Real Analysis Exchange
Vol. 20(2), 1994/5, pp. 580–592

A. B. Kharazishvili, Institute of Applied Mathematics, University of Tbilisi, University str. 2, 380043 Tbilisi 43, Georgia

SOME QUESTIONS CONCERNING INVARIANT EXTENSIONS OF LEBESGUE MEASURE

Abstract

A number of invariant extensions of the Lebesgue measure are constructed. In connection with such extensions the class of negligible sets and the class of absolutely negligible sets are investigated. Also the Steinhaus property for these extensions is considered.

In this paper we present some results and some problems concerning invariant extensions of the classical Lebesgue measure l on the real line $\mathbb{R}$. As far as we know one of the first works devoted to (countably additive) invariant extensions of the Lebesgue measure is the paper of Szpilrajn (Marczewski) [1], where several constructions of such extensions are discussed (see also another paper [2] by the same author, in which a list of problems in measure theory is given and, in particular, invariant extensions of the Lebesgue measure are touched). Then two papers of Kakutani and Oxtoby [3] and Kodaira and Kakutani [4] appeared, in which two essentially different constructions of nonseparable invariant extensions of the Lebesgue measure were presented. Afterwards the mentioned theme became a subject of investigations of various authors (see, for instance, [5] – [15]). Notice that this theme is closely connected with other branches of mathematics, namely, with the geometry of Euclidean space, with group theory, with foundations of set theory etc. In our opinion one of the central questions of this subject is the following: what kind of sets can be used to construct invariant extensions of the Lebesgue measure? In other words, the problem is to find an appropriate characterization of all subsets X of $\mathbb{R}$ such that X is measurable with respect to some (certainly, depending on X) invariant extension of the Lebesgue measure l. Up to now there exists no general approach to this problem. We shall consider below several

Key Words: invariant measure, extension of measure, negligible set, absolutely negligible set, Steinhaus property
Mathematical Reviews subject classification: Primary 28A05, Secondary 28A20
Received by the editors March 15, 1994

examples of subsets of $\mathbb{R}$ to which the measure l can be extended preserving the properties of countable additivity and invariance. It is worth noting here that among such subsets of $\mathbb{R}$ are well known classical sets of reals: Hamel base, Bernstein set, Sierpinski set and others. Notice also that some important properties of the Lebesgue measure (for instance, metrical transitivity, density points property, Steinhaus property etc.) fail or require an essential modification for certain invariant extensions of l. Namely, we shall construct below an invariant extension of l which does not have the Steinhaus property.

In the present paper we use the following standard notation.

ω - the cardinality of the set of all natural numbers;

ω_1 - the first uncountable cardinal;

$\mathbf{c}$ - the cardinality continuum;

$|X|$ - the cardinality of a given set X.

Let E be a basic set and let Γ be a group of transformations of E. If $\{X_i : i \in I\}$ is an arbitrary family of subsets of E, then the symbol $\sigma(\Gamma, \{X_i : i \in I\})$ denotes the smallest Γ-invariant σ-algebra in E containing the family $\{X_i : i \in I\}$. In particular, if $|I| \leq \omega$, then we say that the mentioned Γ-invariant σ-algebra is countably generated (let us remark that if the group Γ contains only the identity transformation of E, then our notion coincides with the usual notion of a countably generated σ-algebra).

G - a subgroup of the additive group of $\mathbb{R}$;

$M(l, G)$ - the class of all G-invariant measures on $\mathbb{R}$ extending Lebesgue measure l;

$M(G)$ - the class of all nonzero σ-finite G-invariant measures on $\mathbb{R}$.

$\mathrm{dom}(\mu)$ - the domain of a given measure μ.

It is clear that $M(l, G) \subset M(G)$. To avoid some trivial cases, when using the notation $M(l, G)$ we assume, as a rule, that G is a dense subgroup of $\mathbb{R}$ and when using the notation $M(G)$ we assume, as a rule, that G is an uncountable (hence, also a dense) subgroup of $\mathbb{R}$.

In the sequel we shall need the following two general notions.

Let E be a basic set and let M be a class of measures defined on E. Let X be a subset of E. We say that X is a negligible set with respect to the class M if

1) there exists a measure $\mu \in M$ such that $X \in \mathrm{dom}(\mu)$;

2) for every measure $\nu \in M$ we have $X \in \text{dom}(\nu) \rightarrow \nu(X) = 0$.

Example 1 Let M be the class of all $\mathbb{R}^2$-invariant measures defined on the Euclidean plane $\mathbb{R}^2$ and extending the two-dimensional Lebesgue measure l^2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$. Then it is not difficult to prove that the graph of f is a negligible set with respect to M. (In this connection see [7] or [9].) Of course, the graph of f may be an l^2-nonmeasurable subset of $\mathbb{R}^2$.

Let again E be a basic set and let M be some class of measures defined on E. Let $Y \subset E$. We say that Y is an absolutely negligible set with respect to the class M if

3) for each measure $\mu \in M$ there exists a measure $\nu \in M$ extending μ such that $Y \in \text{dom}(\nu)$ and $\nu(Y) = 0$.

Example 2 Every Hamel base in $\mathbb{R}$ is absolutely negligible with respect to the class $M(\mathbb{R})$ and also with respect to the class $M(l, \mathbb{R})$. (In this connection see [7], [9] or [11], Th. 3.3.) Notice that a more general fact can be proved. Namely, let H be a Hamel base in $\mathbb{R}$ and let n be any natural number. Denote by Z_n the set of all real numbers whose representation in a form of rational linear combination of elements of H contains at most n nonzero coefficients. Then it is not difficult to show, using the arguments from [7] or [9], that the set Z_n is absolutely negligible with respect to the class $M(\mathbb{R})$. From this fact we can also deduce that there exists a countable covering of $\mathbb{R}$ by sets absolutely negligible with respect to $M(\mathbb{R})$.

Example 3 There exists a function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that its graph is not absolutely negligible with respect to the class of all $\mathbb{R}^2$-invariant extensions of the Lebesgue measure l^2 (see [7]).

Now let us consider some sets with "very bad" properties with respect to the Lebesgue measure l. We mean for instance, the so called Sierpinski sets. Let us recall that $X \subseteq \mathbb{R}$ is a Sierpinski set if X is uncountable and for each Lebesgue measure zero set $Y \subseteq \mathbb{R}$ we have $|X \cap Y| \leq \omega$. It is well known that if the Continuum Hypothesis holds, then there exist Sierpinski sets in $\mathbb{R}$. Also it is easy to see that $X \subseteq \mathbb{R}$ is a Sierpinski set if and only if X is uncountable and each uncountable subset of X is Lebesgue nonmeasurable.

Observe that from the point of view of invariant extensions of the Lebesgue measure the Sierpinski sets are not too bad (at least they are not so bad as Vitali sets which do not belong to the domain of any $\mathbb{R}$-invariant extension of l). Indeed the family of all Sierpinski sets is an $\mathbb{R}$-invariant σ-ideal of subsets of $\mathbb{R}$ and the inner Lebesgue measure of each member of this σ-ideal is equal to zero. From this fact it follows immediately that there exists an $\mathbb{R}$-invariant extension μ of l such that all Sierpinski sets are of μ-measure zero (see [1] or

[2]). On the other hand, not every Sierpinski set is negligible with respect to the class $M(l,\mathbb{R})$. Indeed, we have the following theorem.

Theorem 1 *Assume that the Continuum Hypothesis holds. Then there exist a measure $\mu \in M(l,\mathbb{R})$ and a Sierpinski set X such that $\mu(\mathbb{R} \setminus X) = 0$.*

PROOF. Since $\mathbf{c} = \omega_1$, we can define an increasing (with respect to inclusion) family $\{G_\xi : \xi < \omega_1\}$ of countable subgroups of $\mathbb{R}$ whose union is equal to $\mathbb{R}$. Let $\{B_\xi : \xi < \omega_1\}$ be a Borel base of the σ-ideal of all l-measure zero subsets of $\mathbb{R}$. Without loss of generality we can assume that this base is also increasing with respect to inclusion. We choose an injective family $\{x_\xi : \xi < \omega_1\}$ of points of $\mathbb{R}$ in such a way that for each ordinal $\xi < \omega_1$ we have $G_\xi(x_\xi) \cap B_\xi = \emptyset$, where the symbol $G_\xi(x_\xi)$ denotes the G_ξ-orbit of a point x_ξ. Then we put $X = \bigcup\{G_\xi(x_\xi) : \xi < \omega_1\}$. It is clear that X is a Sierpinski set in $\mathbb{R}$. Notice also that the set $\mathbb{R} \setminus X$ is almost $\mathbb{R}$-invariant in $\mathbb{R}$ and has inner Lebesgue measure zero. Hence, it is possible to extend the measure l to an $\mathbb{R}$-invariant measure μ such that $\mu(\mathbb{R} \setminus X) = 0$. □

Using Theorem 1, applying the classical Vitali's construction and taking into account that every uncountable subset of a Sierpinski set is also a Sierpinski set we obtain the following proposition.

Theorem 2 *If the Continuum Hypothesis holds, then there exist a measure $\mu \in M(l,\mathbb{R})$ and a Sierpinski set Y such that Y does not belong to the domain of any $\mathbb{R}$-invariant measure extending μ.*

Remark 1 *Results analogous to Theorems 1 and 2 can be established also under Martin's Axiom if we consider so called generalized Sierpinski sets.*

If instead of the class $M(l,G)$ we take the wider class $M(G)$, then we can prove the following theorem.

Theorem 3 *Let G be an arbitrary uncountable subgroup of $\mathbb{R}$, let $\mu \in M(G)$ and let X be a μ-measurable set with $\mu(X) > 0$. Then there exists a set $Y \subseteq X$ nonmeasurable with respect to μ and absolutely negligible with respect to $M(G)$.*

PROOF. Here we need a result from [7]. According to this result there exists a countable covering $\{X_i : i \in I\}$ of $\mathbb{R}$ by sets absolutely negligible with respect to $M(G)$. Notice that in [7] only the case $G = \mathbb{R}$ is considered but the same result is true for every uncountable subgroup G of $\mathbb{R}$. (Also see [12] where a more general assertion is proved.) Obviously we can write $X = \cup\{X \cap X_i : i \in I\}$. Since we have $\mu(X) > 0$, we can find an index $i \in I$ such that the set $X \cap X_i$ is not μ-measurable. But we also have $X \cap X_i \subseteq X_i$. Hence, the set $X \cap X_i$ is absolutely negligible with respect to $M(G)$. So we can put $Y = X \cap X_i$. □

Theorem 3 shows, in particular, that if $G \subseteq \mathbb{R}$ is an uncountable group, μ is a measure from the class $M(G)$ and X is a μ-measurable set with $\mu(X) > 0$, then X contains a μ-nonmeasurable subset Y which is "very good" from the point of view of all G-invariant extensions of μ. Namely, for each G-invariant extension μ' of μ there exists a G-invariant extension μ'' of μ' such that $\mu''(Y) = 0$. In connection with this fact a natural question arises: does a given set X contain "very bad" subsets with respect to all G-invariant extensions of μ? More exactly: does there exist a subset Z of X such that Z is nonmeasurable with respect to each G-invariant extension of μ? This natural question was open for almost twenty years. Moreover it was open even in the particular case where $G = \mathbb{R}$. As a referee informed us, this question recently was solved in a positive way by S. Solecki in his paper *On sets nonmeasurable with respect to invariant measures* which will be published in the Proc. Amer. Math. Soc.

Notice also that for measures from the class $M(l, G)$, where G is a dense subgroup of $\mathbb{R}$, the well known Vitali construction easily gives us a positive answer to the analogous question.

It is known (see [7] or [9]) that if a given group $G \subseteq \mathbb{R}$ is uncountable, then the family of all absolutely negligible sets with respect to the class $M(G)$ is an ideal (but not a σ-ideal). The family of all negligible sets with respect to the same class is not even an ideal. Moreover, it can be proved (see [7]) that there exist two subsets X_1 and X_2 of $\mathbb{R}$ satisfying the following conditions:

a) X_1 and X_2 are negligible with respect to the class $M(l, G)$ and also with respect to the class $M(G)$;

b) the set $X_1 \cup X_2$ is absolutely nonmeasurable with respect to the class $M(G)$, i.e. this set does not belong to the domain of any measure from $M(G)$ and, therefore, it does not belong to the domain of any measure from the class $M(l, G)$.

In particular, if $G = \mathbb{R}$, then, using the existence of these two sets, it is not difficult to deduce the next proposition.

Theorem 4 *There exist two σ-algebras S_1 and S_2 of subsets of $\mathbb{R}$ satisfying the following conditions:*

1) both S_1 and S_2 are countably generated $\mathbb{R}$-invariant σ-algebras and each of them contains the family of all one-element subsets of $\mathbb{R}$;

2) there exist a nonzero σ-finite continuous $\mathbb{R}$-invariant measure μ_1 on S_1 and a nonzero σ-finite continuous $\mathbb{R}$-invariant measure μ_2 on S_2;

3) there exists no nonzero σ-finite

continuous $\mathbb{R}$-invariant measure on the $\mathbb{R}$-invariant σ-algebra generated by the family $S_1 \cup S_2$.

PROOF. Indeed let $\{V_i : i \in I\}$ be a countable base of the Euclidean topology on $\mathbb{R}$. Then it is sufficient to put

$$S_1 = \sigma(\mathbb{R}, \{V_i : i \in I\} \cup \{X_1\}),$$
$$S_2 = \sigma(\mathbb{R}, \{V_i : i \in I\} \cup \{X_2\}),$$

where X_1 and X_2 are two negligible sets with respect to the class $M(l, \mathbb{R})$, whose union is an absolutely nonmeasurable set with respect to the class $M(\mathbb{R})$.

It is obvious that in the same way Theorem 4 can be established for each uncountable subgroup G of $\mathbb{R}$. In this connection notice that for ordinary (non invariant) measures and σ-algebras the result analogous to the theorem just formulated certainly cannot be proved in theory **ZFC**, since this result would imply the nonexistence of a real-valued measurable cardinal $\leq$ c. (See, for example, A.Pelc, K.Prikry, *On a problem of Banach*, Proc. Amer. Math. Soc., **89** (1983), 608–610, where a much stronger assertion is established.) But the result mentioned can easily be proved if we assume the Continuum Hypothesis or the much weaker Martin's Axiom. In fact, for a proof of this result it is sufficient to construct a generalized Luzin subset of $\mathbb{R}$, which is a dual object to a generalized Sierpinski subset of $\mathbb{R}$. (See, for example, E.Grzegorek, *Remarks on σ-fields without continuous measures*, Colloq. Math., **1** (1978), 103–108.)

Note also that if G is a countable dense subgroup of $\mathbb{R}$, then a result similar to Theorem 4 can be proved for G, also. Indeed, consider the partition of $\mathbb{R}$ into G-orbits and denote this partition by $\{Z_i : i \in I\}$. Using a well known "zero-one law" it is not difficult to find a partition $\{I_1, I_2\}$ of the set I such that both sets $\cup\{Z_i : i \in I_1\}$, and $\cup\{Z_i : i \in I_2\}$ are nonmeasurable with respect to the measure l. Let X be a selector of the family $\{Z_i : i \in I\}$. Since X is a Vitali type set for G, it is clear that X is absolutely nonmeasurable with respect to the class $M(l, G)$. Now, if we put

$$X_1 = X \cap (\bigcup\{Z_i : i \in I_1\}),$$
$$X_2 = X \cap (\bigcup\{Z_i : i \in I_2\}),$$

then it is easy to see that the sets X_1 and X_2 are negligible with respect to $M(l, G)$ and their union coincides with the whole set X. From these facts we deduce that there are two countably generated G-invariant σ-algebras S_1 and S_2 of subsets of $\mathbb{R}$ satisfying the following conditions:

a) there exist a G-invariant measure μ_1 on S_1 and a G-invariant measure μ_2 on S_2 such that the completions of μ_1 and μ_2 both extend the measure l;

b) there exists no measure on the G-invariant σ-algebra generated by $S_1 \cup S_2$ with completion belonging to the class $M(l, G)$.

Now we want to consider the following question: how small are negligible sets? Let X_1 and X_2 be any two disjoint sets negligible with respect to the class $M(\mathbb{R})$. Then it is obvious that the set $X_1 \cup X_2$ does not coincide with the whole real line $\mathbb{R}$. We shall see below that there exists a partition of $\mathbb{R}$ into three sets negligible with respect to $M(\mathbb{R})$.

Lemma 1 *Let E be a basic set, let Γ be a group of transformations of E and let μ be a σ-finite Γ-invariant (or, more generally, Γ-quasi-invariant) measure on E. If X is any μ-measurable set then there exists a countable family $\{h_i : i \in I\}$ of elements of Γ such that the set $Z = \bigcup\{h_i(X) : i \in I\}$ is almost Γ-invariant with respect to the measure μ, i.e. for each element $h \in \Gamma$ we have $\mu(h(Z) \triangle Z) = 0$, where $\triangle$ denotes the operation of symmetric difference.*

This lemma is well known. The required set Z can be easily constructed by the method of transfinite recursion, since a given measure μ satisfies the Suslin condition (i.e. countable chain condition).

In the sequel we need an auxiliary notion. Let G be a subgroup of $\mathbb{R}$ and let X be a subset of $\mathbb{R}$. We say that X is finite (countable) with respect to G if for every $x \in \mathbb{R}$ the set $X \cap (G + x)$ is finite (countable).

Lemma 2 *Let G be an uncountable subgroup of $\mathbb{R}$ and let X be a subset of $\mathbb{R}$ finite with respect to G. Then the set X is negligible with respect to the class $M(\mathbb{R})$.*

PROOF. It is not difficult to construct a nonzero σ-finite $\mathbb{R}$-invariant measure ν on $\mathbb{R}$ such that $X \in \text{dom}(\nu)$ and $\nu(X) = 0$. Now let μ be any σ-finite $\mathbb{R}$-invariant (or, more generally, $\mathbb{R}$-quasi-invariant) measure on $\mathbb{R}$ such that the set X is μ-measurable. We must show that $\mu(X) = 0$. Suppose to the contrary that $\mu(X) > 0$. According to Lemma 1, there exists a countable family $\{h_i : i \in I\}$ of elements of G such that the set $Z = \cup\{h_i + X : i \in I\}$ is almost G-invariant with respect to μ. Let $\{g_j : j \in J\}$ be a countable family of elements of G satisfying the following condition: for any two distinct indices j and k from J the difference $g_j - g_k$ does not belong to the group generated by the countable family $\{h_i : i \in I\}$. Then, taking into account that the set X is finite with respect to G, it is not difficult to check that $\cap\{g_j + Z : j \in J\} = \emptyset$.

But, on the other hand, we must have $\mu(\cap\{g_j + Z : j \in J\}) > 0$, since the set Z is almost G-invariant with respect to μ and $\mu(Z) > 0$. This contradiction completes the proof. □

Remark 2 *Let G be an uncountable subgroup of $\mathbb{R}$ and let X be a subset of $\mathbb{R}$ countable with respect to G. Notice that in general the set X is not negligible with respect to the class $M(\mathbb{R})$. Moreover, it may happen that there exists a measure μ from this class such that X is μ-measurable and $\mu(\mathbb{R} \setminus X) = 0$ (see [7]). Notice also that if Martin's Axiom and the negation of the Continuum Hypothesis hold then there always exists an $\mathbb{R}$-invariant extension ν of the Lebesgue measure l for which the relations $X \in \mathrm{dom}(\nu)$ and $\nu(X) = 0$ are satisfied (the proof of this fact is similar to the argument used in the proof of Lemma 2).*

Example 4. Let $n > 1$ be a natural number, let $\mathbb{R}^n$ be n dimensional Euclidean space, let G be a one dimensional vector subspace of $\mathbb{R}^n$ and let X be an arbitrary subset of $\mathbb{R}^n$ finite with respect to G. Then, taking into account the fact that $\mathbb{R}$ and $\mathbb{R}^n$ are isomorphic as abstract groups and using Lemma 2, it is easy to check that X is negligible set with respect to the class of all nonzero σ-finite $\mathbb{R}^n$-invariant measures on $\mathbb{R}^n$ and also with respect to the class of all $\mathbb{R}^n$-invariant extensions of n dimensional Lebesgue measure l^n.

Theorem 5 *There exists a partition $\{A, B, C\}$ of $\mathbb{R}$ into three sets negligible with respect to the class $M(\mathbb{R})$. In particular for each measure μ from this class at least one of the sets A, B and C is μ-nonmeasurable.*

PROOF. Let us consider $\mathbb{R}$ as a vector space over the field **Q** of all rational numbers. Then it is clear that $\mathbb{R}$ can be represented as a direct sum $\mathbb{R} = G_1 + G_2 + G_3 + G$ of four subgroups G_1, G_2, G_3 and G, where $|G_1| = |G_2| = |G_3| = \omega_1$. Now we apply to the group $G_1 + G_2 + G_3$ the classical construction of Sierpinski (see, for instance, [16]). Notice that this construction was made by Sierpinski for the three dimensional Euclidean space under the Continuum Hypothesis, but it can be carried out in **ZFC** for an arbitrary direct sum of three groups each of cardinality ω_1. In particular the mentioned construction gives us a partition $\{A', B', C'\}$ of the group $G_1 + G_2 + G_3$ into three sets in such a way that the set A' is finite with respect to G_1, the set B' is finite with respect to G_2 and the set C' is finite with respect to G_3. Afterwards we put $A = A' + G$, $B = B' + G$ and $C = C' + G$. Obviously for the sets A, B and C, considered as subsets of $\mathbb{R}$, the same property holds, namely, the set A is finite with respect to G_1, the set B is finite with respect to G_2 and the set C is finite with respect to G_3. Hence, according to Lemma 2, the sets A, B and C are negligible with respect to the class $M(\mathbb{R})$. It is evident also that $\{A, B, C\}$ is a partition of $\mathbb{R}$. □

Remark 3 *In an analogous way it can be proved that for any uncountable subgroup G of $\mathbb{R}$ there exists a partition of $\mathbb{R}$ into three sets negligible with respect to the class $M(G)$. The proof of this more general result is based on the fact that an uncountable group $G \subseteq \mathbb{R}$ always contains a direct sum of some three of its subgroups G_1, G_2 and G_3 such that $|G_1| = |G_2| = |G_3| = \omega_1$. In addition since the real line $\mathbb{R}$ and n dimensional Euclidean space $\mathbb{R}^n$, where $n > 0$, are isomorphic as abstract groups, the same result is true for the space $\mathbb{R}^n$ and for each uncountable subgroup G of $\mathbb{R}^n$.*

The last theorem shows, in particular, that negligible sets in fact are not very small subsets of $\mathbb{R}$. The problem of geometrical (i.e. in terms of a given uncountable group G) characterization of negligible sets with respect to the class $M(G)$ is still open. Notice here that for absolutely negligible sets with respect to the same class an analogous problem is solved (see [7] or [9], where a geometrical characterization of absolutely negligible sets is established). Notice also that it is not known a characterization of subsets of $\mathbb{R}$ negligible with respect to the class $M(l, G)$.

It is worth remarking that, as a rule, in the case of invariant extensions of Lebesgue measure l constructed by means of l-nonmeasurable negligible sets or l-nonmeasurable absolutely negligible sets many good properties of l are preserved. Other constructions of invariant extensions of l may sometimes essentially change or completely destroy those properties. Here we consider only one property of this type — the so called Steinhaus property, which is connected with small translations of measurable sets. First of all we introduce this notion in a general form.

Let E be a basic set, let Γ be a topological group of transformations of E and let μ be a σ-finite Γ-invariant measure defined on some σ-algebra of subsets of E. We say that this measure has the Steinhaus property if for every μ-measurable set X we have $\lim_{h \to id} \mu(h(X) \cap X) = \mu(X)$, where id denotes the identity transformation of E. In particular, if μ has the Steinhaus property, then for every μ-measurable set X with $\mu(X) > 0$ there exists a neighborhood V of id such that the relation $h(X) \cap X \neq \emptyset$ holds for each transformation $h \in V$.

It is well known that the classical Lebesgue measure has the Steinhaus property. More generally every Haar measure on a σ-compact locally compact topological group $(\Gamma, \cdot)$ has this property. Notice that the Steinhaus property is important in various questions of analysis. Here we mention only the following three assertions which are directly implied by this property:

1) the metrical transitivity of a Haar measure on $(\Gamma, \cdot)$ with respect to any dense subgroup of Γ;

2) the continuity of each additive measurable functional defined on $(\Gamma, \cdot)$;

3) no Hamel base in $\mathbb{R}$ is an analytic set (an old result of Sierpinski).

Observe also that the Steinhaus property has some analog in terms of Baire category. The corresponding result is well known in general theory of topological groups and is usually called the Banach-Kuratowski-Pettis theorem (see, for instance, [17]). This result states that if X is a non first category set with the Baire property in an arbitrary topological group $(\Gamma, \cdot)$, then the set $X \cdot X^{-1}$ is a neighborhood of the unit element of Γ. As we have said above, a Haar measure on a σ-compact locally compact group has the Steinhaus property and, what is more important, this fact can be proved without using uncountable forms of the Axiom of Choice, for instance, in theory $(\mathbf{ZF}) \wedge (\mathbf{DC})$. Notice that if a topological group $(\Gamma, \cdot)$ satisfies the first countability axiom (or, if Γ is a topological vector space over the field $\mathbf{Q}$), then the mentioned Banach-Kuratowski-Pettis theorem can be proved for this group in $(\mathbf{ZF}) \wedge (\mathbf{DC})$, too. But in the general case of an arbitrary topological group $(\Gamma, \cdot)$ the standard proof of this theorem is based on the classical result of Banach about first category open sets and hence uses some uncountable form of the Axiom of Choice.

Returning to invariant extensions of the Lebesgue measure l we must notice that in most situations they have the Steinhaus property. In particular if we extend the Lebesgue measure using negligible or absolutely negligible subsets of $\mathbb{R}$, then the obtained extensions will necessarily have the Steinhaus property. But there exists an invariant extension of l which does not have this property. To establish this fact we need the following auxiliary proposition.

Lemma 3 *Let* $\mathbf{T}$ *be the unit one-dimensional torus on the Euclidean plane considered as a commutative compact topological group and equipped with the invariant probability Haar measure* λ. *Then there exists a group homomorphism* $\phi : \mathbb{R} \to \mathbf{T}$ *such that its graph* $\{(x, \phi(x)) : x \in \mathbb{R}\}$ *is an* $(l \times \lambda)$*-massive subset (i.e. an* $(l \times \lambda)$*-thick subset, according to the terminology of [18]) of the product space* $\mathbb{R} \times \mathbf{T}$. *In particular* ϕ *is discontinuous everywhere on* $\mathbb{R}$.

This lemma is well known (cf. [4]). The required homomorphism ϕ can be constructed in a standard way, by the method of transfinite recursion, if we consider $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$.

Now let ϕ be as in Lemma 3. We recall here how one can construct, starting with ϕ, a certain invariant extension of the measure l (cf. [4] or [8]). For each set Z from $\mathrm{dom}(l \times \lambda)$ we put $Z' = \{x \in \mathbb{R} : (x, \phi(x)) \in Z\}$. Then we also put $S = \{Z' : Z \in \mathrm{dom}(l \times \lambda)\}$. It is not difficult to check that the family S is a σ-algebra of subsets of $\mathbb{R}$ invariant under the group of all isometrical transformations of $\mathbb{R}$. Moreover we have the inclusion $\mathrm{dom}(l) \subset S$ and if we put $\mu(Z') = (l \times \lambda)(Z)$ $(Z \in \mathrm{dom}(l \times \lambda))$, then it is not difficult to prove that the last formula correctly defines a measure μ on S which strictly extends the Lebesgue measure l and is invariant under the group of all isometrical transformations of $\mathbb{R}$.

Theorem 6 *The measure μ constructed above does not have the Steinhaus property.*

PROOF. Since homomorphism ϕ is discontinuous everywhere, there exist a real number $\epsilon > 0$ and a sequence $\{x_0, x_1, ..., x_n, ...\}$ of points of $\mathbb{R}$ such that $lim_n x_n = 0$ and $|\phi(x_n) - 1| > 2\epsilon$ for all n. Let $V \subset \mathbf{T}$ be an open neighborhood of 1 with diameter less than ϵ. Put $Z = \mathbb{R} \times V$, $Z' = \{x \in \mathbb{R} : (x, \phi(x)) \in Z\}$. For each n also let $h_n = (x_n, \phi(x_n))$. Then it is easy to check that $h_n(Z) \cap Z = \emptyset$, $(Z' + x_n) \cap Z' = \emptyset$. Now if we suppose that our measure μ has the Steinhaus property, then we must have $\lim_n \mu((Z' + x_n) \cap Z') = \mu(Z') = +\infty$. But this is impossible, so the formulated theorem is proved. Moreover, we see that the Steinhaus property fails in a very strong form for the given measure μ.

One can also prove for this measure that

a) the completion of μ has the uniqueness property on its domain, i.e. every σ-finite $\mathbb{R}$-invariant measure defined on the domain of the completion of μ is proportional to this completion;

b) there exists a μ-measurable set with exactly one density point with respect to the standard Vitali system in $\mathbb{R}$ consisting of all open subintervals of $\mathbb{R}$.

In connection with assertions a) and b) see [19].

Finally, notice that it would be interesting to find some appropriate conditions under which an $\mathbb{R}$-invariant extension of the Lebesgue measure has the Steinhaus property.

I am indebted to Professor Z.Lipecki and the referee for their helpful remarks.

References

[1] E. Szpilrajn (E. Marczewski), *Sur l'extension de la mesure lebesguienne*, Fund. Math. **25** (1935), 551–558.

[2] E. Szpilrajn (E. Marczewski), *On problems of the theory of measure*, Uspekhi Mat. Nauk, **1 (2) (12)** (1946), 179–188 (in Russian).

[3] S. Kakutani and J. Oxtoby, *Construction of a nonseparable invariant extension of the Lebesgue measure space*, Ann. Math. **52** (1950), 580–590.

[4] K. Kodaira and S. Kakutani, *A nonseparable translation invariant extension of the Lebesgue measure space*, Ann. Math. **52** (1950), 574–579.

[5] S. Pkhakadze, *Theory of the Lebesgue measure*, Trudy Tbilis. Mat. Inst. im. Razmadze Akad. Nauk Gruz. SSR **25** (1958), p. 3–272 (in Russian).

[6] A. Hulanicki, *Invariant extensions of the Lebesgue measure*, Fund. Math. **51** (1962), 111–115.

[7] A. B. Kharazishvili, *Some questions of Set theory and Measure theory*, Tbil. Gos. Univ., Tbilisi, 1978 (in Russian).

[8] A. Hulanicki and C. Ryll-Nardzewski, *Invariant extensions of the Haar measure*, Colloq. Mathem. **42** (1979), 223–227.

[9] A. B. Kharazishvili, *Invariant extensions of the Lebesgue measure*, Tbil. Gos. Univ., Tbilisi, 1983 (in Russian).

[10] K. Ciesielski and A. Pelc, *Extensions of invariant measures on Euclidean spaces*, Fund. Math. **125** (1985), 1–10.

[11] A. Pelc, *Invariant measures and ideals on discrete groups*, Dissertationes Math. 255 (1986).

[12] K. Ciesielski, *Algebraically invariant extensions of σ-finite measures on Euclidean spaces*, Trans. Amer. Math. Soc. **318** (1990), 261–273.

[13] Zakrzewski, *Extensions of isometrically invariant measures on Euclidean spaces*, Proc. Amer. Math. Soc. **110** (1990), p. 325–331.

[14] A. Krawczyk and Zakrzewski, *Extensions of measures invariant under countable groups of transformations*, Trans. Amer. Math. Soc. **326** (1991), 211–226.

[15] A. B. Kharazishvili, *Vitali's theorem and its generalizations*, Tbil. Gos. Univ., Tbilisi, 1991 (in Russian).

[16] W. Sierpinski, *Cardinal and Ordinal numbers*, Monogr. Matem. **34**, PWN, Warszawa 1965.

[17] J. L. Kelley, *General Topology, D. Van Nostrand*, New York 1955.

[18] R. Halmos, *Measure theory, D. Van Nostrand*, New York 1950.

[19] A. B. Kharazishvili, *Some remarks on density points and the uniqueness property for invariant extensions of the Lebesgue measure*, Proceedings of the 22-nd Winter School on Abstract Analysis, Topology Section (January 22–29, 1994), Acta Universitatis Carolinae, Mathematics and Physics, 1995 (to appear).