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ON DENSITY POINTS

J. Foran in [2] has presented a new condition under which a measurable function $f : \mathbb{R} \to \mathbb{R}$ is approximately continuous. In [4] a similar condition was formulated for I-approximate continuity. We shall deal with points of density, strong density, I-density and strong I-density of plane sets. For the definitions see [3] and [1].

Let $A \subset \mathbb{R}^2$ be a Lebesque measurable set and f its characteristic function. We are interested in the following condition (compare [2]) concerning a decreasing sequence $\{t_n\}_{n\in\mathbb{N}}$ convergent to 0.

($\star\star$) Whenever $f(t_n x, t_n y)$ converges to 1 for almost every (x, y) in $(-1, 1)^2$, then $(0, 0)$ is a point of density of A.

Theorem 1 *In order that ($\star\star$) holds for each measurable set $A \subset \mathbb{R}^2$, it is necessary and sufficient that there is an $r > 0$ so that for each $n \in \mathbb{N}$, $t_{n+1} > rt_n$.*

In the case of strong density we shall need two decreasing sequences $\{t_n\}_{n\in\mathbb{N}}$, $\{s_n\}_{n\in\mathbb{N}}$ convergent to 0.

($\star\star_S$) Whenever $f(t_n x, s_k y)$ converges to 1 (as a double sequence) for almost every (x, y) in $(-1, 1)^2$, then $(0, 0)$ is a point of strong density of A.

Theorem 2 *In order that ($\star\star_s$) holds for each measurable set $A \subset R^2$ it is necessary and sufficient that there is an $r > 0$ so that for each $n \in \mathbb{N}$ $t_{n+1} > rt_n$ and $s_{n+1} > rs_n$.*

In the case of I-density and strong I-density the results are completely analogous.

However, the both cases, linear and planar, for each sequence $\{t_n\}_{n\in\mathbb{N}}$ decreasing to zero there exists a set $A \subset R$ (or $A \subset R^2$) such that 0 (or $(0, 0)$, respectively) is a point of density of A, but it is not true that $f(t_n x)$ ($f(t_n x, t_n y)$, respectively) converges to 1 for almost every $x \in (-1, 1)$ ($(x, y) \in (-1, 1)^2$, respectively).

For complete proofs see [5].

References

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[3] Saks,S., *Theory of the Integral*, Warsaw 1937.

[4] Wilczynski,W., *Sequence conditions which imply I-approximate continuity*, Tatra Mountains Math. Publ., 2 (1992), 1-5.

[5] Wilczynski, W. Wojdowski,W, *Sequence conditions which imply approximate and I-approximate continuities of functions of two variables*, submitted to Tatra Mountains Math. Publ.