - \int_a^x f(t)dt$. However we have not lost any generality because $F(x) = G(x) - G(a)$ for any primitive G of f. We state the following two Theorems here to give some insight into the result of this paper.

Theorem 1.1 *(The AP Controlled Convergence Theorem, [5]).*

Let f and F be functions on $[a,b]$.

Given a sequence of AP-integrable functions $\{f_n\}$ and f on $[a,b]$, denote the sequence of primitives of f_n by $\{F_n\}$. If

(1) *$f_n \to f$ almost everywhere on $[a,b]$ as $n \to \infty$,*

(2) *F_n are ACG^*_{ap} uniformly in n, and*

(3) *F_n are convergent to F everywhere on $[a,b]$,*

then f is AP-integrable on $[a,b]$ and

$$F(x) = AP - \int_a^x f.$$

Theorem 1.2 *(The first equivalent of the AP Controlled Convergence Theorem).*

The following combination of conditions is equivalent to the combination of (1), (2), (3) in Theorem 1.

(1) *$f_n \to f$ almost everywhere on $[a,b]$ as $n \to \infty$,*

(2) *F_n satisfy ASL on $[a,b]$ uniformly in n,*

(3) *For every $\varepsilon > 0$, there exists closed $E \subset [a,b]$, $|[a,b] \setminus E| < \varepsilon$ such that F_n are $AC(E)$ uniformly in n (for $AC(E)$, see [4], [7]).*

(4) *$F_n \to F$ everywhere on $[a,b]$.*

2 Fundamental Definitions

Definition 2.1. Let $F : [a,b] \to \mathbb{R}$ and $E \subset [a,b]$ be given. For every choice S on E, let

$$V_{ap}(F;S;E) = \sup \sum |F(u,v)|$$

where the supremum is taken over all partial divisions $\{[u,v];x\}$ of S on E. We define the approximate variation of F on E by

$$V_{ap}(F;E) = \inf_S V_{ap}(F;S;E)$$

the infimum taken over all choices S on E.

If $V_{ap}(F;E) < \infty$, we say F is of *ap* bounded variation on E, denoted by $F \in BV^*_{ap}(E)$.

Proposition 2.1 *F is of ap bounded variation if and only if there exists a choice S on E such that for any $\varepsilon > 0$, there exists $\delta(\varepsilon) : E \to (0,+\infty)$ such that*

$$V_{ap}(F;S_\delta;E) - V_{ap}(F;E) < \varepsilon$$

PROOF. By Lemma 3.4 of [5]. □

The following functions $F_n F$ could be replaced by $F_n - F_n(a), F - F(a)$ respectively without losing generality, hence we will assume them to have values 0 at point a.

Definition 2.2. Let F and F_n be real valued functions on $[a,b]$.

(a) A sequence of functions $\{F_n\}$ is said to be *ap* variational (*apv*) convergent to F on $E \subset [a,b]$, if there exists a choice of *ap* neighbourhoods S on E such that for any given $\varepsilon > 0$, there is a $\delta : E \to (0,+\infty)$ and an $N \in \mathbb{N}$ such that $V_{ap}(F_n - F; S_\delta; E) < \varepsilon$ for any $n \geq N$.

(b) A sequence of functions $\{F_n\}$ is said to be generalized *ap* variational (*apvg*) convergent to F on $E \subset [a,b]$, if there exists $E_i \subset E$ $i = 1,2,\ldots, \bigcup_{i=1}^{\infty} E_i = E$ such that $\{F_n\}$ is *apv* convergent to F on E_i, $i = 1,2,\ldots$.

*Definition 2.2**. Let F and F_n be real valued functions on $[a,b]$.

(a) A sequence of functions $\{F_n\}$ is said to be an *apv* Cauchy sequence on $E \subset [a,b]$ if there exists a choice S on E such that for any $\varepsilon > 0$, there is a $\delta : E \to (0,+\infty)$ and an $N \in \mathbb{N}$ such that

$$V_{ap}(F_m - F_n; S_\delta; E) < \varepsilon$$

for any $n, m > N$.

(b) A sequence of functions $\{F_n\}$ is said to be *apvg* Cauchy sequence on $E \subset [a,b]$ if there exist $E_i \subset E$ $i = 1,2,\ldots, \bigcup_{i=1}^{\infty} E_i = E$ such that $\{F_n\}$ is *apv* Cauchy sequence on E_i, $i = 1,2,\ldots$.

Proposition 2.2 *A sequence of functions $\{F_n\}$ is apvg convergent to some F on $[a,b]$ if and only if $\{F_n\}$ is an apvg Cauchy sequence.*

PROOF. If: Let $x \in [a,b]$ be given, and let $E_k^x = E_k \cap [a,x]$ for $k = 1,2,\ldots$. For any $\varepsilon > 0$, we have $\delta_k : E_k \to (0,+\infty)$ and $N_k \in \mathbb{N}$ such that $V_{ap}(F_m - F_n; S_{\delta_k}; E_k^x \setminus \bigcup_{\ell=1}^{k-1} E_\ell^x) < \frac{\varepsilon}{2^k}$ for any $m,n \geq N_k$. Now let $\delta(t) = \delta_k(t)$ when $t \in E_k^x \setminus \bigcup_{\ell=1}^{k-1} E_\ell^x$, defining $E_0^x = \emptyset$. Then S_δ determines an AFC of $[a,x]$, hence we have a δ-fine partition on $[a,x]$, $\{[u_i,v_i];x_i\}$ (cf. [1]), and

$$|F_m(x) - F_n(x)| \leq \sum_i |(F_m - F_n)(u_i,v_i)| < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon,$$

whenever $m,n \geq \max_i\{N_k; \exists x_i \in E_k^x \setminus \bigcup_{\ell=1}^{k-1} E_\ell^X\}$. So $\{F_m(x) - F_n(x)\}$ is a Cauchy sequence of real numbers. Hence $\lim_{n\to\infty} F_n(x)$ exists for every $x \in$

$[a,b]$ and we denote it by $F(x)$. Now for any $\varepsilon > 0$, we have $\delta_k : E_k \to (0, +\infty)$ and $N_k \in \mathbb{N}$ such that

$$V_{ap}(F_m - F_n; S_{\delta_k}; E_k) < \varepsilon$$

whenever $m, n \geq N_k$, and hence

$$V_{ap}(F_m - F; S_{\delta_k}; E_k) \leq \varepsilon,$$

i.e. $\{F_n\}$ is *apvg* convergent to F on $[a,b]$.

Only if: For any $\varepsilon > 0$, we have $\delta_k : E_k \to (0, +\infty)$ and $N_k \in \mathbb{N}$ such that

$$V_{ap}(F_m - F_n; S_{\delta_k}; E_k) \leq V_{ap}(F_m - F; S_{\delta_k}; E_k) + V_{ap}(F - F_n; S_{\delta_k}; E_k) < 2\varepsilon,$$

hence $\{F_n\}$ is an *apvg* Cauchy sequence. □

Definition 2.3. A sequence of functions $\{f_n\}$ is said to be *ap* equi-integrable on $[a,b]$ if f_n, $n = 1, 2, \ldots$, are *AP*-integrable on $[a,b]$ with the same δ and S in the definition of $(\delta)R^*_{ap}$.

3 Some Convergence Theorems Equivalent to that of Controlled Convergence and the Problem of the Independence of the Conditions for the *AP* Controlled Convergence Theorem

We begin investigating the problem of equivalency between the condition of *AP* Controlled-Convergence Theorems and other types of conditions using *apvg* convergence or equi-integrability.

The condition of equi-integrability is especially related to the values of the integrands f_n. Generally speaking, if we change the values of f_n even only at one point, it maybe enough to destroy the equi-integrability of $\{f_n\}$. As a trivial example, let $f_n(x) = 0$ for every $x \in [0,1]$, $n = 1, 2, \ldots$, then $\{f_n\}$ is equi-integrable, but if we replace the values of f_n at any point $a \in [0,1]$ by $f_n(a) = n$, then $\{f_n\}$ is no more equi-integrable. So it is wrong if we think that once $\{f_n\}$ is equi-integrable than any $\{g_n\}$ with $g_n(x) = f_n(x)$ almost everywhere is also equi-integrable. But, however, we have the following.

Lemma 3.1 *Given a sequence of AP-integrable functions $\{f_n\}$ on $[a,b]$ such that*

(1) *$f_n \to f$ almost everywhere;*

(2) *$\{f_n\}$ is ap equi-integrable on $[a,b]$,*

then for any sequence of functions $\{g_n\}$ with $g_n(x) = f_n(x)$ almost everywhere on $[a,b]$, $\{g_n\}$ is ap equi-integrable if and only if $\{g_n(x)\}$ is bounded for every $x \in [a,b]$.

Particularly, if $g_n \to f$ everywhere, then $\{g_n\}$ is ap equi-integrable.

PROOF. We first point out that the primitives of g_n and f_n are the same, so we can denote the sequence of primitives of $\{g_n\}$ by $\{F_n\}$.

Only if: It is obvious that $g_n(x)$ is bounded at any point x with $g_n(x) \to f(x)$, so we only need to prove $g_n(x)$ is bounded at any point x with $g_n(x) \not\to f(x)$.

Let $\varepsilon > 0$ and $S\delta$ are given as in the definition of *ap* equi-integrable. Suppose there is $x \in [a,b]$ such that $g_n(x) \not\to f(x)$ and $\{g(x)\}$ is unbounded. We take a $t > 0$ be such that for any u, v with $x \in [u,v] \subset [x-t, x+t]$ we have $|[u,v] \cap S_{\delta,x}| \geq \frac{3}{4}|u-v|$. Let

$$H = \{y : g_n(y) \to f(y),\ y \in (x-t, x+t) \cap S_{\delta,x}\}.$$

Then by condition (1) we have $|H| \geq \frac{3}{4} \cdot 2t = \frac{3}{2}t$.

For every $y \in H$, we can define $\delta^*(y)$ satisfying:

(a) $\delta^*(y) \leq \delta(y)$,

(b) $x \notin [y - \delta^*(y), y + \delta^*(y)] \supset [x-t, x+t]$,

(c) for any $[\alpha, \beta]$ satisfying $y \in [\alpha,\beta] \supset [y-\delta^*(y), y+\delta^*(y)]$ we have

$$|[\alpha,\beta] \cap S_{\delta^*,y}| \geq \frac{3}{4}(\beta - \alpha).$$

Then the choice S_{δ^*} on H determine an *AFC* covering H in Vitali's sense. Hence there exists a sequence $\{[\alpha_i, \beta_i]; y_i\}$, $i = 1, 2, \ldots, p$, with $y_i \in [\alpha_i, \beta_i] \supset [y_i - \delta^*(y_i), y_i + \delta^*(y_i)]$, and $[\alpha_i, \beta_i]$ non-overlapping, such that

$$\sum |[\alpha_i, \beta_i] \cap S_{\delta^*, y_i}| > \frac{3}{4}|H| \geq \frac{1}{2}x\frac{3}{2}t = \frac{3}{4}t.$$

But $|[x-t, x+t] \cap S_{\delta,x}| \geq \frac{3}{2}t$, hence there at least exists a $y \in H$ and $\alpha,]b(\alpha < \beta)$ such that $y \in [\alpha, \beta]$ and $\alpha, \beta \in [y - \delta^*(y), y + \delta^*(y)] \cap S_{\delta^*,y} \cap S_{\delta^*,y} \cap S_{\delta,x}$ (it is possible that one of α, β is y itself).

Hence

$$\begin{aligned}|F_n(\beta) - F_n(\alpha)| &\leq |F_n(\beta) - F_n(y) - g_n(y)(\beta - y)| \\ &\quad + |F_n(\alpha) - F_n(y) - g_n(y)(\alpha - y)| + |g_n(y)||\beta - \alpha| < M\end{aligned}$$

for some $M > 0$ because $g_n(y)$ is convergent. On other hand,

$$|F_n(\beta) - F_n(\alpha)| \geq |g_n(x)||\beta - \alpha| - |F_n(\beta) - F_n(x) - g_n(x)(\beta - x)| \\ - |F_n(x) - F_n(\alpha) - g_n(x)(x - \alpha)| \geq |g_n(x)||\beta - \alpha| - 2\varepsilon.$$

If $\{g_n(x)\}$ is unbounded, so is $|F_n(\beta) - F_n(\alpha)|$. Thus we have got a contradiction. Hence $\{g_n(x)\}$ is bounded for every $x \in [a, b]$.

If: We point out that because $\{f_n\}$ is *ap* equi-integrable, so by the above proof, $\{f_n\}$ is bounded everywhere on $[a, b]$. Now suppose $\{g_n\}$ is bounded at every $x \in [a, b]$. Let $K = \{x : g_n(x) \neq f_n(x)\}$, it is easy to see that for any $\varepsilon > 0$ there exists $\delta_1 : [a, b] \to (0, +\infty)$ satisfying:

(a) Let S be given by the equi-integrability of f_n. For any partial division $\{[u', v']; x'\}$ of S_{δ_1} on $[a, b]$, we have $|\sum(f_n(u'v') - f_n(x')(v' - u')| < \varepsilon$, for all n, and

(b) for any partial division of S_{δ_1} on k, $\{[u, v]; x\}$, we have $|\sum g_n(x)(v-u)| < \varepsilon$, $|\sum f_n(x)(v - u)| \leq \varepsilon$. This can be satisfied since $f_n(x)$ and $g_n(x)$ are bounded and $|K| = 0$.

Hence if we replace $f_n(x)$ by $g_n(x)$ on K, we will have for any partition of S_{δ_1} on $[a, b]$,

$$|\sum(F_n(u', v') - g_n(x')(v' - u'))| \\ = |\sum(f_n(u', v') - f_n(x')(v' - u'))| + \sum(f_n(x') - g_n(x'))(v' - u')| < 3\varepsilon.$$

i.e., $\{g_n\}$ is *ap* equi-integrable.

In particular if the sequence $\{g_n\}$ in Lemma 1 satisfies $g_n \to f$ everywhere, then $\{g_n\}$ is bounded everywhere on $[a, b]$, and hence $\{g_n\}$ is *ap* equi-integrable. □

Theorem 3.1 *Given a sequence of AP-integrable functions $\{f_n\}$ on $[a, b]$ such that:*

(1) *$f_n \to f$ almost everywhere;*

(2) *$\{f_n\}$ is ap equi-integrable on $[a, b]$,*

then f is AP-integrable on $[a, b]$ and

$$\lim_{n \to \infty} AP - \int_a^x f_n = AP - \int_a^x f.$$

PROOF. By Lemma 1, without losing generality, we can assume that $f_n \to f$ everywhere.

By the definition of equi-integrable, there exists a choice of *ap* neighbourhoods S such that, for any given $\varepsilon > 0$, there is $\delta(\varepsilon) : [a,b] \to (0,+\infty)$ such that for any $x \in [a,b]$, and any δ-find partition $\{|u_i, v_i]; x_i\}$ of S_δ on $[a,x]$, we have

$$|\sum(F_n(u_i, v_i) - f_n(x_i)(v_i - u_i))| < \varepsilon.$$

Here F_n is the primitive of f_n. On the other hand, there is an $N \in \mathbb{N}$ such that whenever $m, n \geq N$, for each of the points x_i we have $|f_m(x_i) - f_n(x_i)| < \varepsilon$. Hence

$$\begin{aligned}|F_m(x) - F_n(x)| \leq |\sum(F_m(u_i, v_i) - f_m(x_i)(v_i - u_i))\\ + \sum(f_n(x_i) - f_m(x_i))(v_i - u_i)|\\ + |\sum(f_n(x_i)(v_i - u_i) - F_n(u_i, v_i))| < \varepsilon[2 + (x - a)].\end{aligned}$$

Hence $\{F_n(x)\}$ is a Cauchy sequence for any $x \in [a,b]$, so $\lim_{n\to\infty} F_n(x) = F(x)$ exists for every $x \in [a,b]$, and $\sum |F(u_i, v_i - f(x_i)(v_i - u_i)| < \varepsilon$ for any partition $\{[u_i, v_i]; x_i\}$ of S_δ on $[a,x]$, i.e. f is AP-integrable on $[a,b]$ and

$$\lim_{n\to\infty} AP - \int_a^x f_n = \lim_{n\to\infty} F_n(x) = F(x) = AP - \int_a^x f. \qquad \square$$

Lemma 3.2 *Given a sequence of AP-integrable functions* $\{f_n\}$ *on* $[a,b]$ *with sequence of primitives* $\{F_n\}$, *if* $f_n \to f$ *everywhere, and* F_n *apvg converges to* F, *then* $\{f_n\}$ *is ap equi-integrable.*

PROOF. Let $\{E_k\}$ be such that $[a,b] = \bigcup_{k=1}^{\infty} E_k$ and $\{F_n\}$ is *apv* convergent to F on E_k, $k = 1, 2, \ldots$. Without loss of generality, let $\{E_k\}$ be pairwise disjoint.

For any given $\varepsilon > 0$, let, for $p \geq 1$,

$$K_p = \{x : |f_n(x) - f_m(x)| < \varepsilon, m, n \geq p\}.$$

Then $K_p \subset K_{p+1}$ for $p = 1, 2, \ldots$. Let

$$H_p = K_p \setminus K_{p-1} \text{ with } p = 2, 3, \ldots, \text{ and } H_1 = K_1,$$

then $[a,b] = \bigcup_{p=1}^{\infty} H_p$. Let S be a choice on $[a,b]$ corresponding to a *apv* convergence of F_n to F on E_k for each k.

On $H_p \cap E_k$, there exist a $\delta_{pk} : H_p \cap E_k \to (0,+\infty)$ and an $N_{p,k}$ such that whenever $m, n \geq N_{p,k}$, for any partial division $\{[u,v]; x\}$ of $S_{\delta_{p,k}}$ on $H_p \cap E_k$, we have

$$\sum |(F_m - F_n)(u,v)| < \frac{\varepsilon}{2^{p+k}},$$

and $|f_m(x) - f_n(x)| < \varepsilon$ everywhere on $H_p \cap E_k$. Diminish $\delta_{p,k}$ on $H_p \cap E_k$ if necessary so that

$$\sum |F_n(u,v) - f_n(x)(u-v)| < \frac{\varepsilon}{2^{p+k}},\ n = 1, 2, \ldots, N_{p,k}$$

for any partial division of $S_{\delta_{p,k}}$ on $H_p \cap E_k$.

Then for any partial division of $S_{\delta_{p,k}}$ on $H_p \cap E_k$, and any $n > N_{p,k}$ we have

$$\begin{aligned}\sum |F_n(u,v) - f_n(x)(v-u)| &\leq \sum |(F_n - F_{N_{p,k}})(u,v)| \\ &+ \sum |F_{N_{p,k}}(u,v) - f_{N_{p+k}}(x)(v-u)| \\ &+ \sum |f_{N_{p,k}}(x) - f_n(x)|(v-u) \leq \frac{\varepsilon}{2^{p+k-1}} + \varepsilon \sum (v-u).\end{aligned}$$

With p fixed and the same process applied to every H_p, $p = 1, 2, \ldots$, then we have a choice S_δ on $[a,b]$ such that for any partial division of it on $[a,b]$ we have

$$\sum |F_n(u,v) - f_n(x)(v-u)| < \sum \frac{\varepsilon}{2^{p-1}} + \varepsilon(b-a) < (2+b-a)\varepsilon,$$

i.e. $\{f_n\}$ is *ap* equi-integrable. □

Theorem 3.2 *Let $\{f_n\}$ be a sequence of AP-integrable functions on $[a,b]$. Denote the sequence of primitives of f_n by $\{F_n\}$. Suppose $\{f_n\}$ converges to a function f almost everywhere on $[a,b]$. The following conditions (A), (B), (C), (D) are mutually equivalent, and any one of (A), (B), (C), (D) implies f is AP-integrable on $[a,b]$ and*

$$\lim_{n\to\infty} AP - \int_a^x f_n = AP - \int_a^x f.$$

(A) *$\{F_n\} \in UACG^*_{ap}([a,b])$;*

(B) (1) *$\{F_n\}$ satisfies $UASL$ on $[a,b]$,*

(2) *for every $\varepsilon > 0$, there exists closed $E \subset [a,b]$, $|[a,b] \setminus E| < \varepsilon$ such that $\{F_n\} \in UAC(E)$ (for $AC(E)$ see [4], [7]);*

(C) *$\{F_n\}$ is apvg convergent to a function F on $[a,b]$;*

(D) *After having redefined $\{f_n(x)\}$ at $x \in \{x : f_n(x) \not\to f(x)\}$ so that $\{f_n(x)\}$ is bounded we have $\{f_n\}$ being ap equi-integrable on $[a,b]$.*

PROOF. The equivalence between (A) and (B): see [5], Corollary 4.3 and its note.

Before we prove "(B) implies (C)", we prove the following Lemma 3 and Lemma 4.

Lemma 3.3 *If an AP-integrable function g has primitive G being $AC(E)$ on a closed $E \subset [a,b]$, then $g\chi_E$ is AP-integrable on $[a,b]$ and*

$$AP - \int_a^b g\chi_E(x)dx = G(b) - G(a) - \sum_{k=1}^{\infty} G(a_k, b_k),$$

where (a_k, b_k) are the components of $(a,b) \setminus E$, and χ_E is the characteristic function of E.

PROOF OF LEMMA 3. Let H be the set of all points of density in E, then $|E \setminus H| = 0$ (see [7]). Let $\varepsilon^* > 0)$ be given. Take η from the definition of $G \in AC(E)$, then take N so that $\sum_{i=N}^{\infty}(b_i - a_i) < \eta$. By the AP-integrabiilty of g there exists a choice of ap neighbourhood S with $S_x = S_x \cap H \subset H$ when $x \in H$, such that for any $\varepsilon^* > 0$, there exists $\delta : [a,b] \to (0, +\infty)$ corresponding to ε^* as in the definition of $(\delta)R^*_{ap}$ integration.

We can choose suitable δ satisfying:

(1) $\delta(x) < \min_{1 \le k \le N}\{\eta, b_k - a_k\}$ for every $x \in [a,b]$, and when $x \in (a_k, b_k)$, $k \in \mathbb{N}$ we have $\delta(x) < \min\{x - a_k, b_k - x\}$,

(2) $\sum_i |G(u_i, v_i) - g(x_i)(v_i - u_i)| < \varepsilon^*$ for any partial division $\{[u_i, v_i]; x_i\}$ of S_δ on $[a,b]$,

(3) for any partial division $\{[u_i, v_i]; x_i\}$ of S_δ on $E \setminus H$, we have

   (a) $\sum(v_i - u_i) < \eta$ (because $|E \setminus H| = 0$),

   (b) $\sum_i |G(u_i, v_i)| < \varepsilon^*$ by G being ASL, and

   (c) $\sum_i |g(x_i)(v_i - u_i)| < \varepsilon^*$ by $g(x)$ being finite everywhere.

For any partition $\{[u_i, v_i], x_i\}$ of S_δ on $[a,b]$, we define

$$\Omega = |\sum_i g\chi_E(x_i)(v_i - u_i) - \{G(b) - G(a) - \sum_{k=1}^{\infty} G(a_k, b_k)\}|.$$

We let $\sum_i$ be the sum for those associated points $x_i \in H$, and $\sum_2$ be for $x_i \in E \setminus H$, giving

$$\begin{aligned}\Omega = |\sum_1 g\chi_E(x_i)(v_i - u_i) + \sum_2 g\chi_E(x_i)(v_i - u_i) \\ - \sum_1 G(u_i, v_i) - \sum_2 G(u_i, v_i) + \sum_1 G(u_i, v_i) \\ + \sum_2 G(u_i v_i) - \{G(b) - G(a) - \sum_{k=1}^{\infty} G(a_k, b_k)\}|.\end{aligned}$$

When $x_i \notin E$ we have $g\chi_E(x_i)(v_i - u_i) = 0$, so these terms disappear from the sum.

Since $S_x \subset H$ when $x \in H$, the u_i, v_i appearing in $\sum_1$ must be in E. The u_i, v_i appearing in $\sum_2$ may be in E or not. For each u_i not in E, u_i must be in (a_k, b_k) for a certain k, and for each v_j not in E, v_j must in (a_ℓ, b_ℓ) for a certain ℓ. If we replace u_i by b_k, and v_j by a_ℓ, we get a sum $\sum_2 G(u_i', v_i')$ with $u_i', v_i' \in E$ and by condition (3)(a) we have $\sum_2(v_i' - u_i') < \sum_2(v_i - u_i) < \eta$, hence $\sum_2 |G(u_i', v_i')| < \varepsilon^*$.

Let $\zeta = \sum_2(G(u_i, v_i) - G(u_i', v_i'))$, then $|\zeta| < 2\varepsilon^*$. Rewrite Ω as:

$$\begin{aligned}\Omega = |\sum_1 g(x_i)(v_i - u_i) + \sum_2 g(x_i)(v_i - u_i) \\ - \sum_1 G(u_i, v_i) - \sum_2 G(u_i, v_i) + \sum_1 G(u_i, v_i) \\ + \sum_2 G(u_i', v_i') + \zeta + \sum_3 G(a_k, b_k) + \sum_4 G(a_k, b_k) - G(b) + G(a)|;\end{aligned}$$

here $\sum_3$ is the sum for those k such that for every (u_i, v_i) appearing in $\sum_1$ or $\sum_2$, $(a_k, b_k \not\subset (u_i, v_i)$. By the condition (1) for δ, $G(a_k, b_k)$ with $k = 1, 2, \ldots, N$ appear in $\sum_3$. And $\sum_4$ is the sum for those k other than those in $\sum_3$. Hence we have

$$\sum_1 G(u_i, v_i) + \sum_2 G(u_i', v_i') + \sum_3 G(a_k, b_k = G(b) - G(a).$$

Also $\sum_4 |G(a_k, b_k)| < \varepsilon^*$. So

$$\begin{aligned}\Omega = |\sum_1 g(x_i)(v_i - u_i) + \sum_2 g(x_i)(v_i - u_i) - \sum_1 G(u_i, v_i) - \sum_2 G(u_i, v_i) \\ + G(b) - G(a) + \zeta + \sum_4 G(a_k, b_k) - G(b) + G(a)| < \varepsilon^* + 2\varepsilon^* + \varepsilon^* = 4\varepsilon^*.\end{aligned}$$

Thus Lemma 3 has been proved. □

Note: In the proof of Lemma 3, we only have used the ASL condition of G, and the quality of η that whenever $\sum_{k\geq N}^{\infty}(b_k - a_k) < \eta$ for some N we have $\sum |G(a_k, b_k)| < \varepsilon^*$. But this quality also holds for $G \in BV(E)$. So we have:

Lemma 3.3* *If we replace $AC(E)$ by $BV(E)$ in the conditions of Lemma 3, then the conclusion of Lemma 3 holds.*

Lemma 3.4 *Let $\{F_n\}$ be a sequence of functions which are uniformly $AC(E)$ where E is a closed subset of $[a,b]$. Let H be the set of all points of density in E. Then there exists a choice S of ap neighbourhoods on $[a,b]$ such that for any $\varepsilon > 0$, there is $\delta : [a,b] \to (0,+\infty)$ such that for any partial division of S_δ on H, $\{[u_i,v_i];x_i\}$, we have*

$$|\sum F_n(u_i,v_i) - \{F_n(b) - F_n(a) - \sum_{k=1}^{\infty} F_n(c_k,d_k)\}| < \varepsilon, \quad n - 1,2,\ldots$$

where (c_k,d_k) are the components of $(a,b) \setminus \{E \cap (\bigcup_i[u_i,v_i])\}$.

PROOF. We first refine S by $S_x \cap H$ when $x \in H$ and after having refined we keep denoting the choice by S. Let $(a,b) \setminus E = \bigcup_{i=1}^{\infty}(a_i,b_i)$ where (a_i,b_i) are nonoverlapping. Take η corresponding ε from the definition of $F_n \in UAC(E)$ and N so that $\sum_{i=N}^{\infty}(b_i - a_i) < \eta$. Let δ be defined as follows. For any $x \in H$, we denote the distance from x to $\bigcup_{k=1}^{N}[a_k, b_k$ by $\rho(x)$. Since very a_k, b_k are not points of density of E, so $\rho(x) > 0$, we define $\delta(x) < \min\{\eta, \rho(x)\}$. Hence for any partial division $\{[u_i,v_i];x_i\}$ of S_δ on H, by the definition of δ, we have: when $k = 1,2,\ldots,N, (a_k,b_k)$ are outside the $\bigcup_i[u_i,v_i]$ and every $c_k,d_k)$ included in one of $[u_i,v_i]$ actually is one of (a_k,b_k) with $k > N$. Hence we have

$$|\sum F_n(u_i,v_i) - \{F_n(b) - F_n(a) - \sum_{k=1}^{\infty} F_n(c_k,d_k)\}| = |\sum_4 F_n(c_k,d_k)| < \varepsilon,$$

where (c_k,d_k) appearing in $\sum_4$ are those included in one of $[u_i,v_i]$. Hence Lemma 4 is proved. □

Corollary 3.1 *Let the conditions of Lemma 4 hold. Further, suppose that each F_n is the primitive of an AP integrable function f_n, $n = 1,2,\ldots$. Then there exists a choice S of ap neighbourhoods on $[a,b]$, such that for any $\varepsilon > 0$, there is $\delta : [a,b] \to (0,+\infty)$ such that for any partial division of S_δ on H, $\{[u_i,v_i];x_i\}$, we have*

$$|\sum F_n(u_i,v_i) - AP - \int_{E^*} f_n| < \varepsilon, \ n = 1,2,\ldots$$

where $E^* = E \cap (\bigcup_i[u_i,v_i])$.

PROOF. By Lemma 3.

$$AP - \int_{E^*} f_n = F_n(b) - F_n(a) - \sum_{k=1}^{\infty} F_n(c_k, d_k),\ n = 1, 2, \ldots$$

where (c_k, d_k) are defined as in Lemma 4. Hence by Lemma 4 we have the Corollary proved. □

Now let us prove "*(B) implies (C)*".

For any given $\varepsilon > 0$, there exists a closed $E \subset [a, b], |[\alpha, b] \setminus E| < \frac{\varepsilon}{2}$ such that $F_n \in AC(E)$ uniformly and by Egoroff's theorem ([4]), we have closed $H \subset E$, $|[a, b] \setminus H| < \varepsilon$ such that $f_n \to f$ uniformly on H. Hence there is a sequence of closed subsets $\{E_k\}$ of $[a, b]$ such that $|[a, b] \setminus \bigcup_{k=1}^{\infty} E_k| = 0$, and $F_n \in AC(E_k)$ uniformly and $f_n \to f$ uniformly on E_k for every $k \in \mathbb{N}$.

Let H_k be the set of points of density in E_k, then we also have $|[a, b] \setminus \bigcup_{k=1}^{\infty} H_k| = 0$. Hence for every $k \in \mathbb{N}$, and $\varepsilon^* > 0$ there is $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \varepsilon^*$ for any $x \in H_k$ when $n, m \geq N$. By Corollary 1 there is a choice S and a $\delta : H_k \to (0, +\infty)$ such that for any partial division $\{[u, v]; x\}$ of S_δ on H_k, we have $|\sum F_n(u, v) - AP - \int_{E^*}$ for $n = 1, 2, \ldots$. Hence for any $n, m \geq N$, we have

$$\begin{aligned} |\sum(F_m(u, v) - F_n(u, v)| &\leq |\sum F_m(u, v) - AP - \int_{E^*} f_m| \\ &+ |\sum F_n(u, v) - AP - \int_{E^*} f_n| + |AP - \int_{E^*} (f_m - f_n)| < 2\varepsilon^* + \varepsilon^*|E^*|, \\ E^* &= H_k \cap \left(\bigcup_i [u_i, v_i]\right). \end{aligned}$$

Since the inequality is for all partial divisions,

$$\sum |F_m(u, v) - F_n(u, v)| < (4 + 2|E^*|)\varepsilon^*.$$

Lastly, since $K = [a, b] \setminus \cup H_k$ is of measure zero, by condition (1) of (B), $\{F_n\}$ satisfy ASL uniformly. Hence for any partial division $\{[u_i, v_i]; x_i\}$ of a suitable S_δ on K and any $m, n \in \mathbb{N}$, we have

$$\sum |(F_m - F_n)(u_i, v_i)| \leq \sum |F_m(u_i, v_i)| + \sum |F_n(u_i, v_i)| < 2\varepsilon^*.$$

So $\{F_n\}$ is an *apvg* Cauchy sequence on $[a, b]$, and hence $\{F_n\}$ satisfies (C).

(C) implies (D): By Lemma 2 and Lemma 1.

(D) implies (A):

By Lemma 1, after redefinition on a subset of measure zero $\{f_n(x)\}$ is bounded for every $x \in [a,b]$. Let

$$E_k = \{x : |f_n(x)| < k,\ n = 1,2,\dots\}$$

then $[a,b] = \bigcup_k E_k$.

For every E_k, given $\varepsilon > 0$ let $\delta_k : [a,b] \to (0,+\infty)$ be such that for every partial division of E_k, $\{[u_i,v_i];x_i\}$, we have

$$|\sum(F_n(u_i,v_i) - f_n(x_i)(v_i - u_i)| < \varepsilon,$$

and hence $\sum|F_n(u_i,v_i) - f_n(x_i)(v_i - u_i)| < 2\varepsilon$.

Let $\eta = \frac{\varepsilon}{k}$, then when $\sum(v_i - u_i) < \eta$, we have

$$\sum|F_n(u_i,v_i)| < 2\varepsilon + |f_n(x_i)|\sum(v_i - u_i) < 2\varepsilon + k\eta = 3\varepsilon.$$

Hence $F_n \in AC^*_{ap}(E_k)$ uniformly, and (A) is true. □

Corollary 3.2 *The condition that $F_n \to F$ everywhere in $[a,b]$ is implied by the other conditions in the Controlled Convergence Theorem (i.e. Theorem 1.1) and likewise by the other conditions in the First Equivalent of the Controlled Convergence Theorem (i.e. Theorem 1.2).*

It is easy to see that the Conditions of (A) together with the Condition $f_n \to f$ almost everywhere in $[a,b]$ are mutually independent. Likewise, the Conditions (or Condition) in any one of (B), (C), (D) together with the Condition $f_n \to f$ almost everywhere in $[a,b]$ are mutually independent. Hence we have got the least number of Conditions for the Controlled Convergence Theorem.

4 Further Weakening of the Conditions for the *AP* Controlled Convergence Theorem and Some Other Convergence Theorems

Now we generalize Theorem 3.2 by weakening the condition that $f_n \to f$ almost everywhere in $[a,b]$. For this purpose, we first make a few remarks about the question of absolute AP-integrability, and then we give a definition of "generalized *ap* mean convergence" and some properties of it.

We call a function f absolutely AP-integrable if $|f|$ is AP-integrable. But we will prove that this definition is nothing but absolutely Henstock integrable. More precisely, we have

Lemma 4.1 *Given a function $f : [a,b] \to \mathbb{R}$, if there exists a choice S_δ on $E \subset [a,b]$ such that*

$$\sup \sum |f(x)|(v-u) < +\infty$$

with the supremum over all partial divisions $\{[u,v];x\}$ of S_δ on E, then f is absolutely Henstock integrable, i.e. Lebesgue integrable, on E, and vice versa.

PROOF. Let $f_N : f_N(x) = \max\{-N, \min(f(x), N)\}$ with $N \in \mathbb{N}$. Then f_N is a bounded measurable function, hence f_N is absolutely Henstock integrable, i.e. Lebesgue integrable, and

$$\int_E |f_N| \leq \sup \sum |f(x)|(v-u).$$

Hence by the Monotone Convergence Theorem of Lebesgue integration, we have $|f|$ being Lebesgue integrable.

The reverse is direct. □

Corollary 4.1 *Given a function f on $E \subset [a,b]$, f is absolutely AP-integrable if and only if f is absolutely Henstock integrable.*

Hence we will not distinguish absolutely AP-integrable from absolutely Henstock integrable, or Lebesgue integrable.

Lemma 1 means that if any question can be reduced into a question of absolutely AP-integrable functions, then it can be solved by means of Lebesgue integration. In particular, we have.

Corollary 4.2 *(The Monotone Convergence Theorem for AP-integration). Let f_n; $n = 1, 2, \ldots$, be AP-integrable functions on $[a,b]$, such that:*

(1) *f_n converge to a function f almost everywhere in $[a,b]$ as $n \to \infty$;*

(2) *$f_1(x) \leq f_2(x) \leq \cdots \leq f_n(x) \leq \cdots$ for almost all $x \in [a,b]$;*

(3) *$AP - \int_a^b f_n < M$ for some $M > 0$.*

Then f is AP-integrable and

$$AP - \int_a^b f = \lim_{n\to\infty} AP - \int_a^b f_n.$$

Corollary 4.3 *(Dominated Convergence Theorem for AP-integration). Let f_n, $n = 1, 2, \ldots$, be AP-integrable function on $[a,b]$. If:*

(1) *f_n converges to a function f almost everywhere in $[a,b]$ as $n \to \infty$, and*

(2) *there exist AP integrable functions $g(x)$ and $h(x)$ on $[a,b]$ such that $g(x) \leq f_n(x) \leq h(x)$, $n = 1, 2, \ldots$, almost everywhere in $[a,b]$,*

then f is AP-integrable on $[a,b]$ and

$$AP - \int_a^b f = \lim_{n\to\infty} AP - \int_a^b f_n \text{ as } n \to \infty.$$

What is important for us is that, by virtue of Lemma 1, the following definition become meaningful.

Definition 4.1. A sequence of functions $f_n : [a,b] \to \mathbb{R}$ is said to be generalized mean (mg) convergent to $f : [a,b] \to \mathbb{R}$ if there exists a sequence of sets $E_k \subset [a,b], [a,b] = \bigcup_{k=1}^{\infty} E_k$ such that

$$\int_{E_k} |f_n - f| \to 0 \quad k = 1, 2, \ldots .$$

We give the following properties of mg convergence, where $\{f_n\}$ is a sequence of functions on $[a,b]$, $\{F_n\}$ is the sequence of primitives if f_n are AP-integrable, f and F are functions on $[a,b]$, $\{E_k\}$ are as in the definition of mg convergence, or of $apvg$ convergence in accordance with the context.

Proposition 4.1 *If $\{f_n\}$ is mg convergent, then its limit function is unique.*

Proposition 4.2 *$\{f_n\}$ is mg convergent if and only if there exists $\{E_k\}$ as above such that*

$$\int_{E_k} |f_n - f_m| \to 0 \text{ as } m, n \to \infty$$

for $k = 1, 2, \ldots$.

Proposition 4.3 *If $\{f_n\}$ is mg convergent to f, then there exists a subsequence of $\{f_n\}, \{f_{n_k}\}$, converging to f almost everywhere in $[a,b]$.*

PROOF. Mean convergence of a sequence $\{f_n\}$ to f on E_k implies that $\{f_n\}$ also converges to f in measure on E_k (cf. e.g. H.L. Royden's Real Analysis, third edition p. 95). Hence there exist subsequences $\{f_{n,k}\}$ of $\{f_n\}$ converging to f on E_k almost everywhere for every $k \in \mathbb{N}$, and $\{f_{n,k}\} \supset \{f_{n,k+1}\}$. So the sequence $\{f_{k,k}\}$ converges to f on $[a,b]$ almost everywhere. □

Proposition 4.4 *If $\{F_n\}$ apvg converges to F, then $\{f_n\}$ is mg convergent to a function f and $F(x) = AP - \int_a^x f$.*

PROOF. By $\{F_n\}$ *apvg* converging to F, for every E_k, given any $\varepsilon > 0$, there exist $N \in \mathbb{N}$ and δ_1 such that for any partial division $\{[u, v]; x\}$ of S_{δ_1} on E_k we have

$$\sum |(F_m - F_n)(u, v)| < \varepsilon \text{ for } m, n \geq N.$$

Let $m, n \geq N$ be temporally fixed; there is δ_2 such that for any partial division $\{[u, v]; x\}$ of S_{δ_2} on E_k we have

$$\sum |F_m(u, v) - f_m(x)(v - u)| < \varepsilon; \quad \sum |F_n(u, v) - f_n(x)(v - u)| < \varepsilon.$$

Hence for any partial division S_δ with $\delta = \min\{\delta_1, \delta_2\}$ we have

$$\sum |(f_m - f_n)(x)|(v - u) < 3\varepsilon \text{ as } m, n \geq N.$$

Hence by Lemma 1, we have

$$\int_{E_k} |f_m - f_n| < 3\varepsilon,$$

and by Proposition 1 and Proposition 2, there exists a function f uniquely such that $f_n \to f(mg)$.

Now we prove $F(x) = \int_a^x f$. Given any $\varepsilon > 0$, there exist $N_k \in \mathbb{N}$ and a choice $S_{\delta_{k,1}}$ such that for any partial division $\{[u, v]; x\}$ of $S_{\delta_{k,2}}$ on E_k we have

$$\sum |(f - f_m)(x)|(v - u) < \frac{1}{3} \cdot \frac{\varepsilon}{2^k},$$

and

$$\sum |F_m(u, v) - f_m(x)(v - u)| < \frac{1}{3} \frac{\varepsilon}{2^k}.$$

Let $\delta_k = \min\{\delta_{k,1}, \delta_{k,2}\}$, we have

$$\sum |F(u, v) - f(x)(v - u)| < \frac{\varepsilon}{2^k}$$

for any partial division $\{[u, v]; x\}$ of S_{δ_k} on E_k.

Letting $\delta : \delta(x) = \delta_k(x)$ when $x \in E_k \setminus \bigcup_{i=0}^{k-1} E_i$ with $E_0 = \emptyset$, $k = 1, 2, \ldots$, we have

$$\sum |F(u, v) - f(x)(v - u)| < \varepsilon$$

for any partial division $\{[u, v]; x\}$ of S_δ on $[a, b]$, i.e.

$$F(x) = AP - \int_a^x f. \qquad \square$$

Proposition 4.5 *If $\{F_n\}$ apvg converges to F, then there exists a subsequence of $\{f_n\}$, $\{f_{n_k}\}$, which converges to a function f almost everywhere in $[a,b]$, and*

$$F(x) = AP - \int_a^x f.$$

PROOF. By Proposition 3 and Proposition 4. □

Now let us turn to setting up a weaker condition, in the presence of the conditions on F_n, to replace the condition that $f_n \to f$ almost everywhere in $[a,b]$ in the Controlled Convergence Theorem.

Lemma 4.2 *Let $f : [a,b] \to \mathbb{R}$ be AP-integrable on $[a,b]$ with primitive F. Then f is absolutely integrable on $E \subset [a,b]$ if and only if $F \in AC^*_{ap}(E)$.*

PROOF. "Only if": We have $G(x) = \int_a^x |f|\chi_E$ absolutely continuous. Hence for any $\varepsilon > 0$, there exists $\eta > 0$ corresponding to ε as in the definition of absolute continuity, and at the same time there exists $\delta_1 : [a,b] \to (0,+\infty)$ such that for any δ_1-fine partial division $\{[u,v];x\}$ on E (see [4])

$$\sum ||f(x)|(v-u) - |G(u,v)|| < \varepsilon.$$

On the other hand, there exists $\delta_2 : [a,b] \to (0,+\infty)$ such that for any partial division $\{[u,v];x\}$ of S_{δ_2} on E we have

$$\sum ||f(x)|(v-u) - |F(u,v)|| < \varepsilon.$$

Letting $\delta = \min\{\delta_1, \delta_2\}$, we have for any partial division $\{[u,v];x\}$ of S_δ on E

$$\sum ||F(u,v)| - |G(u,v)|| < 2\varepsilon,$$

hence

$$\sum |F(u,v)| < \sum |G(u,v)| + 2\varepsilon.$$

Particularly if $\sum(v-u) < \eta$, we have $\sum |F(u,v)| < 3\varepsilon$, i.e. $F \in AC^*_{ap}(E)$.

"If": $F \in AC^*_{ap}(E)$, so there exists a choice S_δ such that for any partial division $\{[u,v];x\}$ of S_δ on E, we have

$$\sum |F(u,v) - f(x)(v-u)| < \varepsilon$$

and $\sum |F(u,v)| < M$ for some $M > 0$. Hence

$$\sum |f(x)|(v-u) < \sum |F(u,v)| + \varepsilon < M + \varepsilon.$$

i.e. $\int_E |f| < +\infty$. □

Lemma 4.3 *If* $\{F_n\}$ *is apvg convergent to* F *on* $[a,b]$, *then* $\{F_n\}$ *is* $UACG^*_{ap}([a,b])$.

PROOF. By Proposition 4 there exists a function f on $[a,b]$ such that $f_n \to f(mg)$ and $F(x) = AP - \int_a^x f$. Hence there exist $E_1, E_2, \ldots, E_n, \ldots$ with $[a,b] = \bigcup_i E_i$, $F \in AC^*_{ap}(E_i)$. On the other hand, there exist $H_1, H_2, \ldots, H_n, \ldots$ such that for any H_j, there is $N_j \in \mathbb{N}$ such that for $n \geq N_j$, $\int_{H_j} |f_n - f| < +\infty$. Hence $F_n - F \in AC^*_{ap}(H_j)$ whenever $n \geq N_j$, and therefore we have $F_n \in AC^*_{ap}(H_j \cap E_i)$ $i = 1, 2, \ldots$.

$$F_k \in AC^*_{ap}(K_{k,\ell}) \quad k = 1, 2, \ldots, N_{ji} \quad \ell = 1, 2, \ldots .$$

Let $K_{\ell_1,\ell_2,\ldots,\ell_{N_j}} = \bigcap_{k=1}^{N_j} K_{k,\ell_k} \ell_k = 1, 2, \ldots$ for any $k = 1, 2, \ldots, N_j$. Hence

$$F_n \in AC^*_{ap}(K_{\ell_1,\ell_2,\ldots,\ell_{N_j}'} \cap H_j \cap E_i) \quad n = 1, 2, \ldots$$

with $\ell_p = 1, 2, \ldots$, for $p = 1, 2, \ldots, N_j$; $j = 1, 2, \ldots$; $i = 1, 2, \ldots$.

Rearrange $K_{\ell_1,\ell_2,\ldots,\ell_{N_j}} \cap H_j \cap E_i$ into a sequence and still denote this sequence by $\{E_i\}$, and we have

$$F_n \in AC^*_{ap}(E_i) \quad n = 1, 2, \ldots, \text{ and } F \in AC^*_{ap}(E_i) \text{ for } i = 1, 2, \ldots .$$

Now we prove $F_n \in UAC^*_{ap}(E_i)$ for $i = 1, 2, \ldots$.

For every $i \in \mathbb{N}$, given $\varepsilon > 0$, we have $N \in \mathbb{N}$, S_{δ_1} and $\eta_1 > 0$ such that for any partial division $\{[u,v]; x\}$ of S_{δ_1} on E_i, we have

$$\sum |(F_n - F)(u,v)| < \varepsilon \text{ for } n \geq N,$$

and whenever $\sum(v-u) < \eta_1$, we have $\sum |F(u,v)| < \varepsilon$. On other hand, there exist $\eta_2 > 0$ and S_{δ_2} such that for any partial division of S_{δ_2} on E_i, whenever $\sum(v-u) < \eta_2$ we have

$$\sum |F_k(u,v)| < \varepsilon \text{ with } k = 1, 2, \ldots, N.$$

Let $\delta = \min\{\delta_1, \delta_2\}$, and $\eta = \min\{\eta_1, \eta_2\}$, then for any partial division of S_δ, $\{[u,v]; x\}$, whenever $\sum(v-u) < \eta$ we have

$$\sum |F_n(u,v)| < 2\varepsilon, \quad n = 1, 2, \ldots .$$

i.e. $\{F_n\} \in UAC^*_{ap}(E_i)$ with $i = 1, 2, \ldots$. Hence $\{F_n\} \in UACG^*_{ap}([a,b])$. □

Now let us propose the following refinement of Theorem 3.2.

Theorem 4.1 *Let $\{f_n\}$ be a sequence of AP-integrable functions on $[a,b]$, and let $\{F_n\}$ be the sequence of the primitives of f_n. The following (A*), (B*), and (C*) are mutually equivalent.*

(A*) (1) *$\{f_n\}$ is mg convergent to a function f,*

(2) *$\{F_n\} \in UACG^*_{ap}([a,b])$;*

(B*) (1) *$\{f_n\}$ is mg convergent to a function f,*

(2) *$\{F_n\}$ satisfies UASL on $[a,b]$,*

(3) *for every $\varepsilon > 0$, there exists closed $E \subset [a,b], |[a,b] \setminus E| < \varepsilon$ such that $\{F_n\} \in UAC(E)$ (i.e. the η in the definition of AC is independent of n);*

(C*) *$\{F_n\}$ is apvg convergent to a function F on $[a,b]$.*

Hence (C) implies $f_n$mg converges to a function f. Any one of (A*), (B*), (C*) implies f is AP-integrable and* $\lim_{n\to\infty} AP - \int_a^x f_n = AP - \int_a^x f$.

Note: Since $\{f_n\}$ being equi-integrable strongly relies on the "pointwise" boundedness of $\{f_n\}$, as we have pointed out at the beginning of paragraph 3, we still have not any equivalent form of it in this theorem.

PROOF. . (A*) is equivalent to (B*): By the equivalent between (2) of (A*) and (2), (3) of (B*) (cf. [5]).

(B*) implies (C*): By investigating the proof for "(B) implies (C)" in the proof of Theorem 3.2, in order that (C*) is true, we actually only use condition (1), (2) of (B), i.e. (2), (3) of (B*), and the following condition:

There exists a sequence of closed subset of $[a,b]$, E_k, such that

$$|[a,b] \setminus \bigcup_{k-1}^{\infty} E_k| = 0 \text{ and } |\int_{E_k^*} (f_m - f_n)| \to 0$$

where $E_k^* = E_k \cap [\bigcup_i [u_i, v_i]$ with $\{u_i, v_i; x_i\}$ being any partial division of some suitable S_δ on the set of all points of density in E_k.

But this is implies by (B*) (1) hence we finish the proof.

(C*) implies (A*): By Lemma 3 and Proposition 4. □

Finally, we point out the relation between Theorem 3.2 and Theorem 4.1 is

Corollary 4.4 *(A), (B), (C) implies (A*), (B*), (C*) representively, and conversely, if $\{f_n\}$ satisfies (A*), (B*), (C*), then there exists a subsequence of $\{f_n\}$ satisfying (A), (B), (C).*

5 Riesz-Type Definition of AP-Integral

Lemma 5.1 *Let $\{F_n\}$ be a sequence of functions on $[a,b]$, If $\{F_n\}$ is UAC on closed sets K_1 and K_2 with $K_1 \cap K_2 = \emptyset$, then $\{F_n\}$ is $UAC(K_1 \cup K_2)$.*

PROOF. For any $\varepsilon > 0$, there are $\eta_i > 0$ with $i = 1,2$ defined as in the definitions of $\{F_n\} \in UAC(K_i)$. Let ρ be the distance from K_1 to K_2, then $\min\{\eta_1, \eta_2, \rho\}$ will satisfies the condition on η as in the definition of $\{F_n\} \in UAC(K_1 \cup K_2)$. □

Lemma 5.2 *If $\{F_n\} \in UACG(E)$, then there exist closed $E_1, E_2, \ldots, E_r, \ldots$ such that $E_1 \subset E_2 \subset \ldots \subset E_r \subset \ldots, E = \bigcup_{r=1}^{\infty} E_r \cup H$ with $|H| = 0$, and $\{F_n\} \in UAC(E_r)$, $r = 1, 2, \ldots$.*

PROOF. First, we prove that for every $\varepsilon > 0$, there is a closed $K \subset E$ such that $\{F_n\} \in UAC(K)$ and $|E \setminus K| < \varepsilon$.

Let H_k, $k = 1, 2, \ldots$ be such that $E = \bigcup_{k=1}^{\infty} H_k$, $\{F_n\} \in UAC(H_k)$, $k = 1, 2, \ldots$. Let $H_0 = \emptyset$, K_k closed be such that $K_k \subset H_k \setminus \bigcup_{i=0}^{k-1} H_i$ with $|H_k \setminus (\bigcup_{i=0}^{k-1} H_i) \setminus K_k| < \frac{\varepsilon}{2^{k+1}}$, $k = 1, 2, \ldots$, and let $N \in \mathbb{N}$ be such that $|E \setminus \bigcup_{k=1}^{N} H_k| < \frac{1}{2}\varepsilon$. Then $K = \bigcup_{k=1}^{N} K_k$ is closed. Since $\{F_n\} \in UAC(K_k)$, and $K_k \cap K_\ell = \emptyset$ for $k \neq \ell$, by Lemma 1 we have $\{F_n\} \in UAC(K)$, and

$$|E \setminus K| < |(E \setminus \bigcup_{k=1}^{N} H_k) \cup \{\bigcup_{k=1}^{N} (H_k \setminus \bigcup_{i=0}^{k-1} H_i \setminus K_k)\}| < \varepsilon.$$

Hence for every $k \in \mathbb{N}$, there exists a closed set L_k such that $\{F_n\} \in UAC(L_k)$ and $|E \setminus L_k| < \frac{1}{2^k}$, $k = 1, 2, \ldots$. Let $E_r = \bigcap_{k=r}^{\infty} L_k$, then $\{E_r\}$ satisfies the conditions of Lemma 2. □

Theorem 5.1 *Given $f : [a,b] \to (-\infty, +\infty)$, the following, i.e. (A), (B), (C), (D), (B′), (C′) and (D′) are mutually equivalent:*

(A) *f is AP-integrable with primitive F.*

(B) *There is a sequence of absolute Henstock integrable functions $\{f_n\}$ (i.e. Lebesgue integrable functions, see [4]) on $[a,b]$, such that:*

  (i) *$f_n \to f$ almost everywhere;*

  (ii) *There exists $E \subset [a,b]$, $|[a,b] \setminus E| = 0$ such that the sequence $\{F_n\}$ of the primitives of f_n with $n = 1, 2, \ldots$ is $UACG(E)$;*

(iii) *F_n converges almost everywhere to a function F satisfying condition ASL.*

(C) *The Conditions in (B) with f_n being Lebesgue integrable replaced by f_n being continuous, $n = 1, 2, \ldots$*

(D) *The conditions in (B) with f_n being Lebesgue integrable replaced by f_n being a step-function, $n = 1, 2, \ldots$.*

(B)′, (C′), (D′): the conditions in (B), (C), (D) with $UACG$ replaced by $UACG^*_{ap}$.

PROOF. (B) implies (A): by condition (ii) and (iii) of (B), there exists $H \subset [a,b]$, $|[a,b] \setminus H| = 0$, such that $\{F_n\} \in UACG(H)$ and $F_n \to F$ on H. Hence $F \in ACG(H)$. By (iii) of (B) F satisfies ASL, and hence if we take F as a sequence with all the elements being F, then by Lemma 2 and Theorem 3.2 we have $F \in ACG^*_{ap}([a,b])$.

By Lemma 2, there exist $E_1 \subset E_2 \subset \ldots \subset E_r \subset \ldots, |[a,b] \setminus \bigcup_{r=1}^{\infty} E_r| = 0$, and $\{F_n\} \in UAC(E_r)$. Let $G_{n,r}$ be defined on $[a,b]$ to be equal to F_n on E_r and linearly extending $G_{n,r}$ from E_r to closed intervals contiguous to E_r. Likewise we define G_r from F. Then $G_{n,r}$, $n = 1, 2, \ldots$ are $UAC(E_r)$. We will prove they are also $UAC([a,b])$.

Given $\varepsilon > 0$, there exists $\eta > 0$ corresponding to ε as in the definition of $UAC(E_r)$. Let $\bigcup_{k=1}^{\infty}(a_k, b_k) = [a,b] \setminus E_r$ where (a_k, b_k) are the intervals contiguous to E_r, and $N \in \mathbb{N}$ is such that $\sum_{k=N}^{\infty}(b_k - a_k) < \eta$.

Since $G_{n,r} \to G_r$ on $[a,b]$, there exists an $L \in \mathbb{N}$ such that

$$\left|\frac{G_{n,r}(b_k) - G_{n,r}(a_k)}{b_k - a_k} - \frac{G_{L,r}(b_k) - G_{L,r}(a_k)}{b_k - a_k}\right| \le$$
$$\left|\frac{G_{n,r}(b_k) - G_{L,r}(b_k) - G_{n,r}(a_k) + G_{L,r}(a_k)}{b_k - a_k}\right| < \epsilon$$

for $k = 1, 2, \ldots N$; and $n \ge L$. Hence,

$$\left|\frac{G_{n,r}(b_k) - G_{n,r}(a_k)}{b_k - a_k}\right| < \left|\frac{G_{L,r}(b_k) - G_{L,r}(a_k)}{b_k - a_k}\right| + \epsilon.$$

Let

$$M_r = \max\{\max_{n \le L}\left|\frac{G_{n,r}(b_k) - G_{n,r}(a_k)}{b_k - a_k}\right|, \left|\frac{G_{L,r}(b_k) - G_{L,r}(a_k)}{b_k - a_k}\right| + \epsilon\}.$$

Then we have $|G_{n,r}(x) - G_{n,r}(y)| \le M_r|x - y|$ for any $x, y \in [a_k, b_k]$, $k = 1, 2, \ldots N$.

Now, let $\nu^* = \min\{\nu, \frac{\varepsilon}{M_r}\}$. Given any partial division on $[a,b]$, $\{[u_i, v_i], i = 1, 2, \dots m\}$, i.e. $a \le u_1 < v_1 \le u_2 < v_2 \cdots \le u_n < v_n \le b$, satisfying $\sum_{i=1}^{\infty}(v_i - u_i) < \nu^*$, we make an appropriate revision to this division as follows. If $(u_i, v_i) \in [a_k, b_k]$ for one k or both $u_i, v_i \in E_r$, we keep the endpoints u_i, v_i unchanged. If $u_i \in (a_k, b_k)$ and $v_i \notin [a_k, b_k]$ we replace $[u_i, v_i]$ by $[u_i, b_k]$, $[b_k, v_i]$. If $v_i \in (a_k, b_k)$ and $u_i \notin [a_k, b_k]$ we replace $[u_i, v_i]$ by $[u_i, a_k]$, $[a_k, v_i]$. After this revision, we denote the partial division by $[u_i', v_i']$. Then we have

$$\sum_{i=1}^{\infty} |G_{n,r}(v_i) - G_{n,r}(u_i)| \le \sum_{i=1}^{\infty} |G_{n,r}(v_i') - G_{n,r}(u_i')| < 2\varepsilon.$$

Hence, $\{G_{n,r}\} \in UAC([a,b])$.

Now, $UAC([a,b])$ is equivalent to $UAC^*([a,b])$, and hence we have that $\{G_{n,r}\} \in UAC^*([a,b])$, $G_{n,r}(x) \to G_r(x)$ on $[a,b]$, and

(1) $G_{n,r}'(x) = (F_N)_{ap}'(x) = f_n(x)$ almost everywhere on E_r, and

(2) $G_{n,r}'(x) \to G_r'(x)$ everywhere on $\cup_{k=1}^{\infty}(a_k, b_k)$.

Hence, $G_{n,r}'(x)$ converges almost everywhere on $[a,b]$. In particular, we have that $G_{n,r}'(x) = f_n(x) = f(x)$ almost everywhere on E_r. Hence, by [4] Corollary 7.7, we have $G_r'(x) = \lim_{n\to\infty} G_{n,r}'(x)$ almost everywhere on $[a,b]$. But $G_r'(x) = F_{ap}'(x)$ almost everywhere on E_r, so we have $F_{ap}'(x) = f(x)$ almost everywhere on E_r. Because $|[a,b] \setminus \cup_{r=1}^{\infty} E_r| = 0$, we have $F_{ap}'(x) = f(x)$ almost everywhere on $[a,b]$, i.e. $F(x) = AP - \int_a^x f$.

(B') implies (A): By means of Corollary 3.10 of [5], we reduce (B') to (B), and hence "(B') implies (A) is true."

(C), (D) are just special cases of (B), and (C'), (D') are special cases of (B'), so any of, (C), (D), (C'), (D') imply (A).

(A) implies (B) and (B').

By Proposition 3.12 of [5] we have F satisfies conditions ASL and there exists $E \subset [a,b]$ with $|[a,b] \setminus E| = 0$ such that $F \in ACG(E)$. Hence by Lemma 2 there exist closed $E_1 \subset E_2 \subset \dots \subset E_r \subset \dots$, $[a,b] = \bigcup_{r=1}^{\infty} E_r \cup H$ with $|H| = 0$, and $F \in AC(E_r)$, $r = 1, 2, \dots$.

Let $F_n = F$ when $x \in E_n$ and linearly extend on the contiguous intervals of E_n as in our proof of "(B)". $F_n \in AC([a,b])$, hence $\dfrac{dF_n}{dx}$ is L-integrable $n = 1, 2, \dots$. Since $F_n(x)$ take the same value for $x \in E_r$ with $r \le n$, so $\{F_n; n \ge r\} \in UAC(E_r)$. Because $F_1, F_2, \dots, F_{r-1}$ are $AC([a,b])$ they are of course $AC(E_r)$, so if we add them into the family $\{F_n(x); n \ge r\}$ we have $\{F_n; n = 1, 2, \dots\} \in UAC(E_r)$, hence $\{F_n\} \in UACG(\bigcup_{r=1}^{\infty} E_r)$, and $|[a,b] \setminus \bigcup_{r=1}^{\infty} E_r| = 0$. Thus the condition (ii) of (B) holds.

On the other hand, let H_r be the set of the points of density in E_r. Then $|E_r \setminus H_r| = 0$ and $\{F_n\} \in UAC^*_{ap}(H_r)$, hence $F_n \in UACG^*_{ap}(\bigcup_{r-1}^{\infty} H_r)$ with $|[a,b] \setminus \bigcup_{r=1}^{\infty} H_r| = 0$, and the Condition (ii) of (B′) holds.

Since $F_n(x) = F(x)$ on E_r when $n \geq r$ for $r = 1, 2, \ldots$, we have $F_n \to F$ almost everywhere on $[a,b]$ as $n \to \infty$, i.e. the Conditions (iii) of (B) and (B′) both hold. Also we have $F_n'(x) = f(x)$ almost everywhere on E_r when $n \geq r$. By $\lim_{r\to\infty} E_r = [a,b] \setminus H$, we have $F_n'(x) \to f(x)$ as $n \to \infty$ almost everywhere on $[a,b]$, i.e. the Condition (i) of (B) and (B′) holds. Hence (B) and (B′) both hold.

(A) implies (C) and (C′).

Let F_n, $n = 1, 2 \ldots$ be defined as in the proof of (A) implying (B) and (B′), and $f_n(x) = F_n'(x)$, $n = 1, 2, \ldots$. Then $\{f_n\}$ is a sequence of Lebesgue integrable functions, so there exists a sequence of continuous functions $\{g_n\}$ such that

$$\int_a^b |f_n(x) - g_n(x)|dx < \frac{1}{2^{2n}}.$$

For every E_r, letting $\varepsilon > 0$ be given, there exists $N \in \mathbb{N}$ such that $N \geq r$ and $\frac{1}{2^{2N}} < \frac{\varepsilon}{2}$, and because $\{F_n\} \in UAC(E_r)$, so there exists $\eta > 0$ corresponding to $\frac{1}{2}\varepsilon$ as in the definition of $UAC(E_r)$. On the other hand, since G_n defined by $G_n(x) = \int_a^x g_n$ $n = 1, 2, \ldots$ are primitives of Lebesgue integrable functions, so they are $AC([a,b])$, hence $AC(E_r)$, so there exist η_n corresponding to G_n with respect to ε as in the definition of $AC(E_r)$. Let $\eta^* = \min\{\eta, \eta_1, \eta_2, \ldots, \eta_N\}$. For any division $[u_i, v_i]$ with $u_i, v_i \in E_r$, $\sum_i (v_i - u_i) < \eta^*$, when $n \leq N$, we have $\sum_i |G_n(v_i) - G_n(u_i)| < \varepsilon$, and when $n \geq N$, we have

$$\sum_i |G_n(v_i) - G_n(u_i)| = \sum_i |\int_{u_i}^{v_i} g_n(x)dx| \leq \sum_i |\int_{u_i}^{v_i} f_n(x)dx| + \frac{\varepsilon}{2} < \varepsilon.$$

Hence $\{G_n\} \in UACG(\bigcup_{r=1}^{\infty} E_r)$ with $|[a,b] \setminus \bigcup_{r=1}^{\infty} E_r| = 0$, and hence Condition (ii) of (C) holds.

The same discussion that showed (A) implied (B) and (B′), shows that we also have $\{G_n\}$ being $UACG^*_{ap}(\bigcup_{r=1}^{\infty} H_r)$ with $|[a,b] \setminus \bigcup_{r=1}^{\infty} H_r| = 0$, i.e. the Condition (ii) of (C′) holds.

Because

$$|G(_n - F_n)(x)| = |\int_a^x (g_n(x) - f_n(x))dx| \to 0 \text{ as } n \to \infty$$

we have $G_n(x) \to F(x)$ almost everywhere on $[a,b]$, i.e. the Condition (iii) of (C) and (C′) holds.

Lastly, since

$$\int_a^b |g_n(x) - f_n(x)| < \frac{1}{2^{2n}},$$

the set

$$K_n = \{x : |g_n(x) - f_n(x)| > \frac{1}{2^n}\}$$

satisfies $|K_n| < \frac{1}{2^n}$. Hence $|\bigcup_{n=N+1}^{\infty} K_n| < \frac{1}{2^N}$ and therefore

$$|\bigcap_{N=1}^{\infty} \bigcup_{n=N+1}^{\infty} K_n| = 0.$$

Since $|g_n(x) - f_n(x)| \to 0$ whenever $x \in [a,b] \setminus \bigcap_{N=1}^{\infty} \bigcup_{n=N+1}^{\infty} K_n$, we have $g_n(x) \to f(x)$ almost everywhere on $[a,b]$, i.e. Condition (i) of (C) and (C′) holds. Thus Conditions (C) and (C′) both hold.

(A) implies (D) and (D′): The same as the proof that (A) implies (C) and (C′). □

Now we give a Riesz-Type Definition of AP-integral.

Definition. Suppose a function $f : [a,b] \to (-\infty, +\infty)$ satisfies:

(i) There is a sequence of step-functions $\{\varphi_n\}$ such that $\varphi_n \to f$ almost everywhere on $[a,b]$,

(ii) the sequence $\{\Phi_n\}$ of primitives of φ_n is $UACG$ or $UACG_{ap}^*$ on a set $E \subset [a,b]$, $|[a,b] \setminus E| = 0$,

(iii) $\Phi_n \to F$ almost everywhere with F satisfying conditions ASL.

Then we say f is Riesz Type AP-integrable, and F is the primitive of f, denoted by

$$F(x) = RP_{ap} - \int_a^x f.$$

By Theorem 1, we have

Corollary 5.1 *RP_{ap}-integrability is equivalent to AP-integrability.*

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